Geometry and topology of cohomology jump loci

Lecture 2: Resonance varieties

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COMMUTATIVE DIFFERENTIAL GRADED ALGEBRAS

- Let $A = (A^\bullet, \mathrm{d})$ be a commutative, differential graded algebra over a field $\Bbbk$ of characteristic 0. That is:
  - $A = \bigoplus_{i \geqslant 0} A^i$, where A^i are $\Bbbk$-vector spaces.
  - The multiplication $\cdot \colon A^i \otimes A^j \to A^{i+j}$ is graded-commutative, i.e., $ab = (-1)^{|a||b|} ba$ for all homogeneous a and b.
  - The differential $\mathrm{d} \colon A^i \to A^{i+1}$ satisfies the graded Leibniz rule, i.e., $\mathrm{d}(ab) = \mathrm{d}(a)b + (-1)^{|a|} a\, \mathrm{d}(b)$.
- A CDGA A is of *finite-type* (or *q-finite*) if
  - it is connected (i.e., $A^0 = \Bbbk \cdot 1$);
  - $\dim_{\Bbbk} A^i$ is finite for $i \leqslant q$.
- Let $H^i(A) = \ker(\mathrm{d} \colon A^i \to A^{i+1}) / \operatorname{im}(\mathrm{d} \colon A^{i-1} \to A^i)$. Then $H^\bullet(A)$ inherits an algebra structure from A.

- A cdga morphism $\varphi\colon A \to B$ is both an algebra map and a cochain map. Hence, it induces a morphism $\varphi^*\colon H^{\bullet}(A) \to H^{\bullet}(B)$.
- A map $\varphi\colon A \to B$ is a *quasi-isomorphism* if φ^* is an isomorphism. Likewise, φ is a q-quasi-isomorphism (for some $q \geqslant 1$) if φ^* is an isomorphism in degrees $\leqslant q$ and is injective in degree $q+1$.
- Two cdgas, A and B, are *(q-)equivalent* ($\simeq_q$) if there is a zig-zag of (q-)quasi-isomorphisms connecting A to B.
- A cdga A is *formal* (or just *q-formal*) if it is (q-)equivalent to $(H^{\bullet}(A), d = 0)$.

RESONANCE VARIETIES

- Since A is connected and $d(1) = 0$, we have $Z^1(A) = H^1(A)$.
- For each $a \in Z^1(A)$, we construct a cochain complex,

$$(A^\bullet, \delta_a)\colon \; A^0 \xrightarrow{\delta_a^0} A^1 \xrightarrow{\delta_a^1} A^2 \xrightarrow{\delta_a^2} \cdots,$$

  with differentials $\delta_a^i(u) = a \cdot u + \mathrm{d}\, u$, for all $u \in A^i$.

- The *resonance varieties* of A are the sets

$$\mathcal{R}_k^i(A) = \{a \in H^1(A) \mid \dim H^i(A^\bullet, \delta_a) \geqslant k\}.$$

  If A is q-finite, then $\mathcal{R}_k^i(A)$ are algebraic varieties for all $i \leqslant q$.

- If A is a CGA (so that $\mathrm{d} = 0$), these varieties are homogeneous subvarieties of $H^1(A) = A^1$.

- Fix a $\Bbbk$-basis $\{e_1, \dots, e_r\}$ for $H^1(A)$, and let $\{x_1, \dots, x_r\}$ be the dual basis for $H_1(A) = (H^1(A))^*$.
- Identify $\mathrm{Sym}(H_1(A))$ with $S = \Bbbk[x_1, \dots, x_r]$, the coordinate ring of the affine space $H^1(A)$.
- Define a cochain complex of free S-modules, $\mathbf{L}(A) := (A^\bullet \otimes_\Bbbk S, \delta)$,

$$\cdots \longrightarrow A^i \otimes S \xrightarrow{\delta^i} A^{i+1} \otimes S \xrightarrow{\delta^{i+1}} A^{i+2} \otimes S \longrightarrow \cdots,$$

  where $\delta^i(u \otimes f) = \sum_{j=1}^n e_j u \otimes f x_j + \mathrm{d}\, u \otimes f$.

- The specialization of $(A \otimes_\Bbbk S, \delta)$ at $a \in A^1$ coincides with (A, δ_a).
- Hence, $\mathcal{R}^i_k(A)$ is the zero-set of the ideal generated by all minors of size $b_i(A) - k + 1$ of the block-matrix $\delta^{i+1} \oplus \delta^i$.
- In particular, $\mathcal{R}^1_k(A) = V(I_{r-k}(\delta^1))$, the zero-set of the ideal of codimension k minors of δ^1.

Example (Exterior algebra)

Let $E = \bigwedge V$, where $V = \Bbbk^n$, and $S = \mathrm{Sym}(V)$. Then $\mathbf{L}(E)$ is the Koszul complex on V. E.g., for $n = 3$:

$$S \xrightarrow{\delta^1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}} S^3 \xrightarrow{\delta^2 = \begin{pmatrix} x_2 & x_3 & 0 \\ -x_1 & 0 & x_3 \\ 0 & -x_1 & -x_2 \end{pmatrix}} S^3 \xrightarrow{\delta^3 = \begin{pmatrix} x_3 & -x_2 & x_1 \end{pmatrix}} S\,.$$

Hence,

$$\mathcal{R}^i_k(E) = \begin{cases} \{0\} & \text{if } k \leqslant \binom{n}{i}, \\ \emptyset & \text{otherwise.} \end{cases}$$

EXAMPLE (NON-ZERO RESONANCE)

Let $A = \bigwedge(e_1, e_2, e_3)/\langle e_1 e_2 \rangle$, and set $S = \Bbbk[x_1, x_2, x_3]$. Then

$$\mathbf{L}(A): \quad S \xrightarrow{\delta^1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}} S^3 \xrightarrow{\delta^2 = \begin{pmatrix} x_3 & 0 & -x_1 \\ 0 & x_3 & -x_2 \end{pmatrix}} S^2 .$$

$$\mathcal{R}^1_k(A) = \begin{cases} \{x_3 = 0\} & \text{if } k = 1, \\ \{0\} & \text{if } k = 2 \text{ or } 3, \\ \emptyset & \text{if } k > 3. \end{cases}$$

EXAMPLE (NON-LINEAR RESONANCE)

Let $A = \bigwedge(e_1, \dots, e_4)/\langle e_1 e_3, e_2 e_4, e_1 e_2 + e_3 e_4 \rangle$. Then

$$\mathbf{L}(A): \quad S \xrightarrow{\delta^1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}} S^4 \xrightarrow{\delta^2 = \begin{pmatrix} x_4 & 0 & 0 & -x_1 \\ 0 & x_3 & -x_2 & 0 \\ -x_2 & x_1 & x_4 & -x_3 \end{pmatrix}} S^3 .$$

$$\mathcal{R}^1_1(A) = \{x_1 x_2 + x_3 x_4 = 0\}$$

EXAMPLE (NON-HOMOGENEOUS RESONANCE)

- Let $A = \bigwedge(a, b)$ with $\mathrm{d}\, a = 0$, $\mathrm{d}\, b = b \cdot a$.
- $H^1(A) = \mathbb{C}$, generated by a. Set $S = \mathbb{C}[x]$. Then:

$$\mathbf{L}(A) : S \xrightarrow{\delta^1 = \binom{0}{x}} S^2 \xrightarrow{\delta^2 = (x-1\ \ 0)} S\,.$$

- Hence, $\mathcal{R}^1(A) = \{0, 1\}$, a non-homogeneous subvariety of $\mathbb{C}$.
- Let A' be the sub-CDGA generated by a. The inclusion map, $A' \hookrightarrow A$, induces an isomorphism in cohomology.
- But $\mathcal{R}^1(A') = \{0\}$, and so the resonance varieties of A and A' differ, although A and A' are quasi-isomorphic.

PROPOSITION

If $A \simeq_q A'$, then $\mathcal{R}^i_k(A)_{(0)} \cong \mathcal{R}^i_k(A')_{(0)}$, for all $i \leqslant q$ and $k \geqslant 0$.

TANGENT CONE INCLUSION

THEOREM (BUDUR–RUBIO, DENHAM–S. 2018)

If A is a connected $\Bbbk$-CDGA A with locally finite cohomology, then

$$\mathrm{TC}_0(\mathcal{R}^i_k(A)) \subseteq \mathcal{R}^i_k(H^\bullet(A)).$$

In general, we cannot replace $\mathrm{TC}_0(\mathcal{R}^i_k(A))$ by $\mathcal{R}^i_k(A)$.

EXAMPLE

- Let $A = \bigwedge(a, b)$ with $\mathrm{d}\, a = 0$ and $\mathrm{d}\, b = b \cdot a$.
- Then $H^\bullet(A) = \bigwedge(a)$, and so $\mathcal{R}^1_1(A) = \{0\}$.
- Hence $\mathcal{R}^1_1(A) = \{0, 1\}$ is *not* contained in $\mathcal{R}^1_1(A)$, though $\mathrm{TC}_0(\mathcal{R}^1(A)) = \{0\}$ is.

In general, the inclusion $\mathrm{TC}_0(\mathcal{R}^i_k(A)) \subseteq \mathcal{R}^i_k(H^\bullet(A))$ is strict.

EXAMPLE

- Let $A = \bigwedge(a, b, c)$ with $\mathrm{d}\, a = \mathrm{d}\, b = 0$ and $\mathrm{d}\, c = a \wedge b$.
- Writing $S = \Bbbk[x, y]$, we have:

$$\mathbf{L}(A):\ S \xrightarrow{\delta^1 = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}} S^3 \xrightarrow{\delta^2 = \begin{pmatrix} y & -x & 1 \\ 0 & 0 & -x \\ 0 & 0 & -y \end{pmatrix}} S^3 .$$

- Hence $\mathcal{R}^1_1(A) = \{0\}$.
- But $H^\bullet(A) = \bigwedge(a, b)/(ab)$, and so $\mathcal{R}^1_1(H^\bullet(A)) = \Bbbk^2$.

ALGEBRAIC MODELS FOR SPACES

- Given any space X, there is an associated Sullivan $\mathbb{Q}$-cdga, $A_{\mathrm{PL}}(X)$, such that $H^\bullet(A_{\mathrm{PL}}(X)) = H^\bullet(X, \mathbb{Q})$.
- We say X is *q-finite* if X has the homotopy type of a connected CW-complex with finite q-skeleton, for some $q \geqslant 1$.
- An *algebraic (q-)model* (over $\Bbbk$) for X is a $\Bbbk$-cgda (A, d) which is (q-) equivalent to $A_{\mathrm{PL}}(X) \otimes_{\mathbb{Q}} \Bbbk$.
- If M is a smooth manifold, then $\Omega_{\mathrm{dR}}(M)$ is a model for M (over $\mathbb{R}$).
- Examples of spaces having finite-type models include:
  - Formal spaces (such as compact Kähler manifolds, hyperplane arrangement complements, toric spaces, etc).
  - Smooth quasi-projective varieties, compact solvmanifolds, Sasakian manifolds, etc.

Germs of jump loci

Theorem (Dimca–Papadima 2014)

Let X be a q-finite space, and suppose X admits a q-finite, q-model A. Then the map $\exp\colon H^1(X,\mathbb{C}) \to H^1(X,\mathbb{C}^*)$ induces a local analytic isomorphism $H^1(A)_{(0)} \to \operatorname{Char}(X)_{(1)}$, which identifies the germ at 0 of $\mathcal{R}^i_k(A)$ with the germ at 1 of $\mathcal{V}^i_k(X)$, for all $i \leqslant q$ and $k \geqslant 0$.

Corollary

If X is a q-formal space, then $\mathcal{V}^i_k(X)_{(1)} \cong \mathcal{R}^i_k(X)_{(0)}$, for $i \leqslant q$ and $k \geqslant 0$.

- A precursor to corollary can be found in work of Green, Lazarsfeld, and Ein on cohomology jump loci of compact Kähler manifolds.
- The case when $q = 1$ was first established in [DPS 2019].

TANGENT CONES AND EXPONENTIAL MAPS

- The map $\exp\colon \mathbb{C}^n \to (\mathbb{C}^\times)^n$, $(z_1,\dots,z_n) \mapsto (e^{z_1},\dots,e^{z_n})$ is a homomorphism taking 0 to 1.
- For a Zariski-closed subset $W = V(I)$ inside $(\mathbb{C}^\times)^n$, define:
  - The *tangent cone* at 1 to W as $\operatorname{TC}_1(W) = V(\operatorname{in}(I))$.
  - The *exponential tangent cone* at 1 to W as

$$\tau_1(W) = \{z \in \mathbb{C}^n \mid \exp(\lambda z) \in W,\ \forall \lambda \in \mathbb{C}\}$$

- These sets are homogeneous subvarieties of $\mathbb{C}^n$, which depend only on the analytic germ of W at 1.
- Both commute with finite unions and arbitrary intersections.
- $\tau_1(W) \subseteq \operatorname{TC}_1(W)$.
  - $=$ if all irred components of W are subtori.
  - $\neq$ in general.
- (DPS 2009) $\tau_1(W)$ is a finite union of rationally defined subspaces.

THE TANGENT CONE THEOREM

Let X be a connected CW-complex with finite q-skeleton.

THEOREM (LIBGOBER 2002, DPS 2009)

For all $i \leqslant q$ and $k \geqslant 0$,

$$\tau_1(\mathcal{V}^i_k(X)) \subseteq \mathrm{TC}_1(\mathcal{V}^i_k(X)) \subseteq \mathcal{R}^i_k(X).$$

THEOREM (DPS-2009, DP-2014)

Suppose X is a q-formal space. Then, for all $i \leqslant q$ and $k \geqslant 0$,

$$\tau_1(\mathcal{V}^i_k(X)) = \mathrm{TC}_1(\mathcal{V}^i_k(X)) = \mathcal{R}^i_k(X).$$

In particular, all irreducible components of $\mathcal{R}^i_k(X)$ are rationally defined linear subspaces of $H^1(X, \mathbb{C})$.

DETECTING NON-FORMALITY

EXAMPLE

Let $\pi = \langle x_1, x_2 \mid [x_1, [x_1, x_2]]\rangle$. Then $\mathcal{V}^1_1(\pi) = \{t_1 = 1\}$, and so

$$\tau_1(\mathcal{V}^1_1(\pi)) = \mathrm{TC}_1(\mathcal{V}^1_1(\pi)) = \{x_1 = 0\}.$$

On the other hand, $\mathcal{R}^1_1(\pi) = \mathbb{C}^2$, and so π is not 1-formal.

EXAMPLE

Let $\pi = \langle x_1, \dots, x_4 \mid [x_1, x_2], [x_1, x_4][x_2^{-2}, x_3], [x_1^{-1}, x_3][x_2, x_4]\rangle$. Then

$$\mathcal{R}^1_1(\pi) = \{z \in \mathbb{C}^4 \mid z_1^2 - 2z_2^2 = 0\}.$$

This is a quadric hypersurface which splits into two linear subspaces over $\mathbb{R}$, but is irreducible over $\mathbb{Q}$. Thus, π is not 1-formal.

Example

Let π be a finitely presented group with $\pi_{\mathrm{ab}} = \mathbb{Z}^3$ and

$$\mathcal{V}^1_1(\pi) = \big\{(t_1, t_2, t_3) \in (\mathbb{C}^*)^3 \mid (t_2 - 1) = (t_1 + 1)(t_3 - 1)\big\},$$

This is a complex, 2-dimensional torus passing through the origin, but this torus does not embed as an algebraic subgroup in $(\mathbb{C}^*)^3$. Indeed,

$$\tau_1(\mathcal{V}^1_1(\pi)) = \{x_2 = x_3 = 0\} \cup \{x_1 - x_3 = x_2 - 2x_3 = 0\}.$$

Hence, π is not 1-formal.

EXAMPLE

- Let $\mathrm{Conf}_n(E)$ be the configuration space of n labeled points of an elliptic curve $E = \Sigma_1$.
- Using the computation of $H^\bullet(\mathrm{Conf}_n(\Sigma_g), \mathbb{C})$ by Totaro (1996), we find that $\mathcal{R}^1_1(\mathrm{Conf}_n(E))$ is equal to

$$\left\{ (x, y) \in \mathbb{C}^n \times \mathbb{C}^n \;\middle|\; \begin{array}{l} \sum_{i=1}^n x_i = \sum_{i=1}^n y_i = 0, \\ x_i y_j - x_j y_i = 0, \text{ for } 1 \leqslant i < j < n \end{array} \right\}$$

- For $n \geqslant 3$, this is an irreducible, non-linear variety (a rational normal scroll). Hence, $\mathrm{Conf}_n(E)$ is not 1-formal.

SPACES WITH FINITE MODELS

THEOREM (EXPONENTIAL AX–LINDEMANN THEOREM)

Let $V \subseteq \mathbb{C}^n$ and $W \subseteq (\mathbb{C}^*)^n$ be irreducible algebraic subvarieties.

1. *Suppose $\dim V = \dim W$ and $\exp(V) \subseteq W$. Then V is a translate of a linear subspace, and W is a translate of an algebraic subtorus.*
2. *Suppose the exponential map $\exp\colon \mathbb{C}^n \to (\mathbb{C}^*)^n$ induces a local analytic isomorphism $V_{(0)} \to W_{(1)}$. Then $W_{(1)}$ is the germ of an algebraic subtorus.*

THEOREM (BUDUR–WANG 2017)

If X is a q-finite space which admits a q-finite q-model, then, for all $i \leqslant q$ and $k \geqslant 0$, the irreducible components of $\mathcal{V}^i_k(X)$ passing through 1 are algebraic subtori of $\mathrm{Char}(X)$.

EXAMPLE

Let G be a f.p. group with $G_{ab} = \mathbb{Z}^n$ and $\mathcal{V}_1^1(G) = \{t \in (\mathbb{C}^\times)^n \mid \sum_{i=1}^n t_i = n\}$. Then G admits no 1-finite 1-model.

THEOREM (PAPADIMA–S. 2017)

Suppose X is $(q+1)$ finite, or X admits a q-finite q-model. Let $\mathfrak{M}_q(X)$ be Sullivan's q-minimal model of X. Then $b_i(\mathfrak{M}_q(X)) < \infty$, $\forall i \leqslant q+1$.

COROLLARY

Let G be a f.g. group. Assume that either G is finitely presented, or G has a 1-finite 1-model. Then $b_2(\mathfrak{M}_1(G)) < \infty$.

EXAMPLE

Let $G = \mathrm{F}_n / \mathrm{F}_n''$ with $n \geqslant 2$. We have $\mathcal{V}_1^1(G) = \mathcal{V}_1^1(\mathrm{F}_n) = (\mathbb{C}^\times)^n$, and so G passes the Budur–Wang test. But $b_2(\mathfrak{M}_1(G)) = \infty$, and so G admits no 1-finite 1-model (and is not finitely presented).

ASSOCIATED GRADED LIE ALGEBRAS

- The *lower central series* of a group G is defined inductively by $\gamma_1 G = G$ and $\gamma_{k+1} G = [\gamma_k G, G]$.
- This forms a filtration of G by characteristic subgroups. The LCS quotients, $\gamma_k G/\gamma_{k+1} G$, are abelian groups.
- The group commutator induces a graded Lie algebra structure on

$$\operatorname{gr}(G, \Bbbk) = \bigoplus_{k\geqslant 1} (\gamma_k G/\gamma_{k+1} G) \otimes_{\mathbb{Z}} \Bbbk.$$

- Assume G is finitely generated. Then $\operatorname{gr}(G)$ is also finitely generated (in degree 1) by $\operatorname{gr}_1(G) = H_1(G, \Bbbk)$.
- For instance, $\operatorname{gr}(F_n)$ is the free graded Lie algebra $\mathbb{L}_n := \operatorname{Lie}(\Bbbk^n)$.

HOLONOMY LIE ALGEBRAS

- Let A be a 1-finite cdga. Set $A_i = (A^i)^* = \operatorname{Hom}_{\Bbbk}(A^i, \Bbbk)$.
- Let $\mu^* \colon A_2 \to A_1 \wedge A_1$ be the dual to the multiplication map $\mu \colon A^1 \wedge A^1 \to A^2$.
- Let $d^* \colon A_2 \to A_1$ be the dual of the differential $d \colon A^1 \to A^2$.
- The *holonomy Lie algebra* of A is the quotient

$$\mathfrak{h}(A) = \operatorname{Lie}(A_1)/\langle \operatorname{im}(\mu^* + d^*)\rangle.$$

- For a f.g. group G, set $\mathfrak{h}(G) := \mathfrak{h}(H^\bullet(G, \Bbbk))$. There is then a canonical surjection $\mathfrak{h}(G) \twoheadrightarrow \operatorname{gr}(G)$, which is an isomorphism precisely when $\operatorname{gr}(G)$ is quadratic.

MALCEV LIE ALGEBRAS

- The group-algebra $\Bbbk G$ has a natural Hopf algebra structure, with comultiplication $\Delta(g) = g \otimes g$ and counit $\varepsilon \colon \Bbbk G \to \Bbbk$. Let $I = \ker \varepsilon$.
- (Quillen 1968) The I-adic completion of the group-algebra, $\widehat{\Bbbk G} = \varprojlim_k \Bbbk G/I^k$, is a filtered, complete Hopf algebra.
- An element $x \in \widehat{\Bbbk G}$ is called *primitive* if $\hat{\Delta} x = x \hat{\otimes} 1 + 1 \hat{\otimes} x$. The set of all such elements, with bracket $[x, y] = xy - yx$, and endowed with the induced filtration, is a complete, filtered Lie algebra.
- We then have $\mathfrak{m}(G) \cong \operatorname{Prim}(\widehat{\Bbbk G})$ and $\operatorname{gr}(\mathfrak{m}(G)) \cong \operatorname{gr}(G)$.
- (Sullivan 1977) G is 1-formal $\iff$ $\mathfrak{m}(G)$ is quadratic, namely:

$$\mathfrak{m}(G) = \widehat{\mathfrak{h}(H^\bullet(G, \Bbbk)}).$$

FINITENESS OBSTRUCTIONS FOR GROUPS

THEOREM (PS 2017)

A f.g. group G admits a 1-finite 1-model A if and only if $\mathfrak{m}(G)$ is the lcs completion of a finitely presented Lie algebra, namely,

$$\mathfrak{m}(G) \cong \widehat{\mathfrak{h}(A)}.$$

THEOREM (PS 2017)

Let G be a f.g. group which has a free, non-cyclic quotient. Then:

- G/G'' *is not finitely presentable.*
- G/G'' *does not admit a 1-finite 1-model.*