Geometry and topology of cohomology jump loci

Lecture 1: Characteristic varieties

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THE CHARACTER GROUP

- Throughout, X will be a connected CW-complex with finite q-skeleton, for some $q \geqslant 1$. We may assume X has a single 0-cell, call it e^0.
- Let $G = \pi_1(X, e^0)$ be the fundamental group of X: a finitely generated group, with generators $x_1 = [e_1^1], \dots, x_m = [e_m^1]$.
- The *character group*,

$$\widehat{G} = \operatorname{Hom}(G, \mathbb{C}^\times) \subset (\mathbb{C}^\times)^m$$

  is a (commutative) algebraic group, with multiplication $\rho \cdot \rho'(g) = \rho(g)\rho'(g)$, and identity $G \to \mathbb{C}^\times$, $g \mapsto 1$.
- Let $G_{\mathrm{ab}} = G/G' \cong H_1(X, \mathbb{Z})$ be the abelianization of G. The projection $\mathrm{ab}\colon G \to G_{\mathrm{ab}}$ induces an isomorphism $\widehat{G}_{\mathrm{ab}} \xrightarrow{\simeq} \widehat{G}$.

- The identity component, $\widehat{G}^0$, is isomorphic to a complex algebraic torus of dimension $n = \operatorname{rank} G_{\mathrm{ab}}$.

- The other connected components are all isomorphic to $\widehat{G}^0 = (\mathbb{C}^\times)^n$, and are indexed by the finite abelian group $\operatorname{Tors}(G_{\mathrm{ab}})$.

- $\operatorname{Char}(X) = \widehat{G}$ is the moduli space of rank 1 local systems on X:

$$\rho \colon G \to \mathbb{C}^\times \quad \rightsquigarrow \quad \mathbb{C}_\rho$$

  the complex vector space $\mathbb{C}$, viewed as a right module over the group ring $\mathbb{Z}G$ via $a \cdot g = \rho(g)a$, for $g \in G$ and $a \in \mathbb{C}$.

THE EQUIVARIANT CHAIN COMPLEX

- Let $p\colon \widetilde{X} \to X$ be the universal cover. The cell structure on X lifts to a cell structure on $\widetilde{X}$.
- Fixing a lift $\tilde{e}^0 \in p^{-1}(e^0)$ identifies $G = \pi_1(X, e^0)$ with the group of deck transformations of $\widetilde{X}$.
- Thus, we may view the cellular chain complex of $\widetilde{X}$ as a chain complex of left $\mathbb{Z}G$-modules,

$$\cdots \longrightarrow C_{i+1}(\widetilde{X},\mathbb{Z}) \xrightarrow{\tilde{\partial}_{i+1}} C_i(\widetilde{X},\mathbb{Z}) \xrightarrow{\tilde{\partial}_i} C_{i-1}(\widetilde{X},\mathbb{Z}) \longrightarrow \cdots .$$

- $\tilde{\partial}_1(\tilde{e}^1_i) = (x_i - 1)\tilde{e}^0$.
- $\tilde{\partial}_2(\tilde{e}^2) = \sum_{i=1}^m (\partial r/\partial x_i)^\phi \cdot \tilde{e}^1_i$, where
  - $r \in F_m = \langle x_1, \dots, x_m \rangle$ is the word traced by the attaching map of e^2;
  - $\partial r/\partial x_i \in \mathbb{Z}F_m$ are the Fox derivatives of r;
  - $\phi\colon \mathbb{Z}F_m \to \mathbb{Z}G$ is the linear extension of the projection $F_m \twoheadrightarrow G$.

- $H_*(X, \mathbb{C}_\rho)$ is the homology of the chain complex of $\mathbb{C}$-vector spaces $\mathbb{C}_\rho \otimes_{\mathbb{Z}G} C_\bullet(\widetilde{X}, \mathbb{Z})$:

$$\cdots \longrightarrow C_{i+1}(X,\mathbb{C}) \xrightarrow{\tilde{\partial}_{i+1}(\rho)} C_i(X,\mathbb{C}) \xrightarrow{\tilde{\partial}_i(\rho)} C_{i-1}(X,\mathbb{C}) \longrightarrow \cdots,$$

  where the evaluation of $\tilde{\partial}_i$ at ρ is obtained by applying the ring homomorphism $\mathbb{Z}G \to \mathbb{C}$, $g \mapsto \rho(g)$ to each entry of $\tilde{\partial}_i$.
- Alternatively, consider the universal abelian cover, X^{ab}, and its equivariant chain complex, $C_\bullet(X^{\mathrm{ab}}, \mathbb{Z}) = \mathbb{Z}G_{\mathrm{ab}} \otimes_{\mathbb{Z}G} C_\bullet(\widetilde{X}, \mathbb{Z})$, with differentials $\partial_i^{\mathrm{ab}} = \mathrm{id} \otimes \tilde{\partial}_i$.
- Then $H_*(X, \mathbb{C}_\rho)$ is computed from the resulting $\mathbb{C}$-chain complex, with differentials $\partial_i^{\mathrm{ab}}(\rho) = \tilde{\partial}_i(\rho)$.
- The identity $1 \in \mathrm{Char}(X)$ yields the trivial local system, $\mathbb{C}_1 = \mathbb{C}$, and $H_*(X, \mathbb{C})$ is the usual homology of X with $\mathbb{C}$-coefficients. Denote by $b_i(X) = \dim_{\mathbb{C}} H_i(X, \mathbb{C})$ the ith Betti number of X.

Characteristic varieties

Definition

The *characteristic varieties* of X are the sets

$$\mathcal{V}^i_k(X) = \{\rho \in \mathrm{Char}(X) \mid \dim_{\mathbb{C}} H_i(X, \mathbb{C}_\rho) \geqslant k\}.$$

- For each i, get stratification $\mathrm{Char}(X) = \mathcal{V}^i_0 \supseteq \mathcal{V}^i_1 \supseteq \mathcal{V}^i_2 \supseteq \cdots$
- $1 \in \mathcal{V}^i_k(X) \iff b_i(X) \geqslant k$.
- $\mathcal{V}^0_1(X) = \{1\}$ and $\mathcal{V}^0_k(X) = \emptyset$, for $k > 1$.
- Define analogously $\mathcal{V}^i_k(X, \Bbbk) \subset \mathrm{Hom}(G, \Bbbk^\times)$, for arbitrary field $\Bbbk$. Then $\mathcal{V}^i_k(X, \Bbbk) = \mathcal{V}^i_k(X, \mathbb{K}) \cap \mathrm{Hom}(G, \Bbbk^\times)$, for any $\Bbbk \subseteq \mathbb{K}$.

Lemma

For each $0 \leqslant i \leqslant q$ and $k \geqslant 0$, the set $\mathcal{V}^i_k(X)$ is a Zariski closed subset of the algebraic group $\widehat{G} = \mathrm{Char}(X)$.

Proof (for $i < q$).

Let $R = \mathbb{C}[G_{\mathrm{ab}}]$ be the coordinate ring of $\widehat{G} = \widehat{G}_{\mathrm{ab}}$. By definition, a character ρ belongs to $\mathcal{V}^i_k(X)$ if and only if

$$\operatorname{rank} \partial^{\mathrm{ab}}_{i+1}(\rho) + \operatorname{rank} \partial^{\mathrm{ab}}_{i}(\rho) \leqslant c_i - k,$$

where $c_i = c_i(X)$ is the number of i-cells of X. Hence,

$$\begin{aligned}\mathcal{V}^i_k(X) &= \bigcap_{r+s=c_i-k+1;\, r,s\geqslant 0} \{\rho \in \widehat{G} \mid \operatorname{rank} \partial^{\mathrm{ab}}_{i+1}(\rho) \leqslant r-1 \text{ or } \operatorname{rank} \partial^{\mathrm{ab}}_{i}(\rho) \leqslant s-1\}\\ &= V\Big(\sum_{r+s=c_i-k+1;\, r,s\geqslant 0} I_r(\partial^{\mathrm{ab}}_{i}) \cdot I_s(\partial^{\mathrm{ab}}_{i+1})\Big),\end{aligned}$$

where $I_r(\varphi) =$ ideal of $r \times r$ minors of φ. $\square$

The characteristic varieties are homotopy-type invariants of a space:

LEMMA

Suppose $X \simeq X'$. There is then an isomorphism $\widehat{G}' \cong \widehat{G}$, which restricts to isomorphisms $\mathcal{V}^i_k(X') \cong \mathcal{V}^i_k(X)$, for all $i \leqslant q$ and $k \geqslant 0$.

PROOF.

Let $f\colon X \to X'$ be a (cellular) homotopy equivalence.

The induced homomorphism $f_\sharp\colon \pi_1(X, e^0) \to \pi_1(X', e'^0)$, yields an isomorphism of algebraic groups, $\hat{f}_\sharp\colon \widehat{G'} \to \widehat{G}$.

Lifting f to a cellular homotopy equivalence, $\tilde{f}\colon \widetilde{X} \to \widetilde{X}'$, defines isomorphisms $H_i(X, \mathbb{C}_{\rho\circ f_\sharp}) \to H_i(X', \mathbb{C}_\rho)$, for each $\rho \in \widehat{G}'$.

Hence, $\hat{f}_\sharp$ restricts to isomorphisms $\mathcal{V}^i_k(X') \to \mathcal{V}^i_k(X)$. $\square$

DEGREE 1 CHARACTERISTIC VARIETIES

- $\mathcal{V}^1_k(X)$ depends only on $G = \pi_1(X)$ (in fact, only on G/G''), so we may write these sets as $\mathcal{V}^1_k(G)$.
- Suppose $G = \langle x_1, \dots, x_m \mid r_1, \dots, r_p \rangle$ is finitely presented
- Away from $1 \in \widehat{G}$, we have that $\mathcal{V}^1_k(G) = V(E_k(\partial^{\mathrm{ab}}_1))$, the zero-set of the ideal of codimension k minors of the Alexander matrix

$$\partial^{\mathrm{ab}}_1 = \big(\partial r_i / \partial x_j\big)^{\mathrm{ab}} \colon \mathbb{Z}G^p_{\mathrm{ab}} \to \mathbb{Z}G^m_{\mathrm{ab}}.$$

- If $\varphi\colon G \twoheadrightarrow Q$ is an epimorphism, then, for each $k \geqslant 1$, the induced monomorphism between character groups, $\varphi^*\colon \widehat{Q} \hookrightarrow \widehat{G}$, restricts to an embedding $\mathcal{V}^1_k(Q) \hookrightarrow \mathcal{V}^1_k(G)$.
- Given any subvariety $W \subset (\mathbb{C}^\times)^n$ defined over $\mathbb{Z}$, there is a finitely presented group G such that $G_{\mathrm{ab}} = \mathbb{Z}^n$ and $\mathcal{V}^1_1(G) = W$.

WARM-UP EXAMPLES

EXAMPLE (THE CIRCLE)

We have $\widetilde{S^1} = \mathbb{R}$.
Identify $\pi_1(S^1, *) = \mathbb{Z} = \langle t \rangle$ and $\mathbb{Z}\mathbb{Z} = \mathbb{Z}[t^{\pm 1}]$. Then:

$$C_\bullet(\widetilde{S^1}) : \ 0 \longrightarrow \mathbb{Z}[t^{\pm 1}] \xrightarrow{t-1} \mathbb{Z}[t^{\pm 1}] \longrightarrow 0$$

For $\rho \in \operatorname{Hom}(\mathbb{Z}, \mathbb{C}^\times) = \mathbb{C}^\times$, we get

$$\mathbb{C}_\rho \otimes_{\mathbb{Z}\mathbb{Z}} C_\bullet(\widetilde{S^1}) : \ 0 \longrightarrow \mathbb{C} \xrightarrow{\rho-1} \mathbb{C} \longrightarrow 0$$

which is exact, except for $\rho = 1$, when $H_0(S^1, \mathbb{C}) = H_1(S^1, \mathbb{C}) = \mathbb{C}$.
Hence:

$$\mathcal{V}^0_1(S^1) = \mathcal{V}^1_1(S^1) = \{1\}$$

$$\mathcal{V}^i_k(S^1) = \emptyset, \quad \text{otherwise.}$$

EXAMPLE (THE n-TORUS)

Identify $\pi_1(T^n) = \mathbb{Z}^n$, and $\operatorname{Hom}(\mathbb{Z}^n, \mathbb{C}^\times) = (\mathbb{C}^\times)^n$. Using the Koszul resolution $C_\bullet(\widetilde{T^n})$ as above, we get

$$\mathcal{V}^i_k(T^n) = \begin{cases} \{1\} & \text{if } k \leqslant \binom{n}{i}, \\ \emptyset & \text{otherwise.} \end{cases}$$

EXAMPLE (NILMANIFOLDS)

More generally, let M be a nilmanifold. An inductive argument on the nilpotency class of $\pi_1(M)$, based on the Hochschild-Serre spectral sequence, yields

$$\mathcal{V}^i_k(M) = \begin{cases} \{1\} & \text{if } k \leqslant b_i(M), \\ \emptyset & \text{otherwise} \end{cases}$$

EXAMPLE (WEDGE OF CIRCLES)

Identify $\pi_1(\bigvee^n S^1) = F_n$, and $\operatorname{Hom}(F_n, \mathbb{C}^\times) = (\mathbb{C}^\times)^n$. Then:

$$\mathcal{V}^1_k(\bigvee^n S^1) = \begin{cases} (\mathbb{C}^\times)^n & \text{if } k < n, \\ \{1\} & \text{if } k = n, \\ \emptyset & \text{if } k > n. \end{cases}$$

EXAMPLE (ORIENTABLE SURFACE OF GENUS $g > 1$)

Write $\pi_1(\Sigma_g) = \langle x_1, \dots, x_g, y_1, \dots, y_g \mid [x_1, y_1] \cdots [x_g, y_g] = 1 \rangle$, and identify $\operatorname{Hom}(\pi_1(\Sigma_g), \mathbb{C}^\times) = (\mathbb{C}^\times)^{2g}$. Then:

$$\mathcal{V}^i_k(\Sigma_g) = \begin{cases} (\mathbb{C}^\times)^{2g} & \text{if } i = 1,\ k < 2g-1, \\ \{1\} & \text{if } i = 1,\ k = 2g-1, 2g; \text{ or } i = 2,\ k = 1, \\ \emptyset & \text{otherwise.} \end{cases}$$

TORIC COMPLEXES AND RAAGS

- Given L simplicial complex on n vertices, define the *toric complex* T_L as the subcomplex of T^n obtained by deleting the cells corresponding to the missing simplices of L:

$$T_L = \bigcup_{\sigma \in L} T^\sigma, \quad \text{where } T^\sigma = \{x \in T^n \mid x_i = * \text{ if } i \notin \sigma\}.$$

- Let $\Gamma = (\mathsf{V}, \mathsf{E})$ be the graph with vertex set the 0-cells of L, and edge set the 1-cells of L. Then $\pi_1(T_L)$ is the *right-angled Artin group* associated to Γ:

$$G_\Gamma = \langle v \in V \mid vw = wv \text{ if } \{v, w\} \in E \rangle.$$

- Properties:
  - $\Gamma = \overline{K}_n \Rightarrow G_\Gamma = F_n$
  - $\Gamma = K_n \Rightarrow G_\Gamma = \mathbb{Z}^n$
  - $\Gamma = \Gamma' \coprod \Gamma'' \Rightarrow G_\Gamma = G_{\Gamma'} * G_{\Gamma''}$
  - $\Gamma = \Gamma' * \Gamma'' \Rightarrow G_\Gamma = G_{\Gamma'} \times G_{\Gamma''}$

- Identify character group $\widehat{G}_\Gamma = \operatorname{Hom}(G_\Gamma, \mathbb{C}^\times)$ with the algebraic torus $(\mathbb{C}^\times)^{\mathsf{V}} := (\mathbb{C}^\times)^n$.
- For each subset $\mathsf{W} \subseteq \mathsf{V}$, let $(\mathbb{C}^\times)^{\mathsf{W}} \subseteq (\mathbb{C}^\times)^{\mathsf{V}}$ be the corresponding coordinate subtorus; in particular, $(\mathbb{C}^\times)^{\varnothing} = \{1\}$.

THEOREM (PAPADIMA–S. 2006/09)

$$\mathcal{V}^i_k(T_L) = \bigcup_{\substack{\mathsf{W}\subseteq\mathsf{V} \\ \sum_{\sigma\in L_{\mathsf{V}\setminus\mathsf{W}}} \dim_{\mathbb{C}} \widetilde{H}_{i-1-|\sigma|}(\mathrm{lk}_{L_{\mathsf{W}}}(\sigma),\mathbb{C})\geqslant k}} (\mathbb{C}^\times)^{\mathsf{W}},$$

where L_{W} is the subcomplex induced by L on W, and $\mathrm{lk}_K(\sigma)$ is the link of a simplex σ in a subcomplex $K \subseteq L$.

In particular:

$$\mathcal{V}^1_1(G_\Gamma) = \bigcup_{\substack{\mathsf{W}\subseteq\mathsf{V} \\ \Gamma_{\mathsf{W}} \text{ disconnected}}} (\mathbb{C}^\times)^{\mathsf{W}}.$$

QUASI-PROJECTIVE MANIFOLDS

- A space M is said to be a quasi-projective variety if M is a Zariski open subset of a projective variety $\overline{M}$ (i.e., a Zariski closed subset of some projective space).
- By resolution of singularities, a connected, smooth, complex quasi-projective variety M can realized as $M = \overline{M} \backslash D$, where $\overline{M}$ is a smooth, complex projective variety, and D is a normal crossing divisor. For short, we say M is a *quasi-projective manifold*.
- When $M = \Sigma$ is a smooth complex curve with $\chi(M) < 0$, we saw that $\mathcal{V}^1_1(M) = \operatorname{Char}(M)$.

THEOREM (GREEN–LAZARSFELD, ..., ARAPURA, ..., BUDUR–WANG)

All the characteristic varieties of a quasi-projective manifold M are finite unions of torsion-translates of subtori of $\operatorname{Char}(M)$, i.e., $\mathcal{V}^i_k(M) = \bigcup_\alpha \rho_\alpha T_\alpha$, where T_α is an algebraic subtorus and $\rho_\alpha^{n_\alpha} = 1$.

- An algebraic map $f\colon M \to \Sigma$ to a smooth complex curve Σ is *admissible* if f is a surjection and has connected generic fiber.
- The homomorphism $f_\sharp\colon \pi_1(M) \to \pi_1(\Sigma)$ is surjective; thus, $\hat{f}_\sharp\colon \operatorname{Char}(\Sigma) \to \operatorname{Char}(M)$ is injective, and $\operatorname{im}(\hat{f}_\sharp)$ is a complex subtorus of $\mathcal{V}^1_1(M)$.
- Up to reparametrization at the target, there is a finite set $\mathcal{E}(M)$ of admissible maps $f\colon M \to \Sigma$ with $\chi(\Sigma) < 0$.

THEOREM (ARAPURA 1997)

The correspondence $f \rightsquigarrow \hat{f}_\sharp \operatorname{Char}(\Sigma)$ defines a bijection between $\mathcal{E}(M)$ and the set of positive-dimensional, irreducible components of $\mathcal{V}^1_1(M)$ passing through 1.

THEOREM (DIMCA–PAPADIMA–S. (2008–09))

If ρT and $\rho' T'$ are two distinct irreducible components of $\mathcal{V}^1_1(M)$, then either $T = T'$ or $T \cap T' = \{1\}$. Hence, distinct components of $\mathcal{V}^1_1(M)$ meet only in a finite set of finite-order characters.

EXAMPLE (ORDERED CONFIGURATION SPACE OF n POINTS IN $\mathbb{C}$)

- Let $\mathrm{Conf}_n(\Sigma) = \{z \in \Sigma^n \mid z_i \neq z_j\}$, and set $M_n = \mathrm{Conf}_n(\mathbb{C})$.
- Then $\pi_1(M_n) = P_n$, and so $\mathrm{Char}(M_n) = (\mathbb{C}^\times)^{\binom{n}{2}}$.
- (D. Cohen–S. 1999) The set of irreducible components of $\mathcal{V}^1_1(M_n)$ passing through 1 consists of the following $\binom{n}{3} + \binom{n}{4} = \binom{n+1}{4}$ subtori of dimension 2:

$$T_{ijk} = \big\{ t_{ij} t_{ik} t_{jk} = 1 \text{ and } t_{rs} = 1 \text{ if } \{r,s\} \not\subset \{i,j,k\} \big\}.$$

$$T_{ijk\ell} = \Big\{ t_{ij} = t_{jk},\ t_{jk} = t_{i\ell},\ t_{ik} = t_{j\ell},\ \prod_{1 \leqslant p < q \leqslant n} t_{pq} = 1, \text{ and } t_{rs} = 1 \text{ if } \{r,s\} \not\subset \{i,j,k,\ell\} \Big\}.$$

EXAMPLE (ORDERED CONFIGURATION SPACE OF $E = \Sigma_1$)

(Dimca 2010) The set of positive-dimensional components of $\mathcal{V}^1_1(\mathrm{Conf}_n(E))$ consists of $\binom{n}{2}$ two-dimensional subtori of $(\mathbb{C}^\times)^{n(n-1)}$, of the form $T_{ij} = \mathrm{im}(\widehat{f_{ij_\sharp}})$, where $f_{ij} \colon E^n \to E \backslash \{1\}$ is given by $z \mapsto z_i z_j^{-1}$.

HOMOLOGY OF FINITE ABELIAN COVERS

The characteristic varieties can be used to compute the homology of finite, abelian, regular covers (work of A. Libgober, E. Hironaka, P. Sarnak–S. Adams, M. Sakuma, D. Matei–A. S. from the 1990s).

THEOREM

Let $Y \to X$ be a regular cover, defined by an epimorphism ν from $G = \pi_1(X)$ to a finite abelian group A. Let $\Bbbk$ be an algebraically closed field of characteristic not dividing the order of A. Then, for each $i \geqslant 0$,

$$\dim_{\Bbbk} H_i(Y, \Bbbk) = \sum_{k \geqslant 1} \left| \operatorname{im}(\hat{\nu}) \cap \mathcal{V}^i_k(X, \Bbbk) \right|.$$

PROOF (SKETCH).

By Shapiro's Lemma and Maschke's Theorem,

$$H_i(Y, \Bbbk) \cong H_i(X, \Bbbk[A]) \cong \bigoplus_{\rho \in \operatorname{im}(\hat{\nu})} H_i(X, \Bbbk_\rho). \qquad \square$$

EXAMPLE

- Let $X = \bigvee_1^n S^1$, and let $Y \to X$ be the 2-fold cover defined by $\nu\colon F_n \to \mathbb{Z}_2$, $x_i \mapsto 1$. (Of course, $Y = \bigvee_1^{2n-1} S^1$.)
- Inside $\mathrm{Char}(X) = (\mathbb{C}^\times)^n$, we have that $\mathrm{im}(\hat{\nu}) = \{1, -1\}$, and $\mathcal{V}_1^1(X) = \cdots = \mathcal{V}_{n-1}^1(X) = (\mathbb{C}^\times)^n$, while $\mathcal{V}_n^1(X) = \{1\}$.
- Hence, $b_1(Y) = n + (n-1) = 2n - 1$.

EXAMPLE

- Let $X = \Sigma_g$ with $g \geqslant 2$, and let $Y \to X$ be an n-fold regular abelian cover. (Of course, $Y = \Sigma_h$, where $h = ng - n + 1$.)
- Inside $\mathrm{Char}(X) = (\mathbb{C}^\times)^{2g}$, we have $\mathcal{V}_1^1(X) = \cdots = \mathcal{V}_{2g-2}^1(X) = (\mathbb{C}^\times)^{2g}$ and $\mathcal{V}_{2g-1}^1(X) = \mathcal{V}_{2g}^1(X) = \{1\}$.
- Hence, $b_1(Y) = 2g + (n-1)(2g-2) = 2(ng - n + 1)$.

DWYER–FRIED SETS

- The characteristic varieties can also be used to determine the homological finiteness properties of free abelian, regular covers.
- For a fixed $r \in \mathbb{N}$, the regular $\mathbb{Z}^r$-covers of a space X are classified by epimorphisms $\nu\colon \pi \twoheadrightarrow \mathbb{Z}^r$.
- Such covers are parameterized by the Grassmannian $\mathrm{Gr}_r(\mathbb{Q}^n)$, where $n = b_1(X)$, via the correspondence

$$\left\{\text{regular } \mathbb{Z}^r\text{-covers of } X\right\} \longleftrightarrow \left\{r\text{-planes in } H^1(X,\mathbb{Q})\right\}$$
$$X^\nu \to X \longleftrightarrow P_\nu := \mathrm{im}(\nu^*\colon \mathbb{Q}^r \to H^1(X,\mathbb{Q}))$$

- The *Dwyer–Fried invariants* of X are the subsets

$$\Omega_r^i(X) = \left\{P_\nu \in \mathrm{Gr}_r(\mathbb{Q}^n) \mid b_j(X^\nu) < \infty \text{ for } j \leqslant i\right\}.$$

- For each $r > 0$, we get a descending filtration,
$$\mathrm{Gr}_r(\mathbb{Q}^n) = \Omega_r^0(X) \supseteq \Omega_r^1(X) \supseteq \Omega_r^2(X) \supseteq \cdots.$$
$\Omega_1^i(X)$ is open, but $\Omega_r^i(X)$ may be non-open for $r > 1$.

THEOREM (DWYER–FRIED 1987, PAPADIMA–S. 2010)

For an epimorphism $\nu\colon \pi_1(X) \twoheadrightarrow \mathbb{Z}^r$*, the following are equivalent:*

- *The vector space* $\bigoplus_{j=0}^{i} H_j(X^\nu, \mathbb{C})$ *is finite-dimensional.*
- *The algebraic torus* $T_\nu = \operatorname{im}\left(\hat{\nu}\colon \widehat{\mathbb{Z}^r} \hookrightarrow \widehat{\pi_1(X)}\right)$ *intersects the variety* $\mathcal{W}^i(X) = \bigcup_{j\leqslant i} \mathcal{V}_1^j(X)$ *in only finitely many points.*

Note that $\exp(P_\nu \otimes \mathbb{C}) = T_\nu$. Thus:

COROLLARY

$$\Omega_r^i(X) = \left\{P \in \operatorname{Gr}_r(H^1(X,\mathbb{Q})) \;\middle|\; \dim\left(\exp(P \otimes \mathbb{C}) \cap \mathcal{W}^i(X)\right) = 0\right\}$$

COROLLARY

- *If* $\mathcal{W}^i(X)$ *is finite, then* $\Omega_r^i(X) = \operatorname{Gr}_r(\mathbb{Q}^n)$*, where* $n = b_1(X)$*.*
- *If* $\mathcal{W}^i(X)$ *is infinite, then* $\Omega_n^q(X) = \emptyset$*, for all* $q \geqslant i$*.*

Duality and abelian duality

- Let X be a connected, finite-type CW-complex, with $G = \pi_1(X)$.
- (Bieri–Eckmann 1978) X is a *duality space* of dimension n if $H^i(X, \mathbb{Z}G) = 0$ for $i \neq n$ and $H^n(X, \mathbb{Z}G) \neq 0$ and torsion-free.
- Let $D = H^n(X, \mathbb{Z}G)$ be the dualizing $\mathbb{Z}G$-module. Given any $\mathbb{Z}G$-module A, we have: $H^i(X, A) \cong H_{n-i}(X, D \otimes A)$.
- (Denham–S.–Yuzvinsky 2016/17) X is an *abelian duality space* of dimension n if $H^i(X, \mathbb{Z}G_{\text{ab}}) = 0$ for $i \neq n$ and $H^n(X, \mathbb{Z}G_{\text{ab}}) \neq 0$ and torsion-free.
- Let $B = H^n(X, \mathbb{Z}G_{\text{ab}})$ be the dualizing $\mathbb{Z}G_{\text{ab}}$-module. Given any $\mathbb{Z}G_{\text{ab}}$-module A, we have: $H^i(X, A) \cong H_{n-i}(X, B \otimes A)$.

THEOREM (DSY)

Let X be an abelian duality space of dimension n. Then:

- $b_1(X) \geqslant n-1$.
- $b_i(X) \neq 0$, *for* $0 \leqslant i \leqslant n$ *and* $b_i(X) = 0$ *for* $i > n$.
- $(-1)^n \chi(X) \geqslant 0$.
- *The characteristic varieties propagate, i.e.,* $\mathcal{V}_1^1(X) \subseteq \cdots \subseteq \mathcal{V}_1^n(X)$.

THEOREM (DENHAM–S. 2018)

Let M be a quasi-projective manifold of dimension n. Suppose M has a smooth compactification $\overline{M}$ for which

1. *Components of $\overline{M} \setminus M$ form an arrangement of hypersurfaces $\mathcal{A}$;*
2. *For each submanifold X in the intersection poset $L(\mathcal{A})$, the complement of the restriction of $\mathcal{A}$ to X is a Stein manifold.*

Then M is both a duality space and an abelian duality space of dimension n.

LINEAR, ELLIPTIC, AND TORIC ARRANGEMENTS

THEOREM (DS18)

Suppose that $\mathcal{A}$ is one of the following:

1. *An affine-linear arrangement in $\mathbb{C}^n$, or a hyperplane arrangement in $\mathbb{CP}^n$;*
2. *A non-empty elliptic arrangement in E^n;*
3. *A toric arrangement in $(\mathbb{C}^\times)^n$.*

Then the complement $M(\mathcal{A})$ is both a duality space and an abelian duality space of dimension $n-r$, $n+r$, and n, respectively, where r is the corank of the arrangement.

This theorem extends several previous results:

1. Davis, Januszkiewicz, Leary, and Okun (2011);
2. Levin and Varchenko (2012);
3. Davis and Settepanella (2013), Esterov and Takeuchi (2018).