Sigma-invariants, cohomology jump loci, and tropicalization

Alex Suciu

Northeastern University

Hyperplane Arrangements and Singularities

Hyper-JARCS

A memorial conference for Ştefan Papadima

University of Tokyo, Japan

December 2, 2019

To the memory of Ştefan Papadima, 1953–2018

Tropical varieties

- Let $\mathbb{K} = \mathbb{C}\{\{t\}\}$ be the field of Puiseux series over $\mathbb{C}$.
- A non-zero element of $\mathbb{K}$ has the form $c(t) = c_1 t^{a_1} + c_2 t^{a_2} + \cdots$, where $c_i \in \mathbb{C}^*$ and $a_1 < a_2 < \cdots$ are rational numbers with a common denominator.
- The (algebraically closed) field $\mathbb{K}$ admits a discrete valuation $v\colon \mathbb{K}^* \to \mathbb{Q}$, given by $v(c(t)) = a_1$.
- Let $v\colon (\mathbb{K}^*)^n \to \mathbb{Q}^n \subset \mathbb{R}^n$ be the n-fold product of the valuation.
- The *tropicalization* of a variety $W \subset (\mathbb{K}^*)^n$, denoted $\operatorname{Trop}(W)$, is the closure of the set $v(W)$ in $\mathbb{R}^n$.
- This is a rational polyhedral complex in $\mathbb{R}^n$. For instance, if W is a curve, then $\operatorname{Trop}(W)$ is a graph with rational edge directions.

- If T be an algebraic subtorus of $(\mathbb{K}^*)^n$, then $\operatorname{Trop}(T)$ is the linear subspace $\operatorname{Hom}(\mathbb{K}^*, T) \otimes \mathbb{R} \subset \operatorname{Hom}(\mathbb{K}^*, (\mathbb{K}^*)^n) \otimes \mathbb{R} = \mathbb{R}^n$.
- Moreover, if $z \in (\mathbb{K}^*)^n$, then $\operatorname{Trop}(z \cdot T) = \operatorname{Trop}(T) + v(z)$.
- For a variety $W \subset (\mathbb{C}^*)^n$, we may define its tropicalization by setting $\operatorname{Trop}(W) = \operatorname{Trop}(W \times_{\mathbb{C}} \mathbb{K})$.
- In this case, the tropicalization is a polyhedral fan in $\mathbb{R}^n$.
- If $W = V(f)$ is a hypersurface, defined by a Laurent polynomial $f \in \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$, then $\operatorname{Trop}(W)$ is the positive-codimensional skeleton of the inner normal fan to the Newton polytope of f.

Exponential tangent cones

- Given a Zariski closed subset $W \subset (\mathbb{C}^*)^n$, let
$$\tau_1(W) = \{z \in \mathbb{C}^n \mid \exp(\lambda z) \in W,\ \forall \lambda \in \mathbb{C}\},$$
where $\exp\colon \mathbb{C}^n \to (\mathbb{C}^*)^n$.
- $\tau_1(W)$ depends only on the analytic germ of W at 1; it is non-empty iff $1 \in W$.
- If $T \cong (\mathbb{C}^*)^r$ is an algebraic subtorus, then $\tau_1(T) = T_1(T) \cong \mathbb{C}^r$.

LEMMA (DIMCA–PAPADIMA–S. 2009; S. 2014)

$\tau_1(W)$ is a finite union of rationally defined linear subspaces.

- Set $\tau_1^{\mathbb{k}}(W) = \tau_1(W) \cap \mathbb{k}^n$, for a subfield $\mathbb{k} \subset \mathbb{C}$.

LEMMA

Let $W \subset (\mathbb{C}^*)^n$ be an algebraic variety. Then $\tau_1^{\mathbb{R}}(W) \subseteq \operatorname{Trop}(W)$.

The Bieri–Neumann–Strebel–Renz invariants

- Let G be a finitely generated group, $n = b_1(G) > 0$. Let $S(G)$ be the unit sphere in $\operatorname{Hom}(G, \mathbb{R}) = \mathbb{R}^n$.

- (Bieri–Neumann–Strebel 1987)

  $$\Sigma^1(G) = \{\chi \in S(G) \mid \mathcal{C}_\chi(G) \text{ is connected}\},$$

  where $\mathcal{C}_\chi(G)$ is the induced subgraph of $\mathrm{Cay}(G)$ on vertex set $G_\chi = \{g \in G \mid \chi(g) \geq 0\}$.

- (Bieri–Renz 1988)

  $$\Sigma^k(G, \mathbb{Z}) = \{\chi \in S(G) \mid \text{the monoid } G_\chi \text{ is of type } \mathrm{FP}_k\},$$

  i.e., there is a projective $\mathbb{Z}G_\chi$-resolution $P_\bullet \to \mathbb{Z}$, with P_i finitely generated for all $i \leq k$. In particular, $\Sigma^1(G, \mathbb{Z}) = \Sigma^1(G)$.

- The BNSR-invariants of form a descending chain of open subsets,

  $$S(G) \supseteq \Sigma^1(G, \mathbb{Z}) \supseteq \Sigma^2(G, \mathbb{Z}) \supseteq \cdots .$$

- The Σ-invariants control the finiteness properties of normal subgroups $N \triangleleft G$ for which G/N is free abelian:

$$N \text{ is of type } \mathrm{FP}_k \iff S(G,N) \subseteq \Sigma^k(G,\mathbb{Z})$$

where $S(G,N) = \{\chi \in S(G) \mid \chi(N) = 0\}$.

- In particular: $\ker(\chi\colon G \twoheadrightarrow \mathbb{Z})$ is f.g. $\iff \{\pm\chi\} \subseteq \Sigma^1(G)$.

- More generally, let X be a connected CW-complex with finite k-skeleton, for some $k \geq 1$.

- Let $G = \pi_1(X, x_0)$. For each $\chi \in S(X) := S(G)$, let

$$\widehat{\mathbb{Z}G}_\chi = \Big\{\lambda \in \mathbb{Z}^G \mid \{g \in \operatorname{supp}\lambda \mid \chi(g) < c\} \text{ is finite, } \forall c \in \mathbb{R}\Big\}$$

be the Novikov–Sikorav completion of $\mathbb{Z}G$.

- (Farber–Geoghegan,–Schütz 2010)

$$\Sigma^q(X,\mathbb{Z}) = \{\chi \in S(X) \mid H_i(X, \widehat{\mathbb{Z}G}_{-\chi}) = 0,\ \forall\, i \leq q\}.$$

- (Bieri 2007) If G is FP_k, then $\Sigma^q(G,\mathbb{Z}) = \Sigma^q(K(G,1),\mathbb{Z})$, $\forall q \leq k$.

The Dwyer–Fried invariants

- The sphere $S(G)$ parametrizes all regular, free abelian covers of X. The Σ-invariants of X keep track of the geometric finiteness properties of these covers.
- Now fix the rank r of the deck-transformation group. Regular $\mathbb{Z}^r$-covers of X are classified by epimorphisms $\nu\colon G \twoheadrightarrow \mathbb{Z}^r$.
- Such covers are parameterized by the Grassmannian $\mathrm{Gr}_r(\mathbb{Q}^n)$, where $n = b_1(X)$, via the correspondence

$$\{\text{regular } \mathbb{Z}^r\text{-covers of } X\} \longleftrightarrow \{r\text{-planes in } H^1(X,\mathbb{Q})\}$$
$$X^\nu \to X \quad \longleftrightarrow \quad P_\nu := \mathrm{im}(\nu^*\colon \mathbb{Q}^r \to H^1(X,\mathbb{Q}))$$

- The *Dwyer–Fried invariants* of X are the subsets

$$\Omega^i_r(X) = \{P_\nu \in \mathrm{Gr}_r(\mathbb{Q}^n) \mid b_j(X^\nu) < \infty \text{ for } j \le i\}.$$

- For each $r > 0$, we get a descending filtration,

$$\mathrm{Gr}_r(\mathbb{Q}^n) = \Omega^0_r(X) \supseteq \Omega^1_r(X) \supseteq \Omega^2_r(X) \supseteq \cdots .$$

Characteristic varieties

- Let $\mathrm{Hom}(G,\mathbb{C}^*) = H^1(X,\mathbb{C}^*)$ be the character group of $G=\pi_1(X)$.
- The *characteristic varieties* of X are the sets

$$\mathcal{V}^i(X) = \{\rho \in \mathrm{Hom}(G,\mathbb{C}^*) \mid H_i(X,\mathbb{C}_\rho) \neq 0\}.$$

- If X has finite k-skeleton, then $\mathcal{V}^i(X)$ is Zariski closed for all $i \leq k$.
- Let $X^{\mathrm{ab}} \to X$ be the maximal abelian cover. View $H_*(X^{\mathrm{ab}},\mathbb{C})$ as a module over $\mathbb{C}[G_{\mathrm{ab}}]$. Then

$$\bigcup_{i\leq q} \mathcal{V}^i(X) = \bigcup_{i\leq q} V\big(\mathrm{ann}\,\big(H_i\big(X^{\mathrm{ab}},\mathbb{C}\big)\big)\big).$$

- Let $\mathcal{W}^i(X) = \mathcal{V}^i(X) \cap \mathrm{Hom}(G,\mathbb{C}^*)^0$. Then

$$\mathcal{W}^1(X) = \{1\} \cup V(\Delta_G),$$

where $\Delta_G = \mathrm{ord}\big(H_1(X^\alpha,\mathbb{C})\big)$ is the Alexander polynomial of G. (Here $X^\alpha \to X$ is the maximal torsion-free abelian cover.)

Resonance varieties

- Let $A = H^*(X, \mathbb{C})$. For each $a \in A^1$, we have that $a^2 = 0$. Thus, there is a cochain complex

$$(A, \cdot a)\colon \ A^0 \xrightarrow{a} A^1 \xrightarrow{a} A^2 \longrightarrow \cdots .$$

- The *resonance varieties* of X are the homogeneous algebraic sets

$$\mathcal{R}^i(X) = \{a \in A^1 \mid H^i(A, a) \neq 0\}.$$

- Identify $A^1 = H^1(X, \mathbb{C})$ with $\mathbb{C}^n$, where $n = b_1(X)$. The map $\exp\colon H^1(X, \mathbb{C}) \to H^1(X, \mathbb{C}^*)$ has image $\operatorname{Hom}(G, \mathbb{C}^*)^0 = (\mathbb{C}^*)^n$.

- (Dimca–Papadima–S. 2009)

$$\tau_1(\mathcal{W}^i(X)) \subseteq \mathcal{R}^i(X).$$

- (DPS-2009, DP-2014) If X is a q-formal space, then, for all $i \leq q$,

$$\tau_1(\mathcal{W}^i(X)) = \mathcal{R}^i(X).$$

Novikov–Betti numbers

- Let $\chi \in S(X)$, and set $\Gamma = \operatorname{im}(\chi)$; then $\Gamma \cong \mathbb{Z}^r$, for some $r \geq 1$.
- A Laurent polynomial $p = \sum_\gamma n_\gamma \gamma \in \mathbb{Z}\Gamma$ is χ-*monic* if the greatest element in $\chi(\operatorname{supp}(p))$ is 0, and $n_0 = 1$.
- Let $\mathcal{R}\Gamma_\chi$ be the Novikov ring, i.e., the localization of $\mathbb{Z}\Gamma$ at the multiplicative subset of all χ-monic polynomials ($\mathcal{R}\Gamma_\chi$ is a PID).
- Let $b_i(X, \chi) = \operatorname{rank}_{\mathcal{R}\Gamma_\chi} H_i(X, \mathcal{R}\Gamma_\chi)$ be the Novikov–Betti numbers.

Bounding the Σ-invariants

THEOREM (PAPADIMA–S. 2010)

Let X be a connected CW-complex with finite k-skeleton, and let $\chi\colon \pi_1(X) \to \mathbb{R}$ be a non-zero character. Then, for all $q \le k$,

- $-\chi \in \Sigma^q(X, \mathbb{Z}) \implies b_i(X, \chi) = 0, \forall i \le q$.
- $\chi \notin \tau_1^{\mathbb{R}}(\bigcup_{i\le q} \mathcal{W}^i(X)) \iff b_i(X,\chi) = 0, \ \forall i \le q$.

COROLLARY

$$\Sigma^q(X, \mathbb{Z}) \subseteq S(X) \setminus S\Big(\tau_1^{\mathbb{R}}\Big(\bigcup_{i\le q} \mathcal{W}^i(X)\Big)\Big)$$

Thus, $\Sigma^q(X, \mathbb{Z})$ is contained in the complement of a finite union of rationally defined great subspheres.

Bounding the Ω-invariants

Theorem (Dwyer–Fried 1987, Papadima–S. 2010)

Let X be a connected CW-complex with finite k-skeleton. For an epimorphism $\nu\colon \pi_1(X) \twoheadrightarrow \mathbb{Z}^r$, the following are equivalent:

- *The vector space $\bigoplus_{i=0}^{k} H_i(X^\nu, \mathbb{C})$ is finite-dimensional.*
- *The algebraic torus $\mathbb{T}_\nu := \operatorname{im}\big(\nu^*\colon \operatorname{Hom}(\mathbb{Z}^r, \mathbb{C}^*) \hookrightarrow \operatorname{Hom}(\pi_1(X), \mathbb{C}^*)\big)$ intersects the variety $\bigcup_{i\le k} \mathcal{V}^i(X)$ in only finitely many points.*

Theorem (S. 2014)

Let $\exp\colon H^1(X,\mathbb{C}) \to H^1(X,\mathbb{C}^*)$. For all $q \le k$ and all $r \ge 1$,

$$\Omega_r^q(X) = \Big\{ P \in \operatorname{Gr}_r(H^1(X,\mathbb{Q})) \;\Big|\; \dim\Big(\exp(P\otimes\mathbb{C}) \cap \Big(\bigcup_{i\le q} \mathcal{W}^i(X)\Big)\Big) = 0 \Big\}.$$

- Let V be a homogeneous variety in $\mathbb{k}^n$. Then the set

$$\sigma_r(V) = \{P \in \mathrm{Gr}_r(\mathbb{k}^n) \mid P \cap V \neq \{0\}\}$$

is Zariski closed.

- If $L \subset \mathbb{k}^n$ is a linear subspace, $\sigma_r(L)$ is the *special Schubert variety* defined by L. If $\operatorname{codim} L = d$, then $\operatorname{codim} \sigma_r(L) = d - r + 1$.

THEOREM (S. 2014)

$$\Omega^q_r(X) \subseteq \mathrm{Gr}_r(H^1(X,\mathbb{Q})) \setminus \sigma_r\Big(\tau_1^{\mathbb{Q}}\Big(\bigcup_{i\le q} \mathcal{W}^i(X)\Big)\Big)$$

- Thus, each set $\Omega^q_r(X)$ is contained in the complement of a finite union of special Schubert varieties.

- If $r = 1$, the inclusion always holds as an equality. In general, though, the inclusion is strict.

Comparing the Σ- and Ω-bounds

Theorem (S. 2012)

Suppose that

$$\Sigma^q(X,\mathbb{Z}) = S(X) \setminus S\Big(\tau_1^{\mathbb{R}}\Big(\bigcup_{i\le q} \mathcal{W}^i(X)\Big)\Big).$$

Then

$$\Omega_r^q(X) = \mathrm{Gr}_r(H^1(X,\mathbb{Q})) \setminus \sigma_r\Big(\tau_1^{\mathbb{Q}}\Big(\bigcup_{i\le q} \mathcal{W}^i(X)\Big)\Big), \text{ for all } r \ge 1.$$

In general, the above implication cannot be reversed.

Example

- Let $G = \langle x_1, x_2 \mid x_1 x_2 = x_2^2 x_1 \rangle$.
- Then $\mathcal{W}^1(G) = \{1, 2\} \subset \mathrm{Hom}(G, \mathbb{C}^*) = \mathbb{C}^*$.
- Thus, $\Omega_1^1(G) = \{\mathrm{pt}\}$, and so $\Omega_1^1(G) = \sigma_1(\tau_1^{\mathbb{Q}}(\mathcal{W}^1(G)))^{\complement}$.
- But $\Sigma^1(G) = \{-1\}$, whereas $S(\tau_1^{\mathbb{Q}}(\mathcal{W}^1(G)))^{\complement} = \{\pm 1\}$.

A tropical bound for the Σ-invariants

- Let X be a connected CW-complex w/ finite k-skeleton.
- For each algebraic variety $W \subset (\mathbb{C}^*)^n$ there is an associated polyhedral fan, $\operatorname{Trop}(W) \subset \mathbb{R}^n$.
- Thus, to each algebraic variety $V \subset H^1(X, \mathbb{C}^*)$ we may associate a polyhedral fan, $\operatorname{Trop}(V) \subset \mathbb{R}^n$, where $n = b_1(X)$.

THEOREM

$$\Sigma^q(X, \mathbb{Z}) \subseteq S(X) \setminus S\Big(\operatorname{Trop}\Big(\bigcup_{i \le q} \mathcal{V}^i(X)\Big)\Big), \quad \forall q \le k.$$

COROLLARY

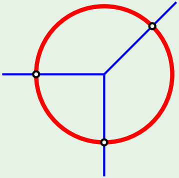
Let G be a finitely generated group. Then:

$$\Sigma^1(G) \subseteq S(G) \setminus S(\operatorname{Trop}(V(\Delta_G))).$$

Two-generator, one-relator groups

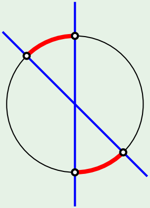
- If $G = \langle x, y \mid r \rangle$, there is a very concrete algorithm for computing $\Sigma^1(G)$ (K. Brown 1987, Friedl–Tillman 2019).

Example



- Let $G = \langle a, b \mid a^{-1}b^2ab^{-1}ab^{-1}a^{-1} \rangle$.
- Then $\Sigma^1(G) = S^1 \setminus \{(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (0, -1), (-1, 0)\}$.
- On the other hand, $\Delta_G = 1 + b - a$.
- Thus, $\Sigma^1(G) = S(\mathrm{Trop}(V(\Delta_G)))^\complement$, though $\tau_1 \mathcal{V}^1(G) = \{0\}$.

Example



- Let $G = \langle a, b \mid a^2ba^{-1}ba^2ba^{-1}b^{-3}a^{-1}ba^2ba^{-1}ba$ $b^{-1}a^{-2}b^{-1}ab^{-1}a^{-2}b^{-1}ab^3ab^{-1}a^{-2}b^{-1}ab^{-1}a^{-1}b \rangle$.
- Then $\Delta_G = (a-1)(ab-1)$, and so $S(\mathrm{Trop}(V(\Delta_G)))$ consists of two pairs of points.
- Yet $\Sigma^1(G)$ consists of two open arcs joining those points.

Kähler manifolds

THEOREM (DELZANT 2010)

Let M be a compact Kähler manifold. Then

$$\Sigma^1(M,\mathbb{Z}) = S(M) \setminus \bigcup_\alpha S(f_\alpha^*(H^1(C_\alpha,\mathbb{R}))),$$

where the union is taken over those orbifold fibrations $f_\alpha\colon M \to C_\alpha$ with the property that either $\chi(C_\alpha) < 0$, or $\chi(C_\alpha) = 0$ and f_α has some multiple fiber.

COROLLARY

$$\Sigma^1(M,\mathbb{Z}) = S(\operatorname{Trop}(\mathcal{V}^1(M))^{\complement}.$$

Hyperplane arrangements

- Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an (essential, central) arrangement of hyperplanes in $\mathbb{C}^d$.
- Its complement, $M(\mathcal{A}) \subset (\mathbb{C}^*)^d$, is a smooth, quasi-projective Stein manifold; thus, it has the homotopy type of a finite, d-dimensional CW-complex.
- $\operatorname{Trop}(M(\mathcal{A}))$ is the 'Bergman fan' of the underlying matroid $L(\mathcal{A})$.
- $H^*(M(\mathcal{A}), \mathbb{Z})$ is the Orlik–Solomon algebra of $L(\mathcal{A})$.
- Let $\mathcal{V}^i(\mathcal{A}) := \mathcal{V}^i(M(\mathcal{A})) \subset (\mathbb{C}^*)^n$ and $\mathcal{R}^i(\mathcal{A}) := \mathcal{R}^i(M(\mathcal{A})) \subset \mathbb{C}^n$.
- $M(\mathcal{A})$ is formal. Thus, $\tau_1(\mathcal{V}^i(\mathcal{A})) = \mathcal{R}^i(\mathcal{A})$ for all i.

THEOREM (KOHNO–PAJITNOV 2015)

Let $S^{-}(\mathcal{A}) := S^{n-1} \cap (\mathbb{R}_{<0})^{n}$. *Then* $S^{-}(\mathcal{A}) \subseteq \Sigma^{q}(\mathcal{A})$, *for all* $q < d$. *In particular,* $S^{-}(\mathcal{A}) \subseteq \Sigma^{1}(\mathcal{A})$.

THEOREM (DENHAM–YUZVINSKY–S. 2016/17)

$M(\mathcal{A})$ *is an "abelian duality space," and hence its characteristic varieties propagate:* $\mathcal{V}^{1}(\mathcal{A}) \subseteq \mathcal{V}^{2}(\mathcal{A}) \subseteq \cdots \subseteq \mathcal{V}^{d}(\mathcal{A})$.

COROLLARY

$$\Sigma^{q}(M(\mathcal{A}), \mathbb{Z}) \subseteq S^{n-1} \setminus S\big(\mathrm{Trop}(\mathcal{V}^{q}(\mathcal{A}))\big), \quad \forall q \le d.$$

QUESTION (S., AT MFO MINIWORKSHOP 2007)

Given an arrangement $\mathcal{A}$, do we have

$$\Sigma^{1}(M(\mathcal{A})) = S(\mathcal{R}^{1}(\mathcal{A}, \mathbb{R}))^{\complement}? \qquad (\star)$$

Example (Koban–McCammond–Meier 2013)

- Let $\mathcal{A}$ be the braid arrangement in $\mathbb{C}^n$, defined by $\prod_{1\leq i<j\leq n}(z_i - z_j) = 0$. Then $M(\mathcal{A}) = \mathrm{Conf}(n, \mathbb{C}) \simeq K(P_n, 1)$.
- Answer to $(\star)$ is yes: $\Sigma^1(M(\mathcal{A}))$ is the complement of the union of $\binom{n}{3} + \binom{n}{4}$ planes in $\mathbb{C}^{\binom{n}{2}}$, intersected with the unit sphere.

Example

- Let $\mathcal{A}$ be the "deleted B_3" arrangement, defined by $z_1 z_2 (z_1^2 - z_2^2)(z_1^2 - z_2^2)(z_2^2 - z_3^2) = 0$.
- (S. 2002) $\mathcal{V}^1(\mathcal{A})$ contains a (1-dimensional) translated torus $\rho \cdot T$.
- Thus, $\mathrm{Trop}(\rho \cdot T) = \mathrm{Trop}(T)$ is a line in $\mathbb{C}^8$ which is *not* contained in $\mathcal{R}^1(\mathcal{A}, \mathbb{R})$. Hence, the answer to $(\star)$ is no.

Question

$$\Sigma^1(M(\mathcal{A})) = S(\mathrm{Trop}(\mathcal{V}^1(\mathcal{A}))^{\complement}? \qquad (\star\star)$$