FINITENESS AND FORMALITY OBSTRUCTIONS

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RESONANCE VARIETIES OF A CDGA

- Let $A = (A^{\bullet}, \mathrm{d})$ be a commutative, differential graded algebra over a field $\Bbbk$ of characteristic 0. That is:
  - $A = \bigoplus_{i \geqslant 0} A^i$, where A^i are $\Bbbk$-vector spaces.
  - The multiplication $\cdot \colon A^i \otimes A^j \to A^{i+j}$ is graded-commutative, i.e., $ab = (-1)^{|a||b|} ba$ for all homogeneous a and b.
  - The differential $\mathrm{d} \colon A^i \to A^{i+1}$ satisfies the graded Leibniz rule, i.e., $\mathrm{d}(ab) = \mathrm{d}(a)b + (-1)^{|a|} a\,\mathrm{d}(b)$.
- We assume A is connected (i.e., $A^0 = \Bbbk \cdot 1$) and of finite-type (i.e., $\dim A^i < \infty$ for all i).
- For each $a \in Z^1(A) \cong H^1(A)$, we have a cochain complex,

$$(A^{\bullet}, \delta_a) \colon \; A^0 \xrightarrow{\delta_a^0} A^1 \xrightarrow{\delta_a^1} A^2 \xrightarrow{\delta_a^2} \cdots,$$

  with differentials $\delta_a^i(u) = a \cdot u + \mathrm{d}(u)$, for all $u \in A^i$.
- The *resonance varieties* of A are the affine varieties

$$\mathcal{R}_s^i(A) = \{ a \in H^1(A) \mid \dim_{\Bbbk} H^i(A^{\bullet}, \delta_a) \geqslant s \}.$$

- Fix a $\Bbbk$-basis $\{e_1, \dots, e_r\}$ for A^1, and let $\{x_1, \dots, x_r\}$ be the dual basis for $A_1 = (A^1)^*$.
- Identify $\mathrm{Sym}(A_1)$ with $S = \Bbbk[x_1, \dots, x_r]$, the coordinate ring of the affine space A^1.
- Build a cochain complex of free S-modules, $\mathbf{L}(A) := (A^\bullet \otimes S, \delta)$:

$$\cdots \longrightarrow A^i \otimes S \xrightarrow{\delta^i} A^{i+1} \otimes S \xrightarrow{\delta^{i+1}} A^{i+2} \otimes S \longrightarrow \cdots,$$

  where $\delta^i(u \otimes f) = \sum_{j=1}^r e_j u \otimes f x_j + \mathrm{d}\, u \otimes f$.

- The specialization of $(A \otimes S, \delta)$ at $a \in Z^1(A)$ is (A, δ_a).
- Hence, $\mathcal{R}^i_s(A)$ is the zero-set of the ideal generated by all minors of size $b_i(A) - s + 1$ of the block-matrix $\delta^{i+1} \oplus \delta^i$.

CHARACTERISTIC VARIETIES

- Let X be a connected, finite-type CW-complex. Then $\pi = \pi_1(X, x_0)$ is a finitely presented group, with $\pi_{ab} \cong H_1(X, \mathbb{Z})$.
- The ring $R = \mathbb{C}[\pi_{ab}]$ is the coordinate ring of the character group, $\mathrm{Char}(X) = \mathrm{Hom}(\pi, \mathbb{C}^*) \cong (\mathbb{C}^*)^r \times \mathrm{Tors}(\pi_{ab})$, where $r = b_1(X)$.
- The *characteristic varieties* of X are the homology jump loci

$$\mathcal{V}^i_s(X) = \{\rho \in \mathrm{Char}(X) \mid \dim_{\mathbb{C}} H_i(X, \mathbb{C}_\rho) \geqslant s\}.$$

- These varieties are homotopy-type invariants of X, with $\mathcal{V}^1_s(X)$ depending only on $\pi = \pi_1(X)$.
- Set $\mathcal{V}^1_1(\pi) := \mathcal{V}^1_1(K(\pi, 1))$; then $\mathcal{V}^1_1(\pi) = \mathcal{V}_1(\pi/\pi'')$.

EXAMPLE

Let $f \in \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ be a Laurent polynomial, $f(1) = 0$. There is then a finitely presented group π with $\pi_{ab} = \mathbb{Z}^n$ such that $\mathcal{V}^1_1(\pi) = V(f)$.

TANGENT CONES

- Let $\exp\colon H^1(X,\mathbb{C}) \to H^1(X,\mathbb{C}^*)$ be the coefficient homomorphism induced by $\mathbb{C} \to \mathbb{C}^*$, $z \mapsto e^z$.
- Let $W = V(I)$, a Zariski closed subset of $\mathrm{Char}(G) = H^1(X,\mathbb{C}^*)$.
- The *tangent cone* at 1 to W is $\mathrm{TC}_1(W) = V(\mathrm{in}(I))$.
- The *exponential tangent cone* at 1 to W:

$$\tau_1(W) = \{z \in H^1(X,\mathbb{C}) \mid \exp(\lambda z) \in W,\ \forall \lambda \in \mathbb{C}\}.$$

- Both tangent cones are homogeneous subvarieties of $H^1(X,\mathbb{C})$; are non-empty iff $1 \in W$; depend only on the analytic germ of W at 1; commute with finite unions and arbitrary intersections.
- $\tau_1(W) \subseteq \mathrm{TC}_1(W)$, with $=$ if all irred components of W are subtori, but $\neq$ in general.
- (Dimca–Papadima–S. 2009) $\tau_1(W)$ is a finite union of rationally defined subspaces.

ALGEBRAIC MODELS FOR SPACES

- A CDGA map $\varphi\colon A \to B$ is a *quasi-isomorphism* if $\varphi^*\colon H^\bullet(A) \to H^\bullet(B)$ is an isomorphism.
- φ is a q-quasi-isomorphism (for some $q \geqslant 1$) if φ^* is an isomorphism in degrees $\leqslant q$ and is injective in degree $q+1$.
- Two CDGAs, A and B, are *(q-) equivalent* if there is a zig-zag of (q-) quasi-isomorphisms connecting A to B.
- A is *formal* (or just *q-formal*) if it is (q-) equivalent to $(H^\bullet(A), d = 0)$.
- A CDGA is *q-minimal* if it is of the form $(\bigwedge V, d)$, where the differential structure is the inductive limit of a sequence of Hirsch extensions of increasing degrees, and $V^i = 0$ for $i > q$.
- Every CDGA A with $H^0(A) = \Bbbk$ admits a *q-minimal model*, $\mathcal{M}_q(A)$ (i.e., a q-equivalence $\mathcal{M}_q(A) \to A$ with $\mathcal{M}_q(A) = (\bigwedge V, d)$ a q-minimal cdga), unique up to iso.

- Given any (path-connected) space X, there is an associated Sullivan $\mathbb{Q}$-cdga, $A_{\mathrm{PL}}(X)$, such that $H^{\bullet}(A_{\mathrm{PL}}(X)) = H^{\bullet}(X, \mathbb{Q})$.
- An *algebraic (q-)model* (over $\mathbb{k}$) for X is a $\mathbb{k}$-cgda (A, d) which is (q-) equivalent to $A_{\mathrm{PL}}(X) \otimes_{\mathbb{Q}} \mathbb{k}$.
- If M is a smooth manifold, then $\Omega_{\mathrm{dR}}(M)$ is a model for M (over $\mathbb{R}$).
- Examples of spaces having finite-type models include:
  - Formal spaces (such as compact Kähler manifolds, hyperplane arrangement complements, toric spaces, etc).
  - Smooth quasi-projective varieties, compact solvmanifolds, Sasakian manifolds, etc.

THE TANGENT CONE THEOREM

Let X be a connected CW-complex with finite q-skeleton. Suppose X admits a q-finite q-model A.

THEOREM

For all $i \leqslant q$ and all s:

- *(DPS 2009, Dimca–Papadima 2014)* $\mathcal{V}^i_s(X)_{(1)} \cong \mathcal{R}^i_s(A)_{(0)}$.
- *(Budur–Wang 2017) All the irreducible components of $\mathcal{V}^i_s(X)$ passing through the origin of $\mathrm{Char}(X)$ are* algebraic subtori.

Consequently,

$$\tau_1(\mathcal{V}^i_s(X)) = \mathrm{TC}_1(\mathcal{V}^i_s(X)) = \mathcal{R}^i_s(A).$$

THEOREM (PAPADIMA–S. 2017)

A f.g. group G admits a 1-finite 1-model if and only if the Malcev Lie algebra $\mathfrak{m}(G)$ is the LCS completion of a finitely presented Lie algebra.

INFINITESIMAL FINITENESS OBSTRUCTIONS

THEOREM

Let X be a connected CW-complex with finite q-skeleton. Suppose X admits a q-finite q-model A. Then, for all $i \leqslant q$ and all s,

- *(Dimca–Papadima 2014)* $\mathcal{V}^i_s(X)_{(1)} \cong \mathcal{R}^i_s(A)_{(0)}$.
  In particular, if X is q-formal, then $\mathcal{V}^i_s(X)_{(1)} \cong \mathcal{R}^i_s(X)_{(0)}$.
- *(Macinic, Papadima, Popescu, S. 2017)* $\operatorname{TC}_0(\mathcal{R}^i_s(A)) \subseteq \mathcal{R}^i_s(X)$.
- *(Budur–Wang 2017) All the irreducible components of $\mathcal{V}^i_s(X)$ passing through the origin of $H^1(X, \mathbb{C}^*)$ are* algebraic subtori.

EXAMPLE

Let G be a f.p. group with $G_{\mathrm{ab}} = \mathbb{Z}^n$ and $\mathcal{V}^1_1(G) = \{t \in (\mathbb{C}^*)^n \mid \sum_{i=1}^n t_i = n\}$. Then G admits no 1-finite 1-model.

THEOREM (PAPADIMA–S. 2017)

Suppose X is $(q+1)$ finite, or X admits a q-finite q-model. Then $b_i(\mathcal{M}_q(X)) < \infty$, for all $i \leqslant q+1$.

COROLLARY

Let G be a f.g. group. Assume that either G is finitely presented, or G has a 1-finite 1-model. Then $b_2(\mathcal{M}_1(G)) < \infty$.

EXAMPLE

- Consider the free metabelian group $G = \mathrm{F}_n / \mathrm{F}_n''$ with $n \geqslant 2$.
- We have $\mathcal{V}^1(G) = \mathcal{V}^1(\mathrm{F}_n) = (\mathbb{C}^*)^n$, and so G passes the Budur–Wang test.
- But $b_2(\mathcal{M}_1(G)) = \infty$, and so G admits no 1-finite 1-model (and is not finitely presented).

LOWER CENTRAL SERIES

- Let G be a group. The *lower central series* $\{\gamma_k(G)\}_{k\geqslant 1}$ is defined inductively by $\gamma_1(G) = G$ and $\gamma_{k+1}(G) = [G, \gamma_k(G)]$.
- Here, if $H, K < G$, then $[H, K]$ is the subgroup of G generated by $\{[a, b] := aba^{-1}b^{-1} \mid a \in H, b \in K\}$. If $H, K \lhd G$, then $[H, K] \lhd G$.
- The subgroups $\gamma_k(G)$ are, in fact, characteristic subgroups of G. Moreover $[\gamma_k(G), \gamma_\ell(G)] \subseteq \gamma_{k+\ell}(G)$, $\forall k, \ell \geqslant 1$.
- $\gamma_2(G) = [G, G]$ is the derived subgroup, and so $G/\gamma_2(G) = G_{\text{ab}}$.
- $[\gamma_k(G), \gamma_k(G)] \lhd \gamma_{k+1}(G)$, and thus the LCS quotients,

  $$\operatorname{gr}_k(G) := \gamma_k(G)/\gamma_{k+1}(G)$$

  are abelian.
- If G is finitely generated, then so are its LCS quotients. Set $\phi_k(G) := \operatorname{rank} \operatorname{gr}_k(G)$.

ASSOCIATED GRADED LIE ALGEBRA

- Fix a coefficient ring $\Bbbk$. Given a group G, we let

$$\mathrm{gr}(G, \Bbbk) = \bigoplus_{k \geqslant 1} \mathrm{gr}_k(G) \otimes \Bbbk.$$

- This is a graded Lie algebra, with Lie bracket $[\ ,\ ]\colon\ \mathrm{gr}_k \times \mathrm{gr}_\ell \to \mathrm{gr}_{k+\ell}$ induced by the group commutator.
- For $\Bbbk = \mathbb{Z}$, we simply write $\mathrm{gr}(G) = \mathrm{gr}(G, \mathbb{Z})$.
- The construction is functorial.
- Example: if F_n is the free group of rank n, then
    - $\mathrm{gr}(F_n)$ is the free Lie algebra $\mathrm{Lie}(\mathbb{Z}^n)$.
    - $\mathrm{gr}_k(F_n)$ is free abelian, of rank $\phi_k(F_n) = \frac{1}{k}\sum_{d|k} \mu(d) n^{\frac{k}{d}}$.

HOLONOMY LIE ALGEBRA

- A quadratic approximation of the Lie algebra $\operatorname{gr}(G,\Bbbk)$, where $\Bbbk$ is a field, is the *holonomy Lie algebra* of G, which is defined as

$$\mathfrak{h}(G,\Bbbk) := \operatorname{Lie}(H_1(G,\Bbbk))/\langle \operatorname{im}(\mu_G^\vee)\rangle,$$

  where
  - $L = \operatorname{Lie}(V)$ the free Lie algebra on the $\Bbbk$-vector space $V = H_1(G;\Bbbk)$, with $L_1 = V$ and $L_2 = V \wedge V$.
  - $\mu_G^\vee\colon H_2(G,\Bbbk) \to V \wedge V$ is the dual of the cup product map $\mu_G\colon H^1(G;\Bbbk) \wedge H^1(G;\Bbbk) \to H^2(G;\Bbbk)$.

- There is a surjective morphism of graded Lie algebras,

$$\mathfrak{h}(G,\Bbbk) \twoheadrightarrow \operatorname{gr}(G;\Bbbk)\,, \qquad (*)$$

  which restricts to isomorphisms $\mathfrak{h}_k(G,\Bbbk) \to \operatorname{gr}_k(G;\Bbbk)$ for $k \leqslant 2$.

ARRANGEMENT GROUPS AND LIE ALGEBRAS

- Let $\mathcal{A} = \{\ell_1, \dots, \ell_n\}$ be an affine line arrangement in $\mathbb{C}^2$, and let $G = G(\mathcal{A})$ be the fundamental group of the complement of $\mathcal{A}$.
- The holonomy Lie algebra $\mathfrak{h}(\mathcal{A}) := \mathfrak{h}(G(\mathcal{A}))$ has (combinatorially determined) presentation

$$\mathfrak{h}(\mathcal{A}) = \Big\langle x_1, \dots, x_n \mid \sum_{k \in P} [x_j, x_k],\ j \in \widehat{P},\ P \in \mathcal{P} \Big\rangle$$

where x_i represents the meridian about the i-th line, $\mathcal{P} \subset 2^{[n]}$ is the set of multiple points, and $\widehat{P} = P \backslash \{\max P\}$ for $P \in \mathcal{P}$.

- Thus, every double point $P = L_i \cap L_j$ contributes a relation $[x_i, x_j]$, each triple point $P = L_i \cap L_j \cap L_k$ contributes two relations, $[x_i, x_j] + [x_i, x_k]$ and $-[x_i, x_j] + [x_j, x_k]$, etc.
- Consequently, $\mathfrak{h}_1(\mathcal{A})$ is free abelian with basis $\{x_1, \dots, x_n\}$, while $\mathfrak{h}_2(\mathcal{A})$ is free abelian of rank $\phi_2 = \binom{n}{2} - \sum_{P \in \mathcal{P}} (|P| - 1)$, with basis $\{[x_i, x_j] :\ i, j \in \widehat{P},\ P \in \mathcal{P}\}$.

- The canonical projection $\mathfrak{h}(G,\mathbb{Q}) \twoheadrightarrow \operatorname{gr}(G,\mathbb{Q})$ is an isomorphism. Thus, the LCS ranks $\phi_k(G)$ are combinatorially determined.
- (Falk–Randell 1985) If $\mathcal{A}$ is *supersolvable*, with exponents $d_1, \dots, d_\ell$, then $G = F_{d_\ell} \rtimes \cdots \rtimes F_{d_2} \rtimes F_{d_1}$ (almost direct product) and
$$\phi_k(G) = \sum_{i=1}^{\ell} \phi_k(F_{d_i}).$$
- (Papadima–S. 2006) If $\mathcal{A}$ is *decomposable*, then $\mathfrak{h}(G) \twoheadrightarrow \operatorname{gr}(G)$ is an isomorphism, and $\operatorname{gr}_k(G)$ is free abelian of rank
$$\phi_k(G) = \sum_{X \in L_2(\mathcal{A})} \phi_k(F_{\mu(X)}) \text{ for } k \geqslant 2.$$
- (S. 2001) For $G = G(\mathcal{A})$, the groups $\operatorname{gr}_k(G)$ may have non-zero torsion. Question: Is that torsion combinatorially determined?
- (Artal Bartolo, Guerville-Ballé, and Viu-Sos 2018): Answer: No!

MALCEV LIE ALGEBRA

- Let $\Bbbk$ be a field of characteristic 0. The group-algebra $\Bbbk G$ has a natural Hopf algebra structure, with comultiplication $\Delta(g) = g \otimes g$ and counit $\varepsilon\colon \Bbbk G \to \Bbbk$.
- Let $I = \ker \varepsilon$. The I-adic completion $\widehat{\Bbbk G} = \varprojlim_k \Bbbk G/I^k$ is a filtered, complete Hopf algebra.
- An element $x \in \widehat{\Bbbk G}$ is called *primitive* if $\hat{\Delta} x = x \hat{\otimes} 1 + 1 \hat{\otimes} x$. The set of all such elements,

$$\mathfrak{m}(G, \Bbbk) = \operatorname{Prim}(\widehat{\Bbbk G}),$$

  with bracket $[x, y] = xy - yx$, is a complete, filtered Lie algebra, called the *Malcev Lie algebra* of G.
- If G is finitely generated, then $\mathfrak{m}(G, \Bbbk) = \varprojlim_k \mathcal{L}(G/\gamma_k(G) \otimes \Bbbk)$, and

$$\operatorname{gr}(\mathfrak{m}(G, \Bbbk)) \cong \operatorname{gr}(G, \Bbbk).$$

Formality and filtered formality

- Let G be a finitely generated group, $\Bbbk$ a field of characteristic 0.
- G is *filtered-formal* (over $\Bbbk$), if there is an isomorphism of filtered Lie algebras,

$$\mathfrak{m}(G;\Bbbk) \cong \widehat{\mathrm{gr}}(G;\Bbbk).$$

- G is 1*-formal* (over $\Bbbk$) if it is filtered formal and the canonical projection $\mathfrak{h}(G,\Bbbk) \twoheadrightarrow \mathrm{gr}(G;\Bbbk)$ is an isomorphism; that is,

$$\mathfrak{m}(G;\Bbbk) \cong \widehat{\mathfrak{h}}(G;\Bbbk).$$

- An obstruction to 1-formality is provided by the Massey products $\langle\alpha_1,\alpha_2,\alpha_3\rangle \in H^2(G,\Bbbk)$, for $\alpha_i \in H^1(G,\Bbbk)$ with $\alpha_1\alpha_2 = \alpha_2\alpha_3 = 0$.

Theorem (S.–Wang)

The above formality properties are preserved under finite direct products and coproducts, split injections, passing to solvable quotients, as well as extension or restriction of coefficient fields.

- Examples of 1-formal groups
  - Fundamental groups of compact Kähler manifolds; e.g., surface groups.
  - Fundamental groups of complements of complex algebraic affine hypersurfaces; e.g., arrangement groups, free groups.
  - Right-angled Artin groups.
- Examples of filtered formal groups
  - Finitely generated, torsion-free, 2-step nilpotent groups with torsion-free abelianization; e.g., the Heisenberg group.
  - Fundamental groups of Sasakian manifolds.
  - Fundamental groups of graphic configuration spaces of surfaces of genus $g \geqslant 1$; e.g., pure braid groups of elliptic curves.
- Examples of non-filtered formal groups
  - Certain finitely generated, torsion-free, 3-step nilpotent groups.

NILPOTENT QUOTIENTS

- Consider the tower of nilpotent quotients of a group G,

$$\cdots \longrightarrow G/\gamma_4(G) \xrightarrow{q_3} G/\gamma_3(G) \xrightarrow{q_2} G/\gamma_2(G) .$$

- We then have central extensions

$$0 \longrightarrow \operatorname{gr}_k(G) \longrightarrow G/\gamma_{k+1}(G) \xrightarrow{q_k} G/\gamma_k(G) \longrightarrow 0 .$$

- Passing to classifying spaces, we obtain commutative diagrams,

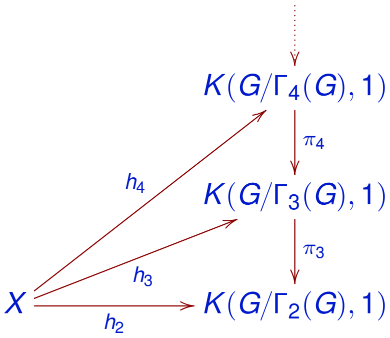
$$\begin{array}{ccc} & & K(G/\gamma_{k+1}(G),1) \\ & \overset{\psi_{k+1}}{\nearrow} & \downarrow \pi_k \\ G & \xrightarrow[\psi_k]{} & K(G/\gamma_k(G),1) \end{array}$$

- The map π_k may be viewed as the fibration with fiber $K(\operatorname{gr}_k(G),1)$ obtained as the pullback of the path space fibration with base $K(\operatorname{gr}_k(G),2)$ via a k-invariant $\chi_k\colon K(G/\gamma_k(G),1) \to K(\operatorname{gr}_k(G),2)$.

- Let X be a connected CW-complex, and let $G = \pi_1(X)$.
- A $K(G,1)$ can be constructed by adding to X cells of dimension 3 or higher. Thus, $H_2(G,\mathbb{Z})$ is a quotient of $H_2(X,\mathbb{Z})$.
- Let $\iota\colon X \to K(G,1)$ be the inclusion, and let

$$h_k = \psi_k \circ \iota\colon X \to K(G/\gamma_k(G),1).$$

- We obtain a Postnikov tower of fibrations,



Injective holonomy and k-invariants

- As noted by Stallings, there is an exact sequence,

$$H_2(X;\mathbb{Z}) \xrightarrow{(h_k)_*} H_2(G/\gamma_k(G);\mathbb{Z}) \xrightarrow{\chi_k} \operatorname{gr}_k(G) \longrightarrow 0 .$$

  In general, this sequence is natural but not split exact.

- The homomorphism

$$(h_2)_* \colon H_2(X;\mathbb{Z}) \longrightarrow H_2(G/\gamma_2(G);\mathbb{Z}) \cong H_1(G;\mathbb{Z}) \wedge H_1(G;\mathbb{Z})$$

  is the *holonomy map* of X (over $\mathbb{Z}$).

- When $H_1(G;\mathbb{Z})$ is torsion-free, set

$$\mathfrak{h}(G) = \operatorname{Lie}(H_1(G;\mathbb{Z}))/\langle \operatorname{im}((h_2)_*)\rangle.$$

- As before, get surjective morphism $\mathfrak{h}(G) \twoheadrightarrow \operatorname{gr}(G)$, which is injective in degrees $k \leqslant 2$.

Suppose $H = H_1(G;\mathbb{Z})$ is a finitely-generated, free abelian group, and the map $(h_2)_*\colon H_2(G;\mathbb{Z}) \to H \wedge H$ is injective.

Theorem (Rybnikov, Porter–S.)

The canonical projection $\mathfrak{h}_3(G) \to \operatorname{gr}_3(G)$ is an isomorphism.

Theorem (Porter–S.)

For each $k \geqslant 3$, there is a split exact sequence,

$$0 \longrightarrow \operatorname{gr}_k(G) \xrightarrow{\ i\ } H_2(G/\gamma_k(G);\mathbb{Z}) \xrightarrow{\ \pi\ } H_2(X;\mathbb{Z}) \longrightarrow 0\,. \qquad (\dagger)$$

with splitting map $\sigma\colon H_2(G/\gamma_k(G);\mathbb{Z}) \to \operatorname{gr}_k(G)$.

Moreover, the k-invariant of the extension from $G/\gamma_k(G)$ to $G/\gamma_{k+1}(G)$,

$$\chi_k \in \operatorname{Hom}(H_2(G/\gamma_k(G)), \operatorname{gr}_k(G)),$$

with respect to the direct sum decomposition defined by σ, is given by $\chi_k(x,c) = x - \lambda(c)$, where $\lambda = \sigma \circ (h_k)_*$.

A HOMOLOGICAL VERSION OF RYBNIKOV'S THEOREM

- Let X_a and X_b be two path-connected spaces with
  - Finitely generated, torsion-free H_1.
  - Injective holonomy map $H_2 \to H_1 \wedge H_1$.
- Let G_a and G_b be the respective fundamental groups.
- A homomorphism $f\colon G_a \to G_b$ induces homomorphisms on nilpotent quotients, $f_k\colon G_a/\gamma_k(G_a) \to G_b/\gamma_k(G_b)$.
- Suppose there is an isomorphism of graded algebras,

$$g\colon H^{\leqslant 2}(X_b) \to H^{\leqslant 2}(X_a).$$

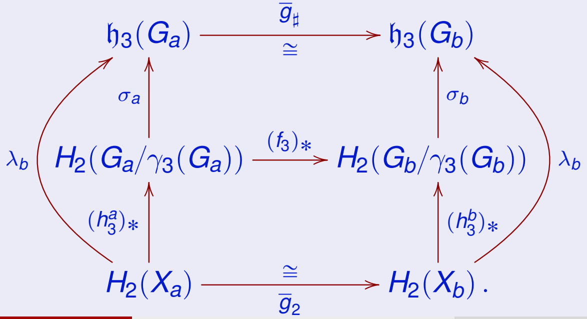
  Set $\overline{g} = g^{\vee}\colon H_{\leqslant 2}(X_a) \to H_{\leqslant 2}(X_b)$.
- There is then an isomorphism $G_a/\gamma_3(G_a) \xrightarrow{\simeq} G_b/\gamma_3(G_b)$.
- Moreover, the isomorphism $\overline{g}_1\colon H_1(X_a) \to H_1(X_b)$ induces an isomorphism $\overline{g}_\sharp\colon \mathfrak{h}_3(G_a) \to \mathfrak{h}_3(G_b)$.

THEOREM (RYBNIKOV, PORTER–S.)

Let $\sigma_b \colon H_2(G_b/\Gamma_3(G_b)) \to \mathfrak{h}_3(G_b)$ *be any left splitting of* $(\dagger)$, *and let* $f_3 \colon G_a/\gamma_3(G_a) \xrightarrow{\cong} G_b/\gamma_3(G_b)$ *be any extension of* $\overline{g}$. *Then* f_3 *extends to an isomorphism*

$$f_4 \colon G_a/\gamma_4(G_a) \xrightarrow{\cong} G_b/\gamma_4(G_b)$$

if and only if there are liftings $h_3^c \colon X_c \to K(G_c/\gamma_3(G_c), 1)$ *for* $c = a$ *and* b *such that the following diagram commutes*



AN EXTENSION TO CHARACTERISTIC p

- Let $p = 0$ or a prime.
- Given a group G, define subgroups $\gamma_k^p(G)$ as $\gamma_1^p(G) = G$ and

$$\gamma_{k+1}^p(G) = \langle gug^{-1}u^{-1}v^p : g \in G,\ u, v \in \gamma_k^p(G)\rangle.$$

- $\{\gamma_k^p(G)\}_{k\geqslant 1}$ is a descending central series of normal subgroups.
- For $p = 0$ it is the LCS; for $p \neq 0$ it is the most rapidly descending central series whose successive quotients are $\mathbb{Z}_p$-vector spaces.
- All the above results work for $p > 0$, by replacing $\gamma_k(G) \rightsquigarrow \gamma_k^p(G)$, $\mathfrak{h}_k(G) \rightsquigarrow \mathfrak{h}_k(G, \mathbb{Z}_p)$, and $H_*(-, \mathbb{Z}) \rightsquigarrow H_*(-, \mathbb{Z}_p)$.
- The entries of the matrices λ_a and λ_b are generalized Massey triple products in $H^2(X_b, \mathbb{Z}_p)$ and $H^2(X_a, \mathbb{Z}_p)$, respectively.

RYBNIKOV'S ARRANGEMENTS

- For groups of hyperplane arrangements, $\mathfrak{h}_2$ and $\mathfrak{h}_3$ are torsion free. Moreover, the holonomy map is injective, and so $\mathfrak{h}_3 \cong \mathrm{gr}_3$.
- The obstruction to extending $\overline{g}$ to an isomorphism from $G/\gamma_4(G_a)$ to $G/\gamma_4(G_b)$ is computed by generalized Massey triple products.
- Rybnikov used the above theorem (with $n = 3$) to show that arrangement groups are not combinatorially determined.
- Starting from a realization $\mathcal{A}$ of the MacLane matroid over $\mathbb{C}$, he constructed a pair of arrangements of 13 planes in $\mathbb{C}^3$, $\mathcal{A}^+$ and $\mathcal{A}^-$, such that
  - $L(\mathcal{A}^+) \cong L(\mathcal{A}^-)$, and thus $G^+/\gamma_3(G^+) \cong G^-/\gamma_3(G^-)$.
  - $G^+/\gamma_4(G^+) \not\cong G^-/\gamma_4(G^-)$.
- Goal: Make explicit the generalized Massey products (over $\mathbb{Z}_3$) that distinguish these two nilpotent quotients.

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