GEOMETRY AND TOPOLOGY OF COHOMOLOGY JUMP LOCI

LECTURE 3: FUNDAMENTAL GROUPS AND JUMP LOCI

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MIMS Summer School: New Trends in Topology and Geometry

Mediterranean Institute for the Mathematical Sciences

Tunis, Tunisia

July 9–12, 2018

1. FUNDAMENTAL GROUPS IN GEOMETRY
   - Fundamental groups of manifolds
   - Kähler groups
   - Quasi-projective groups
   - Complements of hypersurfaces
   - Line arrangements
   - Artin groups

2. COMPARING CLASSES OF GROUPS
   - Kähler groups vs other groups
   - Quasi-projective groups vs other groups

FUNDAMENTAL GROUPS OF MANIFOLDS

- Every finitely presented group π can be realized as $\pi = \pi_1(M)$, for some smooth, compact, connected manifold M^n of dim $n \geqslant 4$.
- M^n can be chosen to be orientable.
- If n even, $n \geqslant 4$, then M^n can be chosen to be symplectic (Gompf).
- If n even, $n \geqslant 6$, then M^n can be chosen to be complex (Taubes).
- Requiring that $n = 3$ puts severe restrictions on the (closed) 3-manifold group $\pi = \pi_1(M^3)$.

KÄHLER GROUPS

- A *Kähler manifold* is a compact, connected, complex manifold, with a Hermitian metric h such that $\omega = \operatorname{im}(h)$ is a closed 2-form.
- Smooth, complex projective varieties are Kähler manifolds.
- A group π is called a *Kähler group* if $\pi = \pi_1(M)$, for some Kähler manifold M.
- The group π is a *projective group* if M can be chosen to be a projective manifold.
- The classes of Kähler and projective groups are closed under finite direct products and passing to finite-index subgroups.
- Every finite group is a projective group. [Serre $\sim$1955]

- The Kähler condition puts strong restrictions on π, e.g.:
  - π is finitely presented.
  - $b_1(\pi)$ is even. [by Hodge theory]
  - π is 1-formal [Deligne–Griffiths–Morgan–Sullivan 1975]
  - π cannot split non-trivially as a free product. [Gromov 1989]
- Problem: Are all Kähler groups projective groups?
- Problem [Serre]: Characterize the class of projective groups.

QUASI-PROJECTIVE GROUPS

- A group π is said to be a *quasi-Kähler group* if $\pi = \pi_1(M \backslash D)$, where M is a Kähler manifold and D is a divisor.
- The group π is a *quasi-projective group* if M can be chosen to be a projective manifold.
- qK/qp groups are finitely presented. The classes of qK/qp groups are closed under finite direct products and passing to finite-index subgroups.
- For a qp group π,
    - $b_1(\pi)$ can be arbitrary (e.g., the free groups F_n).
    - π may be non-1-formal (e.g., the Heisenberg group).
    - π can split as a non-trivial free product (e.g., $F_2 = \mathbb{Z} * \mathbb{Z}$).
- Problem: Are all quasi-Kähler groups quasi-projective groups?

RESONANCE OF QUASI-KÄHLER MANIFOLDS

THEOREM (DIMCA–PAPADIMA–S. 2009)

Let X be a quasi-Kähler manifold, and $G = \pi_1(X)$. Let $\{L_\alpha\}_\alpha$ be the non-zero irreducible components of $\mathcal{R}^1_1(G)$. If G is 1-formal, then

- *Each L_α is a linear subspace of $H^1(G, \mathbb{C})$.*
- *Each L_α is p-isotropic (i.e., restriction of $\cup_G$ to L_α has rank p), with $\dim L_\alpha \geqslant 2p + 2$, for some $p = p(\alpha) \in \{0, 1\}$.*
- *If $\alpha \neq \beta$, then $L_\alpha \cap L_\beta = \{0\}$.*
- $\mathcal{R}^1_k(G) = \{0\} \cup \bigcup_{\alpha : \dim L_\alpha > k + p(\alpha)} L_\alpha$.

Furthermore,

- *If X is compact, then G is 1-formal, and each L_α is 1-isotropic.*
- *If $W_1(H^1(X, \mathbb{C})) = 0$, then G is 1-formal, and each L_α is 0-isotropic.*

COMPLEMENTS OF HYPERSURFACES

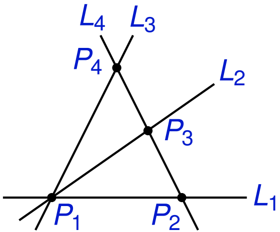
- A subclass of quasi-projective groups consists of fundamental groups of complements of hypersurfaces in $\mathbb{CP}^n$,
$$\pi = \pi_1(\mathbb{CP}^n \backslash \{f=0\}), \quad f \in \mathbb{C}[z_0, \dots, z_n] \text{ homogeneous.}$$
- All such groups are 1-formal. [Kohno 1983]
- By the Lefschetz hyperplane sections theorem, $\pi = \pi_1(\mathbb{CP}^2 \backslash \mathcal{C})$, for some plane algebraic curve $\mathcal{C}$.
- Zariski asked Van Kampen to find presentations for such groups.
- Using the Alexander polynomial, Zariski showed that π is *not* determined by the combinatorics of $\mathcal{C}$ (number and type of singularities), but also depends on the position of its singularities.

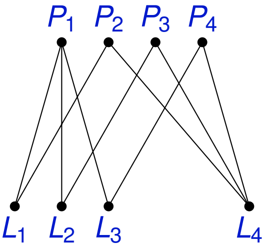
PROBLEM (ZARISKI)

Is $\pi = \pi_1(\mathbb{CP}^2 \backslash \mathcal{C})$ residually finite*, i.e., is the map to the profinite completion,* $\pi \to \pi^{\text{alg}} := \varprojlim_{G \triangleleft_{\text{f.i.}} \pi} \pi/G$*, injective?*

HYPERPLANE ARRANGEMENTS

- Even more special are the *arrangement groups*, i.e., the fundamental groups of complements of complex hyperplane arrangements (or, equivalently, complex line arrangements).
- Let $\mathcal{A}$ be an *arrangement of lines* in $\mathbb{CP}^2$, defined by a polynomial $f = \prod_{L \in \mathcal{A}} f_L$, with f_L linear forms so that $L = \mathbb{P}(\ker(f_L))$.
- The combinatorics of $\mathcal{A}$ is encoded in the *intersection poset*, $\mathcal{L}(\mathcal{A})$, with $\mathcal{L}_1(\mathcal{A}) = \{\text{lines}\}$ and $\mathcal{L}_2(\mathcal{A}) = \{\text{intersection points}\}$.





- Let $U(\mathcal{A}) = \mathbb{CP}^2 \setminus \bigcup_{L \in \mathcal{A}} L$. The group $\pi = \pi_1(U(\mathcal{A}))$ has a finite presentation with
  - Meridional generators $x_1, \dots, x_n$, where $n = |\mathcal{A}|$, and $\prod x_i = 1$.
  - Commutator relators $x_i \alpha_j (x_i)^{-1}$, where $\alpha_1, \dots \alpha_s \in P_n \subset \operatorname{Aut}(F_n)$, and $s = |\mathcal{L}_2(\mathcal{A})|$.
- Let $\gamma_1(\pi) = \pi, \quad \gamma_2(\pi) = \pi' = [\pi, \pi], \quad \gamma_k(\pi) = [\gamma_{k-1}(\pi), \pi]$, be the lower central series of π. Then:
  - $\pi_{\text{ab}} = \pi/\gamma_2$ equals $\mathbb{Z}^{n-1}$.
  - π/γ_3 is determined by $L(\mathcal{A})$.
  - π/γ_4 (and thus, π) is *not* determined by $L(\mathcal{A})$ (G. Rybnikov).

PROBLEM (ORLIK)

Is π torsion-free?

- Answer is yes if $U(\mathcal{A})$ is a $K(\pi, 1)$. This happens if the cone on $\mathcal{A}$ is a simplicial arrangement (Deligne), or supersolvable (Terao).

ARTIN GROUPS

- Let $\Gamma = (V, E)$ be a finite, simple graph, and let $\ell\colon E \to \mathbb{Z}_{\geqslant 2}$ be an edge-labeling. The associated *Artin group*:

$$A_{\Gamma,\ell} = \langle v \in V \mid \underbrace{vwv\cdots}_{\ell(e)} = \underbrace{wvw\cdots}_{\ell(e)}, \text{ for } e = \{v, w\} \in E \rangle.$$

- If (Γ, ℓ) is Dynkin diagram of type A_{n-1} with $\ell(\{i, i+1\}) = 3$ and $\ell(\{i, j\}) = 2$ otherwise, then $A_{\Gamma,\ell}$ is the braid group B_n.
- If $\ell(e) = 2$, for all $e \in E$, then

$$A_{\Gamma} = \langle v \in \mathrm{V} \mid vw = wv \text{ if } \{v, w\} \in \mathrm{E} \rangle.$$

  is the *right-angled Artin group* associated to Γ.
- $\Gamma \cong \Gamma' \Leftrightarrow A_{\Gamma} \cong A_{\Gamma'}$

  [Kim–Makar-Limanov–Neggers–Roush 80 / Droms 87]

The corresponding *Coxeter group*,

$$W_{\Gamma,\ell} = A_{\Gamma,\ell}/\langle v^2 = 1 \mid v \in V \rangle,$$

fits into exact sequence $1 \to P_{\Gamma,\ell} \to A_{\Gamma,\ell} \to W_{\Gamma,\ell} \to 1$.

THEOREM (BRIESKORN 1971)

If $W_{\Gamma,\ell}$ is finite, then $G_{\Gamma,\ell}$ is quasi-projective.

Idea: let

- $\mathcal{A}_{\Gamma,\ell} =$ reflection arrangement of type $W_{\Gamma,\ell}$ (over $\mathbb{C}$)
- $X_{\Gamma,\ell} = \mathbb{C}^n \setminus \bigcup_{H \in \mathcal{A}_{\Gamma,\ell}} H$, where $n = |\mathcal{A}_{\Gamma,\ell}|$
- $P_{\Gamma,\ell} = \pi_1(X_{\Gamma,\ell})$

then:

$$A_{\Gamma,\ell} = \pi_1(X_{\Gamma,\ell}/W_{\Gamma,\ell}) = \pi_1(\mathbb{C}^n \setminus \{\delta_{\Gamma,\ell} = 0\})$$

THEOREM (KAPOVICH–MILLSON 1998)

There exist infinitely many (Γ, ℓ) such that $A_{\Gamma,\ell}$ is not *quasi-projective.*

Kähler groups vs other groups

Question (Donaldson–Goldman 1989)

Which 3-manifold groups are Kähler groups?

Reznikov gave a partial solution in 2002.

Theorem (Dimca–S. 2009)

Let G be the fundamental group of a closed 3-manifold. Then G is a Kähler group $\iff$ π is a finite subgroup of $\mathrm{O}(4)$, acting freely on S^3.

- Idea of our proof: compare the resonance varieties of 3-manifolds to those of Kähler manifolds.
- By passing to a suitable index-2 subgroup of G, we may assume that the closed 3-manifold is orientable.

PROPOSITION

Let M be a closed, orientable 3-manifold. Then:

1. *$H^1(M,\mathbb{C})$ is not 1-isotropic.*
2. *If $b_1(M)$ is even, then $\mathcal{R}^1_1(M) = H^1(M,\mathbb{C})$.*

On the other hand, it follows from a previous theorem that:

PROPOSITION

Let M be a compact Kähler manifold with $b_1(M) \neq 0$. If $\mathcal{R}^1_1(M) = H^1(M,\mathbb{C})$, then $H^1(M,\mathbb{C})$ is 1-isotropic.

- If G is a Kähler, then $b_1(G)$ even.
- Thus, if G is both a 3-mfd group and a Kähler group $\Rightarrow b_1(G) = 0$.
- Using work of Fujiwara (1999) and Reznikov (2002) on Kazhdan's property (T), as well as Perelman (2003), it follows that G is a finite subgroup of $\mathrm{O}(4)$.

- Alternative proofs have later been given by Kotschick (2012) and Biswas, Mj, and Seshadri (2012).

THEOREM (FRIEDL–S. 2014)

Let N be a 3-manifold with non-empty, toroidal boundary. If $\pi_1(N)$ is a Kähler group, then $N \cong S^1 \times S^1 \times I$.

- Subsequent generalization by Kotschick (dropping the toroidal boundary assumption): If G is both an infinite 3-manifold group and a Kähler group, then G is a surface group.

THEOREM (DPS 2009)

Let Γ be a finite simple graph, and le A_Γ be the corresponding RAAG. The following are equivalent:

1. *A_Γ is a Kähler group.*
2. *A_Γ is a free abelian group of even rank.*
3. *Γ is a complete graph on an even number of vertices.*

THEOREM (S. 2011)

Let $\mathcal{A}$ be an arrangement of lines in $\mathbb{CP}^2$, with group $\pi = \pi_1(U(\mathcal{A}))$. The following are equivalent:

1. *π is a Kähler group.*
2. *π is a free abelian group of even rank.*
3. *$\mathcal{A}$ consists of an odd number of lines in general position.*

QUASI-PROJECTIVE GROUPS VS OTHER GROUPS

THEOREM (DIMCA–PAPADIMA–S. 2011)

Let π be the fundamental group of a closed, orientable 3-manifold. Assume π is 1-formal. Then the following are equivalent:

1. $\mathfrak{m}(\pi) \cong \mathfrak{m}(\pi_1(X))$, *for some quasi-projective manifold* X.
2. $\mathfrak{m}(\pi) \cong \mathfrak{m}(\pi_1(N))$, *where* N *is either* S^3, $\#^n S^1 \times S^2$, *or* $S^1 \times \Sigma_g$.

THEOREM (FRIEDL–S. 2014)

Let N be a 3-mfd with empty or toroidal boundary. If $\pi_1(N)$ is a quasi-projective group, then all prime components of N are graph manifolds.

In particular, the fundamental group of a hyperbolic 3-manifold with empty or toroidal boundary is never a qp-group.

THEOREM (DPS 2009)

A right-angled Artin group A_Γ is a quasi-projective group if and only if Γ is a complete multipartite graph $K_{n_1,\dots,n_r} = \overline{K}_{n_1} * \cdots * \overline{K}_{n_r}$, in which case $A_\Gamma = F_{n_1} \times \cdots \times F_{n_r}$.

THEOREM (S. 2011)

Let $\pi = \pi_1(U(\mathcal{A}))$ be an arrangement group. The following are equivalent:

1. *π is a RAAG.*
2. *π is a finite direct product of finitely generated free groups.*
3. *$\mathcal{G}(\mathcal{A})$ is a forest.*

Here $\mathcal{G}(\mathcal{A})$ is the 'multiplicity' graph, with

- vertices: points $P \in \mathcal{L}_2(\mathcal{A})$ with multiplicity at least 3;
- edges: $\{P, Q\}$ if $P, Q \in L$, for some $L \in \mathcal{A}$.