Ab-exact extensions and Milnor fibrations of arrangements

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Lower central series

- The *lower central series* of a group G is defined inductively by $\gamma_1(G) = G$, $\gamma_2(G) = G'$, and $\gamma_{k+1}(G) = [G, \gamma_k(G)]$.
- It is an "N-series," i.e., $[\gamma_k(G), \gamma_\ell(G)] \subseteq \gamma_{k+\ell}(G), \quad \forall k, \ell \geqslant 1$.
- The terms are fully invariant subgroups (i.e., $\varphi\colon G \to H$ morphism $\Rightarrow \varphi(\gamma_k(G)) \subseteq \gamma_k(H)$), and thus, normal subgroups.
- The LCS quotients, $\operatorname{gr}_k(G) := \gamma_k(G)/\gamma_{k+1}(G)$, are abelian.
- Associated graded Lie algebra: $\operatorname{gr}(G) = \bigoplus_{k\geqslant 1} \operatorname{gr}_k(G)$, with Lie bracket $[\ ,\ ]\colon \operatorname{gr}_k \times \operatorname{gr}_\ell \to \operatorname{gr}_{k+\ell}$ induced by the group commutator.
- The factor groups $G/\gamma_{k+1}(G)$ are the maximal k-step nilpotent quotients of G.
- $G/\gamma_2(F) = G_{\mathrm{ab}}$, while $G/\gamma_3(G)$ is determined by $H^{\leqslant 2}(G, \mathbb{Z})$.

Derived series and Alexander invariants

- The *derived series* of G is defined inductively by $G^{(0)} = G$, $G^{(1)} = G'$, $G^{(2)} = G''$, and $G^{(r)} = [G^{(r-1)}, G^{(r-1)}]$.
- Its terms are fully invariant (thus, normal) subgroups.
- Successive quotients: $G^{(r-1)}/G^{(r)} = \big(G^{(r-1)}\big)_{\mathrm{ab}}$.
- $G/G^{(\ell)}$ is the maximal solvable quotient of G of length ℓ.
- *Alexander invariant*: $B(G) := G'/G''$, viewed as a $\mathbb{Z}G_{\mathrm{ab}}$-module via $gG' \cdot xG'' = gxg^{-1}G''$ for $g \in G$ and $x \in G'$.
- Assume now that G is finitely generated. Then $\mathbb{T}_G := \mathrm{Hom}(G, \mathbb{C}^*)$ is an algebraic group. Clearly, $\mathbb{T}_G = \mathbb{T}_{G_{\mathrm{ab}}}$.
- Characteristic varieties: $\mathcal{V}_k(G) := \{\rho \in \mathbb{T}_G \mid \dim H^1(G, \mathbb{C}_\rho) \geqslant k\}$. For a space X, set $\mathcal{V}_k(X) := \mathcal{V}_k(\pi_1(X))$.
- $\mathcal{V}_1(G) = V(\mathrm{ann}(B(G) \otimes \mathbb{C}))$, away from 1.

The complement of a hyperplane arrangement

- Let $\mathcal{A}$ be a central arrangement of n hyperplanes in $\mathbb{C}^d$. For each $H \in \mathcal{A}$ let α_H be a linear form with $\ker(\alpha_H) = H$; set $f = \prod_{H\in\mathcal{A}} \alpha_H$.
- The complement, $M(\mathcal{A}) := \mathbb{C}^d \setminus \bigcup_{H\in\mathcal{A}} H$, is a Stein manifold, and so has the homotopy type of a (connected) CW-complex of dim d.
- In fact, M has a minimal cell structure. Consequently, $H_*(M, \mathbb{Z})$ is torsion-free (and finitely generated).
- In particular, $H_1(M, \mathbb{Z}) = \mathbb{Z}^n$, generated by the meridians $\{x_H\}_{H\in\mathcal{A}}$.
- The cohomology ring $H^*(M, \mathbb{Z})$ is determined solely by the intersection lattice, $L(\mathcal{A})$. [Orlik–Solomon]
- The quasi-projective variety M admits a *pure* mixed Hodge structure, and so M is $\mathbb{Q}$-formal (albeit not $\mathbb{Z}_p$-formal, in general).

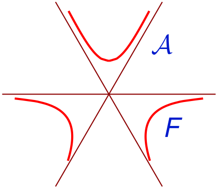
Fundamental groups of arrangements

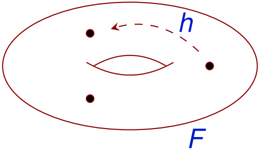
- For an arrangement $\mathcal{A}$, the group $G(\mathcal{A}) = \pi_1(M(\mathcal{A}))$ admits a finite presentation, with generators $\{x_H\}_{H\in\mathcal{A}}$ and commutator-relators.
- $\mathcal{V}_k(M)$ is a finite union of torsion-translated subtori of $\mathbb{T}_G = (\mathbb{C}^*)^n$.
- $G/\gamma_2(G)$ and $G/\gamma_3(G)$ are determined by $L_{\leqslant 2}(\mathcal{A})$.
- $G/\gamma_4(G)$—and thus G—is not necessarily determined by $L_{\leqslant 2}(\mathcal{A})$.
- If $\mathcal{A}$ is decomposable, though, *all* nilpotent quotients are combinatorially determined [Porter–S.]
- Since $M = M(\mathcal{A})$ is formal, $G = G(\mathcal{A})$ is 1-formal, i.e., its pronilpotent completion, $\mathfrak{m}(G)$, is quadratic.
- Hence, $\operatorname{gr}(G)\otimes\mathbb{Q} = \operatorname{gr}(\mathfrak{m}(G))$ is determined by $L_{\leqslant 2}(\mathcal{A})$.
- Let $\mathfrak{h}(G) = \operatorname{Lie}(G_{\rm ab})/\operatorname{im}(H_2(G,\mathbb{Z}) \xrightarrow{\cup^\vee} G_{\rm ab}\wedge G_{\rm ab})$ be the quadratic *(holonomy) Lie algebra associated to* $H^{\leqslant 2}(G,\mathbb{Z})$.

- Then $\mathfrak{h}(G) \twoheadrightarrow \mathrm{gr}(G)$ (always), and $\mathfrak{h}(G) \otimes \mathbb{Q} \xrightarrow{\simeq} \mathrm{gr}(G) \otimes \mathbb{Q}$ (since G is 1-formal).
- $U(\mathfrak{h}(G) \otimes \mathbb{Q}) = \mathrm{Ext}^1_A(\mathbb{Q}, \mathbb{Q}) = \overline{A}^!$, where $\overline{A}$ is the quadratic closure of $A = H^*(M, \mathbb{Q})$.
- An explicit combinatorial formula is lacking in general for the LCS ranks $\phi_k := \mathrm{rank}\,\mathrm{gr}_k(G)$, although such formulas are known when
  - $\mathcal{A}$ is supersolvable $\Rightarrow H^*(M, \mathbb{Q})$ is Koszul
  - $\mathcal{A}$ is decomposable ($\mathrm{gr}_3(G)$ is as predicted by $\mu\colon L_2(\mathcal{A}) \to \mathbb{Z}$)
  - $\mathcal{A}$ is a graphic arrangement

  and in some more cases just for ϕ_3.
- $\mathrm{gr}_k(G)$ may have torsion (at least for $k \geqslant 4$), but the torsion is not necessarily determined by $L_{\leqslant 2}(\mathcal{A})$.
- The map $\mathfrak{h}_3(G) \to \mathrm{gr}_3(G)$ is an isomorphism [Porter–S.], but it is not known whether $\mathfrak{h}_3(G)$ is torsion-free.
- The Chen ranks $\theta_k(G) := \mathrm{rank}\,\mathrm{gr}_k(G/G'')$ are also combinatorially determined.

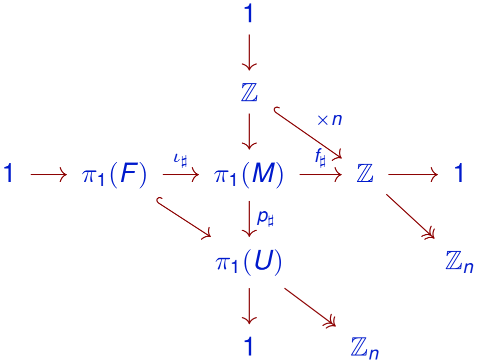
Milnor fibration





- The map $f\colon \mathbb{C}^d \to \mathbb{C}$ restricts to a smooth fibration, $f\colon M \to \mathbb{C}^*$, called the *Milnor fibration* of $\mathcal{A}$.
- The *Milnor fiber* is $F(\mathcal{A}) := f^{-1}(1)$. The monodromy, $h\colon F \to F$, is given by $h(z) = e^{2\pi i/n}z$, where $n = |\mathcal{A}|$.
- F is a Stein manifold. It has the homotopy type of a finite CW-complex of dimension $d-1$ (connected if $d > 1$).
- F is the regular, $\mathbb{Z}_n$-cover of $U = \mathbb{P}(M)$, classified by the projection $\pi_1(U) \twoheadrightarrow \mathbb{Z}_n$, $x_H \mapsto 1$.
- To understand $\pi_1(F)$, we may assume wlog that $d = 3$.

- Let $\iota\colon F \hookrightarrow M$ be the inclusion. Induced maps on π_1:



- $b_1(F) \geqslant n-1$, and may be computed from $\mathcal{V}^1_k(U)$. Combinatorial formulas are known in some cases (e.g., if $\mathbb{P}(\mathcal{A})$ has only double or triple points [Papadima–S.]), but not in general.
- MHS on F may not be pure; $\pi_1(F)$ may be non-1-formal [Zuber].
- $H_1(F,\mathbb{Z})$ may have torsion [Yoshinaga].

Exact sequences and lower central series

- A short exact sequence of groups,

$$1 \longrightarrow K \xrightarrow{\iota} G \xrightarrow{\pi} Q \longrightarrow 1 \qquad (*)$$

  yields
  - A representation $\varphi\colon Q \to \mathrm{Out}(K)$.
  - A "monodromy" representation $\bar{\varphi}\colon Q \to \mathrm{Aut}(K_{\mathrm{ab}})$.
- If (*) admits a splitting, $\sigma\colon Q \to G$, then $G = K \rtimes_{\varphi} Q$, where $\varphi\colon Q \to \mathrm{Aut}(K)$, $x \mapsto$ conjugation by $\sigma(x)$.
- (*) is *ab-exact* if $0 \longrightarrow K_{\mathrm{ab}} \xrightarrow{\iota_{\mathrm{ab}}} G_{\mathrm{ab}} \xrightarrow{\pi_{\mathrm{ab}}} Q_{\mathrm{ab}} \longrightarrow 0$ is also exact; equivalently, Q acts trivially on K_{ab} and ι_{ab} is injective.

THEOREM (FALK-RANDELL)

Let $G = K \rtimes_{\varphi} Q$. If Q acts trivially on K_{ab}, then
- $\gamma_k(G) = \gamma_k(K) \rtimes_{\varphi} \gamma_k(Q)$, *for all* $k \geqslant 1$.
- $\mathrm{gr}(G) = \mathrm{gr}(K) \rtimes_{\bar{\varphi}} \mathrm{gr}(Q)$.

Theorem

Let $1 \to K \xrightarrow{\iota} G \to Q \to 1$ *be a split-exact and* ab*-exact sequence. Assume* Q *is abelian. Then*

- $K' = G'$.
- $B(\iota)\colon B(K) \to B(G)$ *is a* $\mathbb{Z}K_{\mathrm{ab}}$*-linear isomorphism.*
- $\iota^*\colon \mathbb{T}_G \twoheadrightarrow \mathbb{T}_K$ *restricts to a surjection* $\iota^*\colon \mathcal{V}_1(G) \twoheadrightarrow \mathcal{V}_1(K)$.
- $\mathrm{gr}'(\iota)\colon \mathrm{gr}'(K) \xrightarrow{\simeq} \mathrm{gr}'(G)$ *and* $\mathrm{gr}'(\bar{\iota})\colon \mathrm{gr}'(K/K'') \xrightarrow{\simeq} \mathrm{gr}'(G/G'')$.

Corollary

If $\iota_*\colon H_1(F,\mathbb{Z}) \to H_1(M,\mathbb{Z})$ *is injective, then*

- $\iota^*\colon \mathbb{T}_M \twoheadrightarrow \mathbb{T}_F$ *restricts to surjection* $\iota^*\colon \mathcal{V}_1(M) \twoheadrightarrow \mathcal{V}_1(F)$.
- $\phi_k(F) = \phi_k(M)$ *for* $k \geqslant 2$.
- $\theta_k(F) = \theta_k(M)$ *for* $k \geqslant 2$.

The rational lower central series

- The *rational lower central series* of G is defined by $\gamma_1^{\mathbb{Q}} G = G$ and $\gamma_{k+1}^{\mathbb{Q}} G = \sqrt{[G, \gamma_k^{\mathbb{Q}} G]}$. [Stallings]
- Its terms are fully invariant subgroups.
- This is the fastest descending N-series whose successive quotients, $\gamma_k^{\mathbb{Q}} G / \gamma_{k+1}^{\mathbb{Q}} G$, are torsion-free abelian
- $G/\gamma_2^{\mathbb{Q}} G = G_{\mathrm{abf}}$, where $G_{\mathrm{abf}} = G_{\mathrm{ab}} / \operatorname{Tors}(G_{\mathrm{ab}})$ is the maximal torsion-free abelian quotient of G.
- Associated graded Lie algebra: $\operatorname{gr}^{\mathbb{Q}}(G) = \bigoplus_{k\geqslant 1} \gamma_k^{\mathbb{Q}} G / \gamma_{k+1}^{\mathbb{Q}} G$.

Theorem

Let $G = K \rtimes_{\varphi} Q$ be a split extension. If Q acts trivially on K_{abf}, then,

- $\gamma_k^{\mathbb{Q}}(G) = \gamma_k^{\mathbb{Q}}(K) \rtimes_{\varphi} \gamma_k^{\mathbb{Q}}(Q)$, *for all* $k \geqslant 1$.
- $\operatorname{gr}^{\mathbb{Q}}(G) = \operatorname{gr}^{\mathbb{Q}}(K) \rtimes_{\bar{\varphi}} \operatorname{gr}^{\mathbb{Q}}(Q)$.

The rational derived series

- The *rational derived series* of G is defined by $G^{(0)}_{\mathbb{Q}} = G$ and $G^{(r)}_{\mathbb{Q}} = \sqrt{\big[G^{(r-1)}_{\mathbb{Q}}, G^{(r-1)}_{\mathbb{Q}}\big]}$. [Stallings, Harvey, Cochran]
- $G^{(r)}_{\mathbb{Q}}/G^{(r+1)}_{\mathbb{Q}} \cong \big(G^{(r)}_{\mathbb{Q}}\big)_{\mathrm{abf}}$. In particular, $G/G'_{\mathbb{Q}} = G_{\mathrm{abf}}$.
- $B_{\mathbb{Q}}(G) := G'_{\mathbb{Q}}/G''_{\mathbb{Q}}$, viewed as a module over $\mathbb{Z}G_{\mathrm{abf}}$.
- $V(\mathrm{ann}(B_{\mathbb{Q}}(G) \otimes \mathbb{C})) = \mathcal{V}_1(G) \cap \mathbb{T}^0_G$ away from 1.

Theorem

Let $1 \to K \xrightarrow{\iota} G \to Q \to 1$ *be a split-exact and* abf*-exact sequence. Assume* Q *is abelian. Then*

- $K'_{\mathbb{Q}} = G'_{\mathbb{Q}}$.
- $B_{\mathbb{Q}}(\iota)\colon B_{\mathbb{Q}}(K) \to B_{\mathbb{Q}}(G)$ *is a* $\mathbb{Z}K_{\mathrm{abf}}$*-linear isomorphism.*
- $\iota^*\colon \mathbb{T}^0_G \twoheadrightarrow \mathbb{T}^0_K$ *restricts to surjection* $\iota^*\colon \mathcal{V}_1(G) \cap \mathbb{T}^0_G \twoheadrightarrow \mathcal{V}_1(K) \cap \mathbb{T}^0_K$.
- $\mathrm{gr}'(\iota)\colon \mathrm{gr}'_{\mathbb{Q}}(K) \xrightarrow{\simeq} \mathrm{gr}'_{\mathbb{Q}}(G)$ *and* $\mathrm{gr}'(\bar{\iota})\colon \mathrm{gr}'_{\mathbb{Q}}(K/K''_{\mathbb{Q}}) \xrightarrow{\simeq} \mathrm{gr}'_{\mathbb{Q}}(G/G''_{\mathbb{Q}})$.

The p-lower central series

- The *p-lower central series* of G is defined by $\gamma_1^p G = G$ and $\gamma_{k+1}^p G = (\gamma_k^p G)^p [G, \gamma_k^p G]$. [Stallings]
- Fastest descending N-series whose successive quotients, $\gamma_k^p G/\gamma_{k+1}^p G$, are $\mathbb{Z}_p$-vector spaces.
- Its terms are fully invariant subgroups, and $G/\gamma_2^p G = H_1(G, \mathbb{Z}_p)$.
- Associated graded Lie algebra: $\mathrm{gr}^p(G) = \bigoplus_{k\geqslant 1} \gamma_k^p G/\gamma_{k+1}^p G$.

Theorem (Bellingeri–Gervais)

Let $G = K \rtimes_\varphi Q$. If Q acts trivially on $H_1(K, \mathbb{Z}_p)$, then,

- $\gamma_k^p(G) = \gamma_k^p(K) \rtimes_\varphi \gamma_k^p(Q)$, *for all* $k \geqslant 1$.
- $\mathrm{gr}^p(G) = \mathrm{gr}^p(K) \rtimes_{\bar\varphi} \mathrm{gr}^p(Q)$.

The derived p-series

- *Derived p-series*: $G_p^{(0)} = G$ and $G_p^{(r)} = (G_p^{(r-1)})^p[G_p^{(r-1)}, G_p^{(r-1)}]$. [Stallings, Cochran–Harvey, Lackenby]
- $G_p^{(r-1)}/G_p^{(r)} \cong H_1(G_p^{(r-1)}, \mathbb{Z}_p)$. In particular, $G/G'_p = H_1(G, \mathbb{Z}_p)$.
- If G is finitely generated, then $G/G_p^{(r)}$ is a finite p-group, with all elements having order dividing p^r.
- $\mathbb{Z}_p$-Alexander invariant: $B_p(G) := G'_p/G''_p$, as $\mathbb{Z}[H_1(G, \mathbb{Z}_p)]$-module.

THEOREM

Let $1 \to K \xrightarrow{\iota} G \to Q \to 1$ be a split-exact and p-exact sequence. Assume Q is abelian and has no p-torsion. Then,

- $K'_p = G'_p$.
- $B_p(\iota)\colon B_p(K) \to B_p(G)$ *is a* $\mathbb{Z}[H_1(K, \mathbb{Z}_p)]$*-linear isomorphism.*
- $\operatorname{gr}'(\iota)\colon \operatorname{gr}'_p(K) \xrightarrow{\simeq} \operatorname{gr}'_p(G)$ *and* $\operatorname{gr}'(\bar{\iota})\colon \operatorname{gr}'_p(K/K''_p) \xrightarrow{\simeq} \operatorname{gr}'_p(G/G''_p)$.

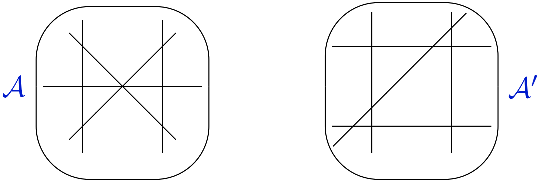
Formality properties

- Let $Y \to X$ be a finite, regular cover, with deck group Γ. If Y is 1-formal, then X is 1-formal, but the converse is not true.
- (Dimca–Papadima) If Γ acts trivially on $H_1(Y, \mathbb{Q})$, then the converse holds.
- Applying to $\mathbb{Z}_n$-cover $F(\mathcal{A}) \to U(\mathcal{A})$: if the Milnor fibration of $\mathcal{A}$ has trivial $\mathbb{Q}$-monodromy, then F is 1-formal.
- (S.–He Wang) Let $1 \to K \to G \to Q \to 1$ be an exact sequence. If G is 1-formal and retracts onto K, then K is also 1-formal.
- (Papadima–S.) Let $1 \to K \to G \to \mathbb{Z} \to 1$ be an exact sequence. Assume G is 1-formal and $b_1(K) < \infty$. Then the eigenvalue 1 of the monodromy action on $H_1(K, \mathbb{C})$ has only 1×1 Jordan blocks.

THEOREM

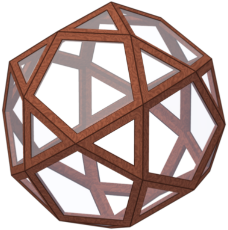
Let $1 \to K \to G \to Q \to 1$ be a split-exact and abf-exact sequence. If G is 1-formal and K is finitely generated, then K is 1-formal.

Falk's pair of arrangements



- Both $\mathcal{A}$ and $\mathcal{A}'$ have 2 triple points and 9 double points, yet $L(\mathcal{A}) \not\cong L(\mathcal{A}')$. Nevertheless, $M(\mathcal{A}) \simeq M(\mathcal{A}')$.
- $\mathcal{V}_1(M)$ and $\mathcal{V}_1(M')$ consist of two 2-dimensional subtori of $(\mathbb{C}^*)^6$, corresponding to the triple points; $\mathcal{V}_2(M) = \mathcal{V}_2(M') = \{1\}$.
- Both Milnor fibrations have trivial $\mathbb{Z}$-monodromy.
- $\mathcal{V}_1(F)$ and $\mathcal{V}_1(F')$ consist of two 2-dimensional subtori of $(\mathbb{C}^*)^5$.
- On the other hand, $\mathcal{V}_2(F) \cong \mathbb{Z}_3$, yet $\mathcal{V}_2(F') = \{1\}$.
- Thus, $\pi_1(F) \not\cong \pi_1(F')$.

Yoshinaga's icosidodecahedral arrangement



- The icosidodecahedron is a quasiregular polyhedron in $\mathbb{R}^3$, with 20 triangular and 12 pentagonal faces, 60 edges, and 30 vertices, given by the even permutations of $(0, 0, \pm 1)$ and $\frac{1}{2}(\pm 1, \pm\phi, \pm\phi^2)$, where $\phi = (1 + \sqrt{5})/2$.
- One can choose 10 edges to form a decagon; there are 6 ways to choose these decagons, thereby giving 6 planes.
- Each pentagonal face has five diagonals; there are 60 such diagonals in all, and they partition in 10 disjoint sets of coplanar ones, thereby giving 10 planes, each containing 6 diagonals.

- These 16 planes form a arrangement $\mathcal{A}_{\mathbb{R}}$ in $\mathbb{R}^3$, whose complexification is the icosidodecahedral arrangement $\mathcal{A}$ in $\mathbb{C}^3$.
- The complement M is a $K(\pi, 1)$. Moreover, $P_U(t) = 1 + 15t + 60t^2$; thus, $\chi(U) = 36$ and $\chi(F) = 576$.
- In fact, $H_1(F, \mathbb{Z}) = \mathbb{Z}^{15} \oplus \mathbb{Z}_2$. Thus, the algebraic monodromy of the Milnor fibration is trivial over $\mathbb{Q}$ and $\mathbb{Z}_p$ ($p > 2$), but not over $\mathbb{Z}$.
- Hence, by the preceding results, $\mathrm{gr}(\pi_1(F)) \cong \mathrm{gr}(\pi_1(U))$, away from the prime 2.
- Computer computation:
  - $\mathrm{gr}_1(\pi_1(F)) = \mathbb{Z}^{15} \oplus \mathbb{Z}_2$
  - $\mathrm{gr}_2(\pi_1(F)) = \mathbb{Z}^{45} \oplus \mathbb{Z}_2^7$
  - $\mathrm{gr}_3(\pi_1(F)) = \mathbb{Z}^{250} \oplus \mathbb{Z}_2^{43}$
  - $\mathrm{gr}_4(\pi_1(F)) = \mathbb{Z}^{1405} \oplus \mathbb{Z}_2^{?}$