Cohomology jump loci of 3-dimensional manifolds

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Resonance varieties

- Let $A = (A^\bullet, d)$ be a graded-commutative, differential graded algebra (cdga) over a field $\Bbbk$. We assume A is connected ($A^0 = \Bbbk$) and of finite-type ($\dim_\Bbbk A^i < \infty$, $\forall i$).
- Since $A^0 = \Bbbk$, we have $Z^1(A) \cong H^1(A)$. Set

$$\mathcal{Q}(A) = \{a \in Z^1(A) \mid a^2 = 0 \in A^2\}.$$

- For each $a \in \mathcal{Q}(A)$, we then have a cochain complex,

$$(A^\bullet, \delta_a)\colon \ A^0 \xrightarrow{\delta_a^0} A^1 \xrightarrow{\delta_a^1} A^2 \xrightarrow{\delta_a^2} \cdots,$$

with differentials $\delta_a^i(u) = a \cdot u + d(u)$, for all $u \in A^i$.

- The *resonance varieties* of A (in degree $i \geqslant 0$ and depth $k \geqslant 0$):

$$\mathcal{R}^i_k(A) = \{a \in \mathcal{Q}(A) \mid \dim_\Bbbk H^i(A^\bullet, \delta_a) \geqslant k\}.$$

- $\mathrm{TC}_0(\mathcal{R}^i_k(A)) \subseteq \mathcal{R}^i_k(H^\bullet(A))$, but not $=$ in general.

Resonance varieties of graded algebras

- Now let A be a graded, graded-commutative $\Bbbk$-algebra (cga). We will assume A is connected and of finite-type (with $d = 0$), and $\operatorname{char} \Bbbk \neq 2$.
- For each $a \in A^1$ we have $a^2 = -a^2$, and so $a^2 = 0$. Thus, $\mathcal{Q}(A) = A^1$, and the differentials in $(A^\bullet, \delta_a)$ are given by $\delta_a^i(u) = a \cdot u$.
- In this case, the resonance varieties $\mathcal{R}^i_k(A)$ are *homogeneous* subvarieties of the affine space A^1.
- An element $a \in A^1$ belongs to $\mathcal{R}^i_k(A)$ if and only if there exist $u_1, \dots, u_k \in A^i$ such that $au_1 = \cdots = au_k = 0$ in A^{i+1}, and the set $\{au, u_1, \dots, u_k\}$ is linearly independent in A^i, for all $u \in A^{i-1}$.
- Set $b_j = b_j(A)$. For each $i \geqslant 0$, we have a descending filtration,

$$A^1 = \mathcal{R}^i_0(A) \supseteq \mathcal{R}^i_1(A) \supseteq \cdots \supseteq \mathcal{R}^i_{b_i}(A) = \{0\} \supset \mathcal{R}^i_{b_{i+1}}(A) = \emptyset.$$

- A linear subspace $U \subset A^1$ is *isotropic* if the restriction of $A^1 \wedge A^1 \xrightarrow{\cdot} A^2$ to $U \wedge U$ is the zero map (i.e., $ab = 0$, $\forall a, b \in U$).
- If $U \subseteq A^1$ is an isotropic subspace of dimension k, then $U \subseteq \mathcal{R}^1_{k-1}(A)$.
- $\mathcal{R}^1_1(A)$ is the union of all isotropic planes in A^1.
- If $\Bbbk \subset \mathbb{K}$ is a field extension, then the $\Bbbk$-points on $\mathcal{R}^i_k(A \otimes_{\Bbbk} \mathbb{K})$ coincide with $\mathcal{R}^i_k(A)$.
- Let $\varphi\colon A \to B$ be a morphism of cgas. If the map $\varphi^1\colon A^1 \to B^1$ is injective, then $\varphi^1(\mathcal{R}^1_k(A)) \subseteq \mathcal{R}^1_k(B)$, for all k.

The BGG correspondence

- Fix a $\Bbbk$-basis $\{e_1, \dots, e_n\}$ for A^1, and let $\{x_1, \dots, x_n\}$ be the dual basis for $A_1 = (A^1)^*$.

- Identify $\mathrm{Sym}(A_1)$ with $S = \Bbbk[x_1, \dots, x_n]$, the coordinate ring of the affine space A^1.

- The BGG correspondence yields a cochain complex of finitely generated, free S-modules, $\mathbf{L}(A) := (A^\bullet \otimes_\Bbbk S, \delta)$,

$$\cdots \longrightarrow A^i \otimes_\Bbbk S \xrightarrow{\delta_A^i} A^{i+1} \otimes_\Bbbk S \xrightarrow{\delta_A^{i+1}} A^{i+2} \otimes_\Bbbk S \longrightarrow \cdots,$$

  where $\delta_A^i(u \otimes s) = \sum_{j=1}^n e_j u \otimes s x_j$.

- The specialization of $(A \otimes_\Bbbk S, \delta)$ at $a \in A^1$ coincides with (A, δ_a), that is, $\delta_A^i\big|_{x_j = a_j} = \delta_a^i$.

- By definition, an element $a \in A^1$ belongs to $\mathcal{R}^i_k(A)$ if and only if

$$\operatorname{rank} \delta_a^{i-1} + \operatorname{rank} \delta_a^i \leqslant b_i(A) - k.$$

- Let $I_r(\psi)$ denote the ideal of $r \times r$ minors of a $p \times q$ matrix ψ with entries in S, where $I_0(\psi) = S$ and $I_r(\psi) = 0$ if $r > \min(p, q)$. Then:

$$\begin{aligned} \mathcal{R}^i_k(A) &= V\Big(I_{b_i(A)-k+1}\big(\delta_A^{i-1} \oplus \delta_A^i\big)\Big) \\ &= \bigcap_{s+t=b_i(A)-k+1} \Big(V\big(I_s(\delta_A^{i-1})\big) \cup V\big(I_t(\delta_A^i)\big)\Big). \end{aligned}$$

- In particular, $\mathcal{R}^1_k(A) = V(I_{n-k}(\delta_A^1))$ $(0 \leqslant k < n)$ and $\mathcal{R}^1_n(A) = \{0\}$.

- The (degree i, depth k) *resonance scheme* $\boldsymbol{\mathcal{R}}^i_k(A)$ is defined by the ideal $I_{b_i(A)-k+1}\big(\delta_A^{i-1} \oplus \delta_A^i\big)$; its underlying set is $\mathcal{R}^i_k(A)$.

Poincaré duality algebras

- Let A be a connected, finite-type $\Bbbk$-cga.
- A is a *Poincaré duality* $\Bbbk$*-algebra* of dimension m if there is a $\Bbbk$-linear map $\varepsilon\colon A^m \to \Bbbk$ (called an *orientation*) such that all the bilinear forms $A^i \otimes_{\Bbbk} A^{m-i} \to \Bbbk$, $a \otimes b \mapsto \varepsilon(ab)$ are non-singular.
- We then have:
  - $b_i(A) = b_{m-i}(A)$, and $A^i = 0$ for $i > m$.
  - ε is an isomorphism.
  - The maps $\mathrm{PD}\colon A^i \to (A^{m-i})^*$, $\mathrm{PD}(a)(b) = \varepsilon(ab)$ are isos.
- Each $a \in A^i$ has a *Poincaré dual*, $a^\vee \in A^{m-i}$, such that $\varepsilon(aa^\vee) = 1$.
- The *orientation class* is $\omega_A := 1^\vee$.
- We have $\varepsilon(\omega_A) = 1$, and thus $aa^\vee = \omega_A$.

The associated alternating form

- Associated to a $\Bbbk$-PD_m algebra there is an alternating m-form,

$$\mu_A \colon \bigwedge^m A^1 \to \Bbbk, \quad \mu_A(a_1 \wedge \cdots \wedge a_m) = \varepsilon(a_1 \cdots a_m).$$

- Assume now that $m = 3$, and set $n = b_1(A)$. Fix a basis $\{e_1, \dots, e_n\}$ for A^1, and let $\{e_1^\vee, \dots, e_n^\vee\}$ be the dual basis for A^2.
- The multiplication in A, then, is given on basis elements by

$$e_i e_j = \sum_{k=1}^{r} \mu_{ijk}\, e_k^\vee, \quad e_i e_j^\vee = \delta_{ij}\omega,$$

where $\mu_{ijk} = \mu(e_i \wedge e_j \wedge e_k)$.

- Let $A_i = (A^i)^*$. We may view μ dually as a trivector,

$$\mu = \sum \mu_{ijk}\, e^i \wedge e^j \wedge e^k \in \bigwedge^3 A_1,$$

which encodes the algebra structure of A.

Classification of alternating forms

- Let V be a $\mathbb{k}$-vector space of dimension n. The group $\mathrm{GL}(V)$ acts on $\bigwedge^m(V^*)$ by $(g\cdot\mu)(a_1\wedge\cdots\wedge a_m)=\mu\left(g^{-1}a_1\wedge\cdots\wedge g^{-1}a_m\right)$.

- The orbits of this action are the equivalence classes of alternating m-forms on V. (We write $\mu\sim\mu'$ if $\mu'=g\cdot\mu$.)

- Over $\overline{\mathbb{k}}$, the closures of these orbits are affine algebraic varieties; there are finitely many orbits only if $m\leqslant 2$ or $m=3$ and $n\leqslant 8$.

- Each complex orbit has only finitely many real forms. When $m=3$, and $n=8$, there are 23 complex orbits, which split into either 1, 2, or 3 real orbits, for a total of 35 real orbits.

- There is a bijection between isomorphism classes of 3-dimensional Poincaré duality algebras and equivalence classes of alternating 3-forms, given by $A \leftrightsquigarrow \mu_A$.

Poincaré duality in orientable manifolds

- Let M be a compact, connected, orientable, m-dimensional manifold. Then the cohomology ring $A = H^{\bullet}(M, \Bbbk)$ is a PD_m algebra over $\Bbbk$.
- Sullivan (1975): for every finite-dimensional $\mathbb{Q}$-vector space V and every alternating 3-form $\mu \in \bigwedge^3 V^*$, there is a closed 3-manifold M with $H^1(M, \mathbb{Q}) = V$ and cup-product form $\mu_M = \mu$.
- Such a 3-manifold can be constructed via "Borromean surgery."
- E.g., 0-surgery on the Borromean rings in S^3 yields $M = T^3$, with $\mu_M = e^1 e^2 e^3$.
- If M is the link of an isolated surface singularity (e.g., if $M = \Sigma(p, q, r)$ is a Brieskorn manifold), then $\mu_M = 0$.

Resonance varieties of PD-algebras

- Let A be a PD_m algebra. For $0 \leqslant i \leqslant m$ and $a \in A^1$, the following diagram commutes up to a sign.

$$\begin{tikzcd}
(A^{m-i})^* \arrow[r, "(\delta_{-a}^{m-i-1})^*"] & (A^{m-i-1})^* \\
A^i \arrow[u, "\mathrm{PD}", "\cong"'] \arrow[r, "\delta_a^i"] & A^{i+1} \arrow[u, "\mathrm{PD}", "\cong"']
\end{tikzcd}$$

- Consequently, $\big(H^i(A,\delta_a)\big)^* \cong H^{m-i}(A,\delta_{-a})$.

- Hence, $\mathcal{R}^i_k(A) = \mathcal{R}^{m-i}_k(A)$ for all i and k. In particular, $\mathcal{R}^m_1(A) = \mathcal{R}^0_1(A) = \{0\}$.

Corollary

Let A be a PD_3 algebra with $b_1(A) = n$. Then $\mathcal{R}^i_k(A) = \emptyset$, except for:

- $\mathcal{R}^i_0(A) = A^1$ *for all* $i \geqslant 0$.
- $\mathcal{R}^3_1(A) = \mathcal{R}^0_1(A) = \{0\}$ *and* $\mathcal{R}^2_n(A) = \mathcal{R}^1_n(A) = \{0\}$.
- $\mathcal{R}^2_k(A) = \mathcal{R}^1_k(A)$ *for* $0 < k < n$.

- A linear subspace $U \subset V$ is 2-*singular* with respect to a 3-form $\mu\colon \bigwedge^3 V \to \Bbbk$ if $\mu(a \wedge b \wedge c) = 0$ for all $a, b \in U$ and $c \in V$.

- The *rank* of $\mu\colon \bigwedge^3 V \to \Bbbk$ is the minimum dimension of a linear subspace $W \subset V$ such that μ factors through $\bigwedge^3 W$. The *nullity* of μ is the maximum dimension of a 2-singular subspace $U \subset V$.

- Clearly, V contains a singular plane if and only if $\operatorname{null}(\mu) \geqslant 2$.

- Let A be a PD_3 algebra. A linear subspace $U \subset A^1$ is 2-singular (with respect to μ_A) if and only if U is isotropic.

- Using a result of A. Sikora [2005], we obtain:

Theorem

Let A be a PD_3 algebra over an algebraically closed field $\Bbbk$ with $\operatorname{char}(\Bbbk) \neq 2$, and let $\nu = \operatorname{null}(\mu_A)$. If $b_1(A) \geqslant 4$, then

$$\dim \mathcal{R}^1_{\nu-1}(A) \geqslant \nu \geqslant 2.$$

In particular, $\dim \mathcal{R}^1_1(A) \geqslant \nu$.

Real forms and resonance

- Sikora made the following conjecture: If $\mu\colon \bigwedge^3 V \to \Bbbk$ is a 3-form with $\dim V \geqslant 4$ and if $\operatorname{char}(\Bbbk) \neq 2$, then $\operatorname{null}(\mu) \geqslant 2$.
- Conjecture holds if $n := \dim V$ is even or equal to 5, or if $\Bbbk = \overline{\Bbbk}$.
- Work of J. Draisma and R. Shaw [2010, 2014] implies that the conjecture does not hold for $\Bbbk = \mathbb{R}$ and $n = 7$. We obtain:

Theorem

Let A be a PD_3 algebra over $\mathbb{R}$. Then $\mathcal{R}^1_1(A) \neq \{0\}$, except when

- $n = 1$, $\mu_A = 0$.
- $n = 3$, $\mu_A = e^1e^2e^3$.
- $n = 7$, $\mu_A = -e^1e^3e^5 + e^1e^4e^6 + e^2e^3e^6 + e^2e^4e^5 + e^1e^2e^7 + e^3e^4e^7 + e^5e^6e^7$.

Sketch: If $\mathcal{R}^1_1(A) = \{0\}$, then the formula $(x \times y) \cdot z = \mu_A(x, y, z)$ defines a cross-product on $A^1 = \mathbb{R}^n$, and thus a division algebra structure on $\mathbb{R}^{n+1}$, forcing $n = 1, 3$ or 7 by Bott–Milnor/Kervaire [1958].

Example

- Let A be the real PD_3 algebra corresponding to octonionic multiplication (the case $n = 7$ above).

- Let A' be the real PD_3 algebra with $\mu_{A'} = e^1e^2e^3 + e^4e^5e^6 + e^1e^4e^7 + e^2e^5e^7 + e^3e^6e^7$.

- Then $\mu_A \sim \mu_{A'}$ over $\mathbb{C}$, and so $A \otimes_{\mathbb{R}} \mathbb{C} \cong A' \otimes_{\mathbb{R}} \mathbb{C}$.

- On the other hand, $A \ncong A'$ over $\mathbb{R}$, since $\mu_A \nsim \mu_{A'}$ over $\mathbb{R}$, but also because $\mathcal{R}^1_1(A) = \{0\}$, yet $\mathcal{R}^1_1(A') \neq \{0\}$.

- Both $\mathcal{R}^1_1(A \otimes_{\mathbb{R}} \mathbb{C})$ and $\mathcal{R}^1_1(A' \otimes_{\mathbb{R}} \mathbb{C})$ are projectively smooth conics, and thus are projectively equivalent over $\mathbb{C}$, but

$$\mathcal{R}^1_1(A \otimes_{\mathbb{R}} \mathbb{C}) = \{x \in \mathbb{C}^7 \mid x_1^2 + \cdots + x_7^2 = 0\}$$

has only one real point ($x = 0$), whereas

$$\mathcal{R}^1_1(A' \otimes_{\mathbb{R}} \mathbb{C}) = \{x \in \mathbb{C}^7 \mid x_1x_4 + x_2x_5 + x_3x_6 = x_7^2\}$$

contains the real (isotropic) subspace $\{x_4 = x_5 = x_6 = x_7 = 0\}$.

Pfaffians and resonance

Let A be a $\Bbbk$-PD_3 algebra with $b_1(A) = n$. The cochain complex $\mathbf{L}(A) = (A \otimes_{\Bbbk} S, \delta_A)$ then looks like

$$A^0 \otimes_{\Bbbk} S \xrightarrow{\delta_A^0} A^1 \otimes_{\Bbbk} S \xrightarrow{\delta_A^1} A^2 \otimes_{\Bbbk} S \xrightarrow{\delta_A^2} A^3 \otimes_{\Bbbk} S\ ,$$

where $\delta_A^0 = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}$ and $\delta_A^2 = (\delta_A^0)^{\top}$, while δ_A^1 is the skew- symmetric matrix whose are entries linear forms in S given by

$$\delta_A^1(e_i) = \sum\nolimits_{j=1}^{n} \sum\nolimits_{k=1}^{n} \mu_{jik} e_k^{\vee} \otimes x_j\ .$$

Theorem

We have $\mathcal{R}^1_{2k}(A) = \mathcal{R}^1_{2k+1}(A) = V(\mathrm{Pf}_{n-2k}(\delta_A^1))$ if n is even and $\mathcal{R}^1_{2k-1}(A) = \mathcal{R}^1_{2k}(A) = V(\mathrm{Pf}_{n-2k+1}(\delta_A^1))$ if n is odd. Moreover, if μ_A has maximal rank $n \geqslant 3$, then

$$\mathcal{R}^1_{n-2}(A) = \mathcal{R}^1_{n-1}(A) = \mathcal{R}^1_{n}(A) = \{0\}.$$

Suppose $\dim_{\Bbbk} V = 2g+1 > 1$. We say $\mu\colon \bigwedge^3 V \to \Bbbk$ is *generic* (in the sense of Berceanu–Papadima [1994]) if there is a $v \in V$ such that the 2-form $\gamma_v \in V^* \wedge V^*$ given by $\gamma_v(a \wedge b) = \mu_A(a \wedge b \wedge v)$ for $a, b \in V$ has rank $2g$, that is, $\gamma_v^g \neq 0$ in $\bigwedge^{2g} V^*$.

Theorem

Let A be a PD_3 algebra with $b_1(A) = n$. Then

$$\mathcal{R}_1^1(A) = \begin{cases} \emptyset & \textit{if } n = 0; \\ \{0\} & \textit{if } n = 1 \textit{ or } n = 3 \textit{ and } \mu \textit{ has rank } 3; \\ V(\mathrm{Pf}(\mu_A)) & \textit{if } n \textit{ is odd, } n > 3, \textit{ and } \mu_A \textit{ is BP-generic;} \\ A^1 & \textit{otherwise,} \end{cases}$$

where $\mathrm{Pf}(\mu_A)$ is the Pffafian of μ_A, as defined by Turaev [2002].

Example

Let $M = \Sigma_g \times S^1$, where $g \geqslant 2$. Then $\mu_M = \sum_{i=1}^g a_i b_i c$ is BP-generic, and $\mathrm{Pf}(\mu_M) = x_{2g+1}^{g-1}$. Hence, $\mathcal{R}_1^1(M) = \{x_{2g+1} = 0\}$. In fact,

$$\mathcal{R}_1^1 = \cdots = \mathcal{R}_{2g-2}^1 \text{ and } \mathcal{R}_{2g-1}^1 = \mathcal{R}_{2g}^1 = \mathcal{R}_{2g+1}^1 = \{0\}.$$

As a corollary, we recover a closely related result, proved by Draisma and Shaw [2010] by very different methods.

Corollary

Let V be a $\Bbbk$-vector space of odd dimension $n \geqslant 5$ and let $\mu \in \bigwedge^3 V^*$. Then the union of all singular planes is either all of V or a hypersurface defined by a homogeneous polynomial in $\Bbbk[V]$ of degree $(n-3)/2$.

For $\mu \in \bigwedge^3 V^*$, there is another genericity condition, due to P. De Poi, D. Faenzi, E. Mezzetti, and K. Ranestad [2017]: $\operatorname{rank}(\gamma_v) > 2$, for all non-zero $v \in V$. We may interpret some of their results, as follows.

Theorem (DFMR)

Let A be a PD_3 algebra over $\mathbb{C}$, and suppose μ_A is generic. Then:

- *If n is odd, then $\mathcal{R}^1_1(A)$ is a hypersurface of degree $(n-3)/2$ which is smooth if $n \leqslant 7$, and singular in codimension 5 if $n \geqslant 9$.*
- *If n is even, then $\mathcal{R}^1_2(A)$ has codim 3 and degree $\frac{1}{4}\binom{n-2}{3}+1$; it is smooth if $n \leqslant 10$, and singular in codimension 7 if $n \geqslant 12$.*

Characteristic varieties of spaces

- Let X be a connected, finite-type CW-complex. Then $G = \pi_1(X, x_0)$ is a finitely presented group, with $G_{\mathrm{ab}} \cong H_1(X, \mathbb{Z})$.
- The ring $R = \mathbb{C}[G_{\mathrm{ab}}]$ is the coordinate ring of the character group, $\mathrm{Char}(X) = \mathrm{Hom}(G, \mathbb{C}^*) \cong (\mathbb{C}^*)^n \times \mathrm{Tors}(G_{\mathrm{ab}})$, where $n = b_1(X)$.
- The *characteristic varieties* of X are the homology jump loci

$$\mathcal{V}^i_k(X) = \{\rho \in \mathrm{Char}(X) \mid \dim H_i(X, \mathbb{C}_\rho) \geqslant k\}.$$

- These varieties are homotopy-type invariants of X, with $\mathcal{V}^1_k(X)$ depending only on $G = \pi_1(X)$.
- Set $\mathcal{V}_1(G) := \mathcal{V}^1_1(K(G,1))$; then $\mathcal{V}_1(G) = \mathcal{V}_1(G/G'')$.
- Let $f \in \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$, $f(1) = 0$. There is then a finitely presented group G with $G_{\mathrm{ab}} = \mathbb{Z}^n$ such that $\mathcal{V}_1(G) = V(f)$.

Tangent cones

- Let $\exp\colon H^1(X,\mathbb{C}) \to H^1(X,\mathbb{C}^*)$ be the coefficient homomorphism induced by $\mathbb{C} \to \mathbb{C}^*$, $z \mapsto e^z$.
- Let $W = V(I)$, a Zariski closed subset of $\mathrm{Char}(G) = H^1(X,\mathbb{C}^*)$.
- The *tangent cone* at 1 to W is $\mathrm{TC}_1(W) = V(\mathrm{in}(I))$.
- The *exponential tangent cone* at 1 to W:

$$\tau_1(W) = \{z \in H^1(X,\mathbb{C}) \mid \exp(\lambda z) \in W,\ \forall \lambda \in \mathbb{C}\}.$$

- Both tangent cones are homogeneous subvarieties of $H^1(X,\mathbb{C})$; are non-empty iff $1 \in W$; depend only on the analytic germ of W at 1; commute with finite unions and arbitrary intersections.
- $\tau_1(W) \subseteq \mathrm{TC}_1(W)$, with $=$ if all irred components of W are subtori, but $\neq$ in general.
- $\tau_1(W)$ is a finite union of rationally defined subspaces.

The tangent cone theorem

- A $\Bbbk$-cdga A is a *model* for a space X is A may be connected through a zig-zag of quasi-isomorphisms to Sullivan's algebra of piecewise polynomial forms $A_{\mathrm{PL}}(X) \otimes_{\mathbb{Q}} \Bbbk$.
- If the maps in the zig-zag are only isomorphisms in $H^{\leqslant q}$ and injective in degree $q+1$, we say A is a q-model.
- A is *formal* (or just q-*formal*) if it is (q-) equivalent to $(H^\bullet(A), 0)$.

Theorem

Let X be a connected CW-complex with finite q-skeleton, and suppose X admits a q-finite q-model A. Then, for all $i \leqslant q$ and all $k \geqslant 0$:

- *(DPS 2009, Dimca–Papadima 2014) $\mathcal{V}^i_k(X)_{(1)} \cong \mathcal{R}^i_k(A)_{(0)}$. In particular, if X is q-formal, then $\mathcal{V}^i_k(X)_{(1)} \cong \mathcal{R}^i_k(X)_{(0)}$.*
- *(Budur–Wang 2020) All the irreducible components of $\mathcal{V}^i_k(X)$ passing through the origin of $\mathrm{Char}(X)$ are* algebraic subtori.

Consequently, $\tau_1(\mathcal{V}^i_k(X)) = \mathrm{TC}_1(\mathcal{V}^i_k(X)) = \mathcal{R}^i_k(A)$.

Alexander polynomials of 3-manifolds

- Let $H = H_1(X, \mathbb{Z})/\mathrm{Tors}$. Let $X^H \to X$ be the maximal torsion-free abelian cover of X, with cellular chain complex $C_\bullet(X^H, \partial^H)$.
- The *Alexander polynomial* $\Delta_X \in \mathbb{Z}[H]$ is the gcd of the codimension 1 minors of the Alexander matrix ∂_1^H.

Proposition

Let λ be a Laurent polynomial in $n \leqslant 3$ variables such that $\bar{\lambda} \doteq \lambda$ and $\lambda(1) \neq 0$. Then λ can be realized as the Alexander polynomial Δ_M of a closed, orientable 3-manifold M with $b_1(M) = n$.

Set $\mathcal{W}_1^1(M) = \mathcal{V}_1^1(M) \cap \mathrm{Char}^0(M)$.

Proposition

Let M be a closed, orientable, 3-dimensional manifold. Then $\mathcal{W}_1^1(M) = V(\Delta_M) \cup \{1\}$. If, moreover, $b_1(M) \geqslant 4$, then $\Delta_M(1) = 0$, and so $\mathcal{W}_1^1(M) = V(\Delta_M)$.

A Tangent Cone theorem for 3-manifolds

Let M be a closed, orientable, 3-manifold, and set $n = b_1(M)$.

Theorem

(1) *If either $n \leqslant 1$, or n is odd, $n \geqslant 3$, and μ_M is BP-generic, then*

$$\mathrm{TC}_1(\mathcal{V}^1_1(M)) = \mathcal{R}^1_1(M).$$

(2) *If n is even, $n \geqslant 2$, then $\mathcal{R}^1(M) = H^1(M, \mathbb{C})$. Moreover,*

$$\mathrm{TC}_1(\mathcal{V}^1_1(M)) = \mathcal{R}^1_1(M) \Longleftrightarrow \Delta_M = 0.$$

Remark

In case (2), the equality $\mathcal{R}^1(M) = H^1(M, \mathbb{C})$ was first proved in [Dimca–S, 2009], where it was used to show that the only 3-manifold groups which are also Kähler groups are the finite subgroups of $\mathrm{O}(4)$.

Theorem

(1) If $n \leqslant 1$, then M is formal, and has the rational homotopy type of S^3 or $S^1 \times S^2$.

(2) If n is even, $n \geqslant 2$, and $\Delta_M \neq 0$, then M is not 1-formal.

(3) If $\Delta_M \neq 0$, yet $\Delta_M(1) = 0$ and $\mathrm{TC}_1(V(\Delta_M))$ is not a finite union of $\mathbb{Q}$-linear subspaces, then M admits no 1-finite 1-model.

Example

Let $M = S^1 \times S^2 \# S^1 \times S^2$; then $\Delta_M = 0$, and so $\mathrm{TC}_1(\mathcal{V}^1_1(M)) = \mathcal{R}^1_1(M) = \mathbb{C}^2$. In fact, M is formal.

Example

- Let M be the Heisenberg 3-d nilmanifold; then $\Delta_M = 1$ and $\mu_M = 0$, and so $\mathrm{TC}_1(\mathcal{V}^1_1(M)) = \{0\}$, whereas $\mathcal{R}^1_1(M) = \mathbb{C}^2$.
- M admits a finite model, namely, $A = \bigwedge(a, b, c)$ with $da = db = 0$ and $dc = ab$, but M is not 1-formal.

Example

Let M be a 3-manifold with $\Delta_M = (t_1+t_2)(t_1t_2+1) - 4t_1t_2$. Then

$$\{0\} = \tau_1(\mathcal{V}^1_1(M)) \subsetneqq \mathrm{TC}_1(\mathcal{V}^1_1(M)) = \{x_1^2+x_2^2=0\}.$$

The latter variety decomposes as the union of two lines defined over $\mathbb{C}$, but not over $\mathbb{Q}$. Hence, M admits no 1-finite 1-model.

The 3d Tangent Cone theorem does not hold in higher depth.

Example

Let M be a 3-manifold with $b_1(M) = 10$ and intersection 3-form

$$\mu_M = e_1e_2e_5 + e_1e_3e_6 + e_2e_3e_7 + e_1e_4e_8 + e_2e_4e_9 + e_3e_4e_{10}.$$

- $\mathcal{R}^1_7(M) \cong \{z \in \mathbb{C}^6 \mid z_1z_6 - z_2z_5 + z_3z_4 = 0\}$, an irreducible quadric with an isolated singular point at 0.
- $\mathcal{V}^1_k(M) \subseteq \{1\}$, for all $k \geqslant 1$.
- Thus, $\mathrm{TC}_1(\mathcal{V}^1_7(M)) \neq \mathcal{R}^1_7(M)$, and so M is not 1-formal.

The Bieri–Neumann–Strebel–Renz invariants

- Let G be a finitely generated group, $n = b_1(G) > 0$. Let $S(G) = S^{n-1}$ be the unit sphere in $\mathrm{Hom}(G, \mathbb{R}) = \mathbb{R}^n$.
- (BNS 1987) $\Sigma^1(G) = \{\chi \in S(G) \mid \mathrm{Cay}_\chi(G) \text{ is connected}\}$, where $\mathrm{Cay}_\chi(G)$ is the induced subgraph of the Cayley graph of G on vertex set the monoid $G_\chi = \{g \in G \mid \chi(g) \geqslant 0\}$.
- (Bieri–Renz 1988) $\Sigma^q(G, \mathbb{Z}) = \{\chi \in S(G) \mid G_\chi \text{ is of type } \mathrm{FP}_q\}$, i.e., there is a projective $\mathbb{Z}G_\chi$-resolution $P_\bullet \to \mathbb{Z}$, with P_i finitely generated for all $i \leqslant q$. Moreover, $\Sigma^1(G, \mathbb{Z}) = -\Sigma^1(G)$.
- The BNSR-invariants of form a descending chain of *open* subsets, $S(G) \supseteq \Sigma^1(G, \mathbb{Z}) \supseteq \Sigma^2(G, \mathbb{Z}) \supseteq \cdots$.
- The Σ-invariants control the finiteness properties of normal subgroups $N \triangleleft G$ for which G/N is free abelian:

$$N \text{ is of type } \mathrm{FP}_q \iff \{\chi \in S(G) \mid \chi(N) = 0\} \subseteq \Sigma^q(G, \mathbb{Z})$$

- In particular: $\ker(\chi\colon G \twoheadrightarrow \mathbb{Z})$ is f.g. $\iff \{\pm\chi\} \subseteq \Sigma^1(G)$.

Novikov–Sikorav homology

- The *Novikov–Sikorav completion* of $\mathbb{Z}G$ at $\chi \in S(G)$ is
$$\widehat{\mathbb{Z}G}_{\chi} = \{\lambda \in \mathbb{Z}^G \mid \{g \in \operatorname{supp}\lambda \mid \chi(g) \geqslant c\} \text{ is finite, } \forall c \in \mathbb{R}\}.$$
- Alternatively, is U_m the additive subgroup of $\mathbb{Z}G$ (freely) generated by $\{g \in G \mid \chi(g) \geqslant m\}$, then $\widehat{\mathbb{Z}G}_{-\chi} = \varprojlim_m \mathbb{Z}G/U_m$.
- Example: Let $G = \mathbb{Z} = \langle t \rangle$ and $\chi(t) = 1$. Then
$$\widehat{\mathbb{Z}G}_{\chi} = \Big\{ \sum_{i \leqslant k} n_i t^i \mid n_i \in \mathbb{Z}, \text{ for some } k \in \mathbb{Z} \Big\}.$$
- Now let X be a connected CW-complex with finite q-skeleton. Write $S(X) := S(G)$ and define (Farber–Geoghegan–Schütz 2010):
$$\Sigma^q(X, \mathbb{Z}) = \{\chi \in S(X) \mid H_i(X, \widehat{\mathbb{Z}G}_{-\chi}) = 0, \ \forall\, i \leqslant q\}.$$
- (Bieri 2007) If G is FP_k, then $\Sigma^q(G, \mathbb{Z}) = \Sigma^q(K(G,1), \mathbb{Z})$, $\forall q \leqslant k$.
- In particular, if G is f.g., the BNS set $\Sigma^1(G) = -\Sigma^1(G, \mathbb{Z})$ consists of those $\chi \in S(G)$ for which both $H_0(G, \widehat{\mathbb{Z}G}_{\chi})$ and $H_1(G, \widehat{\mathbb{Z}G}_{\chi})$ vanish.

Tropical varieties

- Let $\mathbb{K} = \mathbb{C}\{\{t\}\} = \bigcup_{n \geqslant 1} \mathbb{C}((t^{1/n}))$ be the field of Puiseux series $/\mathbb{C}$.
- A non-zero element of $\mathbb{K}$ has the form $c(t) = c_1 t^{a_1} + c_2 t^{a_2} + \cdots$, where $c_i \in \mathbb{C}^*$ and $a_1 < a_2 < \cdots$ are rational numbers with a common denominator.
- The (algebraically closed) field $\mathbb{K}$ admits a valuation $v \colon \mathbb{K}^* \to \mathbb{Q}$, $v(c(t)) = a_1$. Let $v \colon (\mathbb{K}^*)^n \to \mathbb{Q}^n \subset \mathbb{R}^n$ be its n-fold product.
- The *tropicalization* of a subvariety $W \subset (\mathbb{K}^*)^n$, denoted $\mathrm{Trop}(W)$, is the closure (in the Euclidean topology) of $v(W)$ in $\mathbb{R}^n$.
- This is a rational polyhedral complex in $\mathbb{R}^n$. For instance, if W is a curve, then $\mathrm{Trop}(W)$ is a graph with rational edge directions.
- If T be an algebraic subtorus of $(\mathbb{K}^*)^n$, then $\mathrm{Trop}(T)$ is the linear subspace $\mathrm{Hom}(\mathbb{K}^*, T) \otimes \mathbb{R} \subset \mathrm{Hom}(\mathbb{K}^*, (\mathbb{K}^*)^n) \otimes \mathbb{R} = \mathbb{R}^n$. Moreover, if $z \in (\mathbb{K}^*)^n$, then $\mathrm{Trop}(z \cdot T) = \mathrm{Trop}(T) + v(z)$.

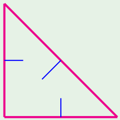
- For a variety $W \subset (\mathbb{C}^*)^n$, we may define its tropicalization by setting $\mathrm{Trop}(W) = \mathrm{Trop}(W \times_{\mathbb{C}} \mathbb{K})$. This is a polyhedral fan in $\mathbb{R}^n$.
- For a polytope P, with (polar) dual P^*, let
  - $\mathcal{F}(P)$ face fan (the set of cones spanned by the faces of P).
  - $\mathcal{N}(P)$ (inner) normal fan.

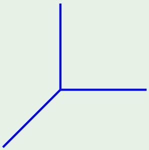
  If $0 \in \mathrm{int}(P)$, then $\mathcal{N}(P) = \mathcal{F}(P^*)$.
- If $W = V(f)$ is a hypersurface defined by $f = \sum_{\mathbf{u} \in A} a_{\mathbf{u}} \mathbf{t}^{\mathbf{u}} \in \mathbb{C}[\mathbf{t}^{\pm 1}]$, and $\mathrm{Newt}(f) = \mathrm{conv}\{\mathbf{u} \mid a_{\mathbf{u}} \neq 0\} \subset \mathbb{R}^n$, then

$$\mathrm{Trop}(V(f)) = \mathcal{N}(\mathrm{Newt}(f))^{\mathrm{codim}>0}.$$

Example

Let $f = t_1 + t_2 + 1$. Then $\mathrm{Newt}(f) = \mathrm{conv}\{(1,0),(0,1),(0,0)\}$ is a triangle, and so $\mathrm{Trop}(V(f))$ is a tripod.





Tropicalizing the characteristic varieties

- Recall $\mathbb{K} = \mathbb{C}\{\{t\}\}$ comes with a valuation map, $v \colon \mathbb{K}^* \to \mathbb{Q}$.
- Let $\nu_X \colon \operatorname{Char}_{\mathbb{K}}(X) \to \mathbb{Q}^n \subset \mathbb{R}^n$ be the composite

$$H^1(X, \mathbb{K}^*) \xrightarrow{v_*} H^1(X, \mathbb{Q}) \longrightarrow H^1(X, \mathbb{R}).$$

- Given an algebraic subvariety $W \subset H^1(X, \mathbb{C}^*)$ we define its *tropicalization* as the closure in $H^1(X, \mathbb{R}) \cong \mathbb{R}^n$ of the image of $W \times_{\mathbb{C}} \mathbb{K} \subset H^1(X, \mathbb{K}^*)$ under ν_X,

$$\operatorname{Trop}(W) := \overline{\nu_X(W \times_{\mathbb{C}} \mathbb{K})}.$$

- Applying this to the characteristic varieties $\mathcal{V}^i(X) := \mathcal{V}^i_1(X)$, and recalling that $\mathcal{V}^i(X, \mathbb{K}) = \mathcal{V}^i(X) \times_{\mathbb{C}} \mathbb{K}$, we have that

$$\operatorname{Trop}(\mathcal{V}^i(X)) = \overline{\nu_X\big(\mathcal{V}^i(X, \mathbb{K})\big)}.$$

Lemma

Let $W \subset (\mathbb{C}^*)^n$ be an algebraic variety. Then $\tau_1^{\mathbb{R}}(W) \subseteq \operatorname{Trop}(W)$.

Sketch of proof.

- Every irreducible component of $\tau_1^{\mathbb{R}}(W)$ is of the form $L \otimes_{\mathbb{Q}} \mathbb{R}$, for some linear subspace $L \subset \mathbb{Q}^n$.
- The complex torus $T := \exp(L \otimes_{\mathbb{Q}} \mathbb{C})$ lies inside W.
- Thus, $\operatorname{Trop}(T) = L \otimes_{\mathbb{Q}} \mathbb{R}$ lies inside $\operatorname{Trop}(W)$. $\square$

Proposition

- $\tau_1^{\mathbb{R}}(\mathcal{V}^i(X)) \subseteq \operatorname{Trop}(\mathcal{V}^i(X))$, *for all* $i \leqslant q$.
- *If there is a subtorus* $T \subset \operatorname{Char}^0(X)$ *such that* $T \not\subset \mathcal{V}^i(X)$, *yet* $\rho T \subset \mathcal{V}^i(X)$ *for some* $\rho \in \operatorname{Char}(X)$, *then* $\tau_1^{\mathbb{R}}(\mathcal{V}^i(X)) \subsetneq \operatorname{Trop}(\mathcal{V}^i(X))$.

A tropical bound for the Σ-invariants

Theorem (PS-2010, S-2021)

Let $\rho\colon \pi_1(X) \to \Bbbk^*$ be a character such that $\rho \in \mathcal{V}^{\leqslant q}(X, \Bbbk)$. Let $v\colon \Bbbk^* \to \mathbb{R}$ be a valuation on $\Bbbk$, and set $\chi = v \circ \rho$. If $\chi\colon \pi_1(X) \to \mathbb{R}$ is non-zero, then $\chi \notin \Sigma^q(X, \mathbb{Z})$.

Theorem (S-2021)

$$\Sigma^q(X, \mathbb{Z}) \subseteq S(\operatorname{Trop}(\mathcal{V}^{\leqslant q}(X)))^{\mathrm{c}}$$

Corollary

$$\Sigma^q(X, \mathbb{Z}) \subseteq S(\operatorname{Trop}(\mathcal{V}^{\leqslant q}(X)))^{\mathrm{c}} \subseteq S(\tau_1^{\mathbb{R}}(\mathcal{V}^{\leqslant q}(X)))^{\mathrm{c}}.$$

$$\Sigma^1(G) \subseteq -S(\operatorname{Trop}(\mathcal{V}^1(G)))^{\mathrm{c}} \subseteq S(\tau_1^{\mathbb{R}}(\mathcal{V}^1(G)))^{\mathrm{c}}.$$

Corollary

If $\mathcal{V}^{\leqslant q}(X)$ contains a component of $\operatorname{Char}(X)$, then $\Sigma^q(X, \mathbb{Z}) = \emptyset$.

BNS invariants of 3-manifolds

- Let M be a compact, connected, orientable 3-manifold with $b_1(M) > 0$.
- The *Thurston norm* $\|\phi\|_T$ of a class $\phi \in H^1(M;\mathbb{Z})$ is the infimum of $-\chi(\hat{S})$, where S runs though all the properly embedded, oriented surfaces in M dual to ϕ, and $\hat{S}$ denotes the result of discarding all components of S which are disks or spheres.
- Thurston showed that $\| - \|_T$ defines a seminorm on $H^1(M;\mathbb{Z})$, which can be extended to a continuous seminorm on $H^1(M;\mathbb{R})$.
- The unit norm ball, $B_T = \{\phi \in H^1(M;\mathbb{R}) \mid \|\phi\|_T \leqslant 1\}$, is a rational polyhedron with finitely many sides and symmetric in the origin.
- *Alexander norm*: $\|\phi\|_A = \mathrm{length}(\phi(\mathrm{Newt}(\Delta_M)))$, where $\mathrm{Newt}(\Delta_M) \subset H_1(M,\mathbb{R})$ is the Newton polytope of Δ_M.
- This defines a semi-norm on $H^1(M,\mathbb{R})$, with unit ball $B_A = \{\phi \in H^1(M;\mathbb{R}) \mid \|\phi\|_A \leqslant 1\}$.

- A non-zero class $\phi \in H^1(M;\mathbb{Z}) = \mathrm{Hom}(\pi_1(M),\mathbb{Z})$ is *fibered* if there is a fibration $p\colon M \to S^1$ such that $\phi = p_*\colon \pi_1(M) \to \pi_1(S^1) = \mathbb{Z}$.
- There are facets of B_T, called the *fibered faces* (coming in antipodal pairs), so that a class $\phi \in H^1(M;\mathbb{Z})$ fibers if and only if it lies in the cone over the interior of a fibered face.
- BNS: If $G = \pi_1(M)$, then $\Sigma^1(G)$ is the projection onto $S(G)$ of the open fibered faces of B_T; in particular, $\Sigma^1(G) = -\Sigma^1(G)$.
- Under some mild assumptions, McMullen showed that $\|\phi\|_A \leqslant \|\phi\|_T$; thus, $B_T \subset B_A$, leading to an upper bound for $\Sigma_1(G)$ in terms of B_A.

THEOREM

Let M be a compact, connected, orientable, 3-manifold with empty or toroidal boundary. Set $G = \pi_1(M)$ and assume $b_1(M) \geqslant 2$. Then

(1) $\operatorname{Trop}\left(\mathcal{V}^1(G) \cap \mathbb{T}^0_G\right)$ *is the positive-codimension skeleton of $\mathcal{F}(B_A)$, the face fan of the unit ball in the Alexander norm.*

(2) (Partly recovers McMullen's theorem) $\Sigma^1(G)$ is contained in the union of the open cones on the facets of B_A.

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