On the topology of Milnor fibrations of hyperplane arrangements

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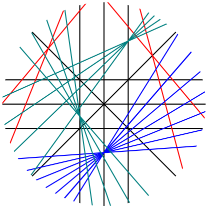
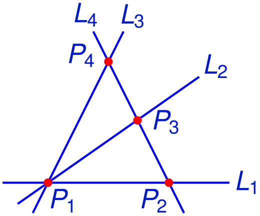
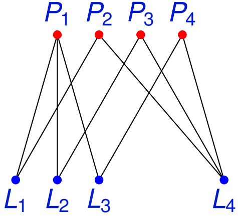
Hyperplane arrangements



- An *arrangement of hyperplanes* is a finite collection, $\mathcal{A}$, of codimension 1 linear (or affine) subspaces in $\mathbb{C}^d$.
- *Intersection lattice* $L(\mathcal{A})$: poset of all intersections of $\mathcal{A}$, ordered by reverse inclusion, and ranked by codimension.





- The *complement* of the arrangement, $M(\mathcal{A}) := \mathbb{C}^d \setminus \bigcup_{H\in\mathcal{A}} H$, is a smooth, complex, quasi-projective variety.
- $M = M(\mathcal{A})$ is also a Stein manifold, and so it has the homotopy type of a finite, connected, CW-complex of dimension at most d.
- In fact, M has a minimal cell structure. Consequently, $H_*(M,\mathbb{Z})$ is torsion-free (and finitely generated).
- In particular, $H_1(M,\mathbb{Z}) = \mathbb{Z}^{|\mathcal{A}|}$, generated by meridians $\{x_H\}_{H\in\mathcal{A}}$.
- $H^*(M,\mathbb{Z})$ is the quotient of the exterior algebra on the duals $e_H = x_H^\vee$ by an ideal determined by $L(\mathcal{A})$. [Orlik–Solomon]
- M admits a *pure* mixed Hodge structure, and so M is $\mathbb{Q}$-formal (albeit not $\mathbb{Z}_p$-formal, in general).

Lower central series

- The *lower central series* of a group G is defined by $\gamma_1(G) = G$, $\gamma_2(G) = G'$, and $\gamma_{k+1}(G) = [G, \gamma_k(G)]$, where $[g, h] = ghg^{-1}h^{-1}$.
- We have: $[\gamma_k(G), \gamma_\ell(G)] \subseteq \gamma_{k+\ell}(G)$, and so $\gamma_k(G) \lhd G$.
- The LCS quotients, $\mathrm{gr}_k(G) := \gamma_k(G)/\gamma_{k+1}(G)$, are abelian.
- Associated graded Lie algebra: $\mathrm{gr}(G) = \bigoplus_{k\geqslant 1} \mathrm{gr}_k(G)$, with Lie bracket $[\ ,\ ]\colon \mathrm{gr}_k \times \mathrm{gr}_\ell \to \mathrm{gr}_{k+\ell}$ induced by the group commutator.
- The factor groups $G/\gamma_{k+1}(G)$ are the maximal k-step nilpotent quotients of G.
- $G/\gamma_2(G) = G_{\mathrm{ab}}$, while $G/\gamma_3(G)$ is determined by $H^{\leqslant 2}(G, \mathbb{Z})$.

Fundamental groups of arrangements

- For an arrangement $\mathcal{A}$, the group $G = \pi_1(M(\mathcal{A}))$ admits a finite presentation, with generators $\{x_H\}_{H\in\mathcal{A}}$ and commutator-relators.
- $G/\gamma_2(G) = \mathbb{Z}^{|\mathcal{A}|}$, while $G/\gamma_3(G)$ is determined by $L_{\leqslant 2}(\mathcal{A})$.
- $G/\gamma_4(G)$—and thus G—is not determined by $L_{\leqslant 2}(\mathcal{A})$. [Rybnikov]
- Since $M = M(\mathcal{A})$ is formal, $G = \pi_1(M)$ is 1-formal, i.e., its $\mathbb{Q}$-pronilpotent completion, $\mathfrak{m}(G)$, is quadratic.
- Hence, $\operatorname{gr}(G)\otimes\mathbb{Q} = \operatorname{gr}(\mathfrak{m}(G))$ is determined by $L_{\leqslant 2}(\mathcal{A})$.
- An explicit combinatorial formula is lacking in general for the LCS ranks $\phi_k := \operatorname{rank}\operatorname{gr}_k(G)$, although such formulas are known when $\mathcal{A}$ is supersolvable, or decomposable, or a graphic arrangement.
- The Chen ranks $\theta_k(G) := \operatorname{rank}\operatorname{gr}_k(G/G'')$ are also combinatorially determined. [Papadima–S.]

- Let $\mathfrak{h}(G) = \text{Lie}(G_{\text{ab}}) / \operatorname{im}(H_2(G,\mathbb{Z}) \xrightarrow{\cup^\vee} G_{\text{ab}} \wedge G_{\text{ab}})$ be the *holonomy Lie algebra* associated to a f.g. group G.
- This is a quadratic Lie algebra, with presentation determined by $H^{\leqslant 2}(G,\mathbb{Z})$. Moreover, $\mathfrak{h}(G) \twoheadrightarrow \text{gr}(G)$.
- If G is 1-formal: $\mathfrak{h}(G) \otimes \mathbb{Q} \xrightarrow{\simeq} \text{gr}(G) \otimes \mathbb{Q}$.
- $\text{gr}_k(G)$ may have torsion (for $k \geqslant 4$), but torsion is not necessarily determined by $L_{\leqslant 2}(\mathcal{A})$. [Artal Bartolo–Guerville–Viu-Sos]
- The map $\mathfrak{h}_3(G) \to \text{gr}_3(G)$ is an isomorphism [Porter–S.], but it is not known whether $\mathfrak{h}_3(G)$ is torsion-free.
- $\mathcal{A}$ is *decomposable* if $\mathfrak{h}_3(G)$ is free abelian of rank $2\sum_{P \in L_2(\mathcal{A})} \binom{m_P}{3}$.
- In this case, $\text{gr}(G) = \mathfrak{h}(G)$ is torsion-free, with ranks computed by an explicit combinatorial formula [Papadima–S.]
- Moreover, all nilpotent quotients $G/\gamma_k(G)$ are combinatorially determined [Porter–S.]

Alexander invariant

- The *derived series* of a group G is defined inductively by $G^{(0)} = G$, $G^{(1)} = G'$, $G^{(2)} = G''$, and $G^{(r)} = [G^{(r-1)}, G^{(r-1)}]$.
- $G/G^{(r)}$ is the maximal solvable quotient of G of length r.
- The *Alexander invariant* of G:

$$B(G) := G'/G'',$$

  a $\mathbb{Z}[G_{\mathrm{ab}}]$-module via $gG' \cdot xG'' = gxg^{-1}G''$ for $g \in G$ and $x \in G'$.

- If X is a connected CW-complex with $\pi_1(X) = G$, then

$$B(G) = H_1(X^{\mathrm{ab}}, \mathbb{Z}) = H_1(X, \mathbb{Z}[G_{\mathrm{ab}}]).$$

- Let $I = I(G_{\mathrm{ab}})$. Then $I^k B(G) = \gamma_{k+2}(G/G'')$, and thus $\mathrm{gr}_k(B) \cong \mathrm{gr}_{k+2}(G/G'')$ for all $n \geqslant 0$. [Massey]

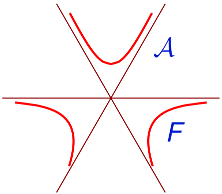
Characteristic varieties

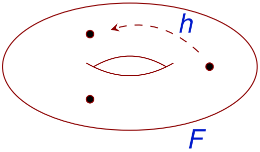
- Let G be a finitely generated group. The character group, $\mathbb{T}_G := \operatorname{Hom}(G, \mathbb{C}^*)$, is an abelian, complex algebraic group.
- $\mathbb{T}_G \cong \mathbb{T}_G^0 \times \operatorname{Tors}(G_{\text{ab}})$, where $\mathbb{T}_G^0 \cong (\mathbb{C}^*)^{b_1(G)}$.
- The *characteristic varieties* of G:

$$\mathcal{V}_j(G) := \{\rho \in \mathbb{T}_G \mid \dim H^1(G, \mathbb{C}_\rho) \geqslant j\}.$$

- $\mathcal{V}_j(G) = V(\operatorname{ann}(\bigwedge^j B(G) \otimes \mathbb{C}))$, at least away from 1.
- Now let $G = \pi_1(M(\mathcal{A}))$. Then $\mathbb{T}_G = (\mathbb{C}^*)^n$, where $n = |\mathcal{A}|$.
- The components of $\mathcal{V}_1(G)$ passing through $1 \in (\mathbb{C}^*)^n$ are combinatorially determined.
- In general, though, there are translated subtori in $\mathcal{V}_1(G) = \mathcal{V}_1(M(\mathcal{A}))$, which are not *a priori* determined by $L(\mathcal{A})$.

Milnor fibration





- The map $f\colon \mathbb{C}^d \to \mathbb{C}$ restricts to a smooth fibration, $f\colon M \to \mathbb{C}^*$, called the *Milnor fibration* of $\mathcal{A}$.
- The *Milnor fiber* is $F(\mathcal{A}) := f^{-1}(1)$. The monodromy, $h\colon F \to F$, is given by $h(z) = e^{2\pi i/n}z$.
- F is a Stein manifold. It has the homotopy type of a finite CW-complex of dimension $d-1$ (connected if $d > 1$).
- MHS on F may not be pure; $\pi_1(F)$ may be non-1-formal [Zuber].

- F is the regular, $\mathbb{Z}_n$-cover of $U = \mathbb{P}(M)$, classified by the projection $\pi_1(U) \twoheadrightarrow \mathbb{Z}_n$, $x_H \mapsto 1$.
- To understand $\pi_1(F)$, we may assume wlog that $d = 3$, so that $\bar{\mathcal{A}} = \mathbb{P}(\mathcal{A})$ is an arrangement of lines in $\mathbb{CP}^2$.
- Let $\iota\colon F \hookrightarrow M$ be the inclusion. Induced maps on π_1:

$$
\begin{array}{ccccccccc}
 & & & & \mathbb{Z} & & & & \\
 & & & & \big\downarrow & \searrow^{\times n} & & & \\
1 & \longrightarrow & \pi_1(F) & \xrightarrow{\iota_\sharp} & \pi_1(M) & \xrightarrow{f_\sharp} & \mathbb{Z} & \longrightarrow & 1 \\
 & & & \searrow & \big\downarrow{\scriptstyle p_\sharp} & & & & \\
 & & & & \pi_1(U) & & & & \\
 & & & & \big\downarrow & \searrow & & & \\
 & & & & 1 & & \mathbb{Z}_n & &
\end{array}
$$

- $b_1(F) \geqslant n - 1$, and may be computed from $\mathcal{V}^1_j(U)$. Combinatorial formulas are known in some cases (e.g., if $\mathbb{P}(\mathcal{A})$ has only double or triple points [Papadima–S.]), but not in general.

Exact sequences and lower central series

- A short exact sequence of groups,

$$1 \longrightarrow K \xrightarrow{\iota} G \xrightarrow{\pi} Q \longrightarrow 1 \qquad (*)$$

  yields
  - A representation $\varphi\colon Q \to \mathrm{Out}(K)$.
  - A "monodromy" representation $\bar{\varphi}\colon Q \to \mathrm{Aut}(K_{\mathrm{ab}})$.

- If (*) admits a splitting, $\sigma\colon Q \to G$, then $G = K \rtimes_{\varphi} Q$, where $\varphi\colon Q \to \mathrm{Aut}(K)$, $x \mapsto$ conjugation by $\sigma(x)$.

- (*) is *ab-exact* if $0 \longrightarrow K_{\mathrm{ab}} \xrightarrow{\iota_{\mathrm{ab}}} G_{\mathrm{ab}} \xrightarrow{\pi_{\mathrm{ab}}} Q_{\mathrm{ab}} \longrightarrow 0$ is also exact; equivalently, Q acts trivially on K_{ab} and ι_{ab} is injective.

Theorem (Falk-Randell)

Let $G = K \rtimes_{\varphi} Q$. If Q acts trivially on K_{ab}, then
- $\gamma_k(G) = \gamma_k(K) \rtimes_{\varphi} \gamma_k(Q)$, *for all* $k \geqslant 1$.
- $\mathrm{gr}(G) = \mathrm{gr}(K) \rtimes_{\bar{\varphi}} \mathrm{gr}(Q)$.

Theorem

Let $1 \to K \xrightarrow{\iota} G \to Q \to 1$ be an ab*-exact sequence, and assume Q is abelian. Then:*

- $\iota\colon K \hookrightarrow G$ *restricts to an equality,* $K' = G'$.
- $B(\iota)$ *factors through a* $\mathbb{Z}[K_{\mathrm{ab}}]$*-linear isomorphism* $B(K) \xrightarrow{\simeq} B(G)_\iota$.
- *If, moreover,* G_{ab} *is f.g., then* $\theta_k(K) \leqslant \theta_k(G)$ *for all* $k \geqslant 1$.
- *If G and K are finitely generated, then the map* $\iota^*\colon \mathbb{T}_G \to \mathbb{T}_K$ *restricts to maps* $\iota^*\colon \mathcal{V}_j(G) \to \mathcal{V}_j(K)$ *for all* $j \geqslant 1$*; furthermore,* $\iota^*\colon \mathcal{V}_1(G) \to \mathcal{V}_1(K)$ *is a surjection.*

Assume now that the sequence is also split-exact. Then:

- $\mathrm{gr}_{\geqslant 2}(K) \xrightarrow{\simeq} \mathrm{gr}_{\geqslant 2}(G)$ *and* $\mathrm{gr}_{\geqslant 2}(K/K'') \xrightarrow{\simeq} \mathrm{gr}_{\geqslant 2}(G/G'')$.
- *If, moreover,* $b_1(G) < \infty$*, then* $\phi_k(K) = \phi_k(G)$ *and* $\theta_k(K) = \theta_k(G)$ *for all* $k \geqslant 2$.

Rational LCS and rational Alexander invariant

- The *rational lower central series* of G is defined by $\gamma_1^{\mathbb{Q}} G = G$ and $\gamma_{k+1}^{\mathbb{Q}} G = \sqrt{[G, \gamma_k^{\mathbb{Q}} G]}$. [Stallings]
- $G/\gamma_2^{\mathbb{Q}} G = G_{\mathrm{abf}}$, where $G_{\mathrm{abf}} = G_{\mathrm{ab}}/\operatorname{Tors}(G_{\mathrm{ab}})$ is the maximal torsion-free abelian quotient of G.
- Quotients $\operatorname{gr}_k^{\mathbb{Q}}(G) := \gamma_k^{\mathbb{Q}} G/\gamma_{k+1}^{\mathbb{Q}} G$ are torsion-free abelian groups.
- Associated graded Lie algebra: $\operatorname{gr}^{\mathbb{Q}}(G) = \bigoplus_{k\geqslant 1} \gamma_k^{\mathbb{Q}} G/\gamma_{k+1}^{\mathbb{Q}} G$.
- The *rational derived series* of G is defined by $G_{\mathbb{Q}}^{(0)} = G$ and $G_{\mathbb{Q}}^{(r)} = \sqrt{\left[G_{\mathbb{Q}}^{(r-1)}, G_{\mathbb{Q}}^{(r-1)}\right]}$. [Stallings, Harvey, Cochran]
- $B_{\mathbb{Q}}(G) := G'_{\mathbb{Q}}/G''_{\mathbb{Q}}$, viewed as a module over $\mathbb{Z}G_{\mathrm{abf}}$.
- Let $\mathcal{W}_k(G) := \mathcal{V}_k(G) \cap \mathbb{T}_G^0$. Then $\mathcal{W}_k(G) = V(\operatorname{ann}(\bigwedge^k B_{\mathbb{Q}}(G) \otimes \mathbb{C}))$, at least away from 1.

Abf-exact sequences

- A sequence $1 \to K \xrightarrow{\iota} G \xrightarrow{\pi} Q \to 1$ is called abf-exact if one of the following equivalent conditions is satisfied.
  - The group Q acts trivially on K_{abf} and the composite $H_2(Q,\mathbb{Z}) \xrightarrow{\delta} H_1(K,\mathbb{Z}) \to H_1(K,\mathbb{Z})/\operatorname{Tors}$ is zero.
  - The sequence $0 \longrightarrow K_{\mathrm{abf}} \xrightarrow{\iota_{\mathrm{abf}}} G_{\mathrm{abf}} \xrightarrow{\pi_{\mathrm{abf}}} Q_{\mathrm{abf}} \longrightarrow 0$ is exact.
- Suppose K_{abf} is finitely generated. Then the extension is abf-exact iff Q acts trivially on $H_1(K,\mathbb{Q})$ and $\delta \otimes \mathbb{Q} = 0$.
- Reverse implication may not hold if K_{abf} not f.g.

LEMMA

Let $1 \to K \to G \to Q \to 1$ be a split exact sequence.

- *The sequence is abf-exact iff Q acts trivially on K_{abf}.*
- *If K_{abf} is finitely generated, then the sequence is abf-exact iff Q acts trivially on $H_1(K,\mathbb{Q})$.*

Theorem

Let $G = K \rtimes_\varphi Q$ be a split extension. If Q acts trivially on K_{abf}, then,

- $\gamma_k^{\mathbb{Q}}(G) = \gamma_k^{\mathbb{Q}}(K) \rtimes_\varphi \gamma_k^{\mathbb{Q}}(Q)$, *for all* $k \geqslant 1$.
- $\mathrm{gr}^{\mathbb{Q}}(G) = \mathrm{gr}^{\mathbb{Q}}(K) \rtimes_{\bar\varphi} \mathrm{gr}^{\mathbb{Q}}(Q)$.

Theorem

Let $1 \to K \xrightarrow{\iota} G \to Q \to 1$ be an abf-exact sequence, and assume Q is torsion-free abelian. Then:

- $K'_{\mathbb{Q}} = G'_{\mathbb{Q}}$ *and* $B_{\mathbb{Q}}(K) \cong B_{\mathbb{Q}}(G)_\iota$.
- *If, moreover,* G_{abf} *is f.g., then* $\theta_n(K) \leqslant \theta_n(G)$ *for all* $n \geqslant 1$.
- *If* G *and* K *are f.g., then* $\iota^* \colon \mathbb{T}_G^0 \twoheadrightarrow \mathbb{T}_K^0$ *restricts to* $\iota^* \colon \mathcal{W}_j(G) \to \mathcal{W}_j(K)$*; furthermore,* $\iota^* \colon \mathcal{W}_1(G) \to \mathcal{W}_1(K)$ *is surjective.*

Assume now that the sequence is also split-exact. Then:

- $\mathrm{gr}^{\mathbb{Q}}_{\geqslant 2}(K) \xrightarrow{\simeq} \mathrm{gr}^{\mathbb{Q}}_{\geqslant 2}(G)$ *and* $\mathrm{gr}^{\mathbb{Q}}_{\geqslant 2}(K/K'') \xrightarrow{\simeq} \mathrm{gr}^{\mathbb{Q}}_{\geqslant 2}(G/G'')$.
- *If* $b_1(G) < \infty$, *then* $\phi_k(K) = \phi_k(G)$ *and* $\theta_k(K) = \theta_k(G)$ *for all* $k \geqslant 2$.

Let $\mathcal{A}$ be an arrangement, with complement M, Milnor fibration $F \xrightarrow{\iota} M \to \mathbb{C}^*$, and split exact sequence $1 \to \pi_1(F) \xrightarrow{\iota} \pi_1(M) \to \mathbb{Z} \to 1$.

Corollary

If the monodromy action on $H_1(F,\mathbb{Z})$ is trivial, then

- $\iota^*\colon H^1(M,\mathbb{C}^*) \twoheadrightarrow H^1(F,\mathbb{C}^*)$ *restricts to* $\iota^*\colon \mathcal{V}_1(M) \twoheadrightarrow \mathcal{V}_1(F)$.
- $\phi_k(\pi_1(F)) = \phi_k(\pi_1(M))$ *for* $k \geqslant 2$.
- $\theta_k(\pi_1(F)) = \theta_k(\pi_1(M))$ *for* $k \geqslant 2$.

Corollary

If the monodromy action on $H_1(F,\mathbb{Q})$ is trivial, then

- $\iota^*\colon H^1(M,\mathbb{C}^*)^0 \twoheadrightarrow H^1(F,\mathbb{C}^*)^0$ *restricts to* $\iota^*\colon \mathcal{W}_1(M) \twoheadrightarrow \mathcal{W}_1(F)$.
- $\phi_k(\pi_1(F)) = \phi_k(\pi_1(M))$ *for* $k \geqslant 2$.
- $\theta_k(\pi_1(F)) = \theta_k(\pi_1(M))$ *for* $k \geqslant 2$.

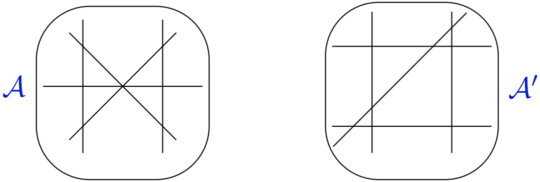
Formality properties

- Let $Y \to X$ be a finite, regular cover, with deck group Γ. If Y is 1-formal, then X is 1-formal, but the converse is not true.
- (Dimca–Papadima) If Γ acts trivially on $H_1(Y, \mathbb{Q})$, then the converse holds.
- Applying to $\mathbb{Z}_n$-cover $F(\mathcal{A}) \to U(\mathcal{A})$: if the Milnor fibration of $\mathcal{A}$ has trivial $\mathbb{Q}$-monodromy, then F is 1-formal.
- (S.–Wang) Let $1 \to K \to G \to Q \to 1$ be an exact sequence. If G is 1-formal and retracts onto K, then K is also 1-formal.
- (Papadima–S.) Let $1 \to K \to G \to \mathbb{Z} \to 1$ be an exact sequence. Assume G is 1-formal and $b_1(K) < \infty$. Then the eigenvalue 1 of the monodromy action on $H_1(K, \mathbb{C})$ has only 1×1 Jordan blocks.

Theorem

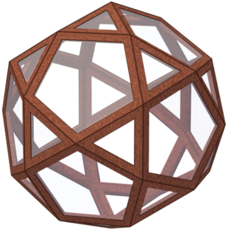
Let $1 \to K \to G \to Q \to 1$ be a split-exact and abf-exact sequence. If G is 1-formal and K is finitely generated, then K is 1-formal.

Falk's pair of arrangements



- Both $\mathcal{A}$ and $\mathcal{A}'$ have 2 triple points and 9 double points, yet $L(\mathcal{A}) \not\cong L(\mathcal{A}')$. Nevertheless, $M(\mathcal{A}) \simeq M(\mathcal{A}')$.
- $\mathcal{V}_1(M)$ and $\mathcal{V}_1(M')$ consist of two 2-dimensional subtori of $(\mathbb{C}^*)^6$, corresponding to the triple points; $\mathcal{V}_2(M) = \mathcal{V}_2(M') = \{1\}$.
- Both Milnor fibrations have trivial $\mathbb{Z}$-monodromy.
- $\mathcal{V}_1(F)$ and $\mathcal{V}_1(F')$ consist of two 2-dimensional subtori of $(\mathbb{C}^*)^5$.
- On the other hand, $\mathcal{V}_2(F) \cong \mathbb{Z}_3$, yet $\mathcal{V}_2(F') = \{1\}$.
- Thus, $\pi_1(F) \not\cong \pi_1(F')$.

Yoshinaga's icosidodecahedral arrangement



- The icosidodecahedron is a quasiregular polyhedron in $\mathbb{R}^3$, with 20 triangular and 12 pentagonal faces, 60 edges, and 30 vertices, given by the even permutations of $(0, 0, \pm 1)$ and $\frac{1}{2}(\pm 1, \pm\phi, \pm\phi^2)$, where $\phi = (1 + \sqrt{5})/2$.
- One can choose 10 edges to form a decagon; there are 6 ways to choose these decagons, thereby giving 6 planes.
- Each pentagonal face has five diagonals; there are 60 such diagonals in all, and they partition in 10 disjoint sets of coplanar ones, thereby giving 10 planes, each containing 6 diagonals.

- These 16 planes form a arrangement $\mathcal{A}_{\mathbb{R}}$ in $\mathbb{R}^3$, whose complexification is the icosidodecahedral arrangement $\mathcal{A}$ in $\mathbb{C}^3$.
- The complement M is a $K(\pi, 1)$. Moreover, $P_U(t) = 1 + 15t + 60t^2$; thus, $\chi(U) = 36$ and $\chi(F) = 576$.
- In fact, $H_1(F, \mathbb{Z}) = \mathbb{Z}^{15} \oplus \mathbb{Z}_2$. Thus, the algebraic monodromy of the Milnor fibration is trivial over $\mathbb{Q}$ and $\mathbb{Z}_p$ ($p > 2$), but not over $\mathbb{Z}$.
- Hence, $\operatorname{gr}(\pi_1(F)) \cong \operatorname{gr}(\pi_1(U))$, away from the prime 2. Moreover,
  - $\operatorname{gr}_1(\pi_1(F)) = \mathbb{Z}^{15} \oplus \mathbb{Z}_2$
  - $\operatorname{gr}_2(\pi_1(F)) = \mathbb{Z}^{45} \oplus \mathbb{Z}_2^7$
  - $\operatorname{gr}_3(\pi_1(F)) = \mathbb{Z}^{250} \oplus \mathbb{Z}_2^{43}$
  - $\operatorname{gr}_4(\pi_1(F)) = \mathbb{Z}^{1405} \oplus \mathbb{Z}_2^{?}$

La Mulți Ani, Mihai!