On the topology of line arrangements

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Complements of hyperplane arrangments

- An *arrangement of hyperplanes* is a finite set $\mathcal{A}$ of codimension 1 linear subspaces in a finite-dimensional $\mathbb{C}$-vector space V.
- The *intersection lattice*, $L(\mathcal{A})$, is the poset of all intersections of $\mathcal{A}$, ordered by reverse inclusion, and ranked by codimension.
- The *complement*, $M(\mathcal{A}) = V \setminus \bigcup_{H \in \mathcal{A}} H$, is a connected, smooth quasi-projective variety, and also a Stein manifold.
- It has the homotopy type of a minimal CW-complex of dimension $\dim V$. In particular, $H_{\bullet}(M(\mathcal{A}), \mathbb{Z})$ is torsion-free.
- The fundamental group $\pi = \pi_1(M(\mathcal{A}))$ admits a finite presentation, with generators x_H for each $H \in \mathcal{A}$.
- Set $U(\mathcal{A}) = \mathbb{P}(M(\mathcal{A}))$. Then $M(\mathcal{A}) \cong U(\mathcal{A}) \times \mathbb{C}^*$.

The abelianization map

- We may assume that $\mathcal{A}$ is essential, i.e., $\bigcap_{H\in\mathcal{A}} H = \{0\}$.
- For each $H \in \mathcal{A}$, let α_H be a linear form s.t. $H = \ker(\alpha_H)$.
- Fix an ordering $\mathcal{A} = \{H_1, \dots, H_n\}$. Since $\mathcal{A}$ is essential, the linear map $\alpha\colon V \to \mathbb{C}^n$, $\; z \mapsto (\alpha_1(z), \dots, \alpha_n(z))$ is injective.
- Let $\mathcal{B}_n$ be the 'Boolean arrangement' of coordinate hyperplanes in $\mathbb{C}^n$, with $M(\mathcal{B}_n) = (\mathbb{C}^*)^n$.
- The map α restricts to an inclusion $\alpha\colon M(\mathcal{A}) \hookrightarrow M(\mathcal{B}_n)$. Thus, $M(\mathcal{A}) = \alpha(V) \cap (\mathbb{C}^*)^n$.
- The induced homomorphism, $\alpha_\sharp\colon \pi_1(M(\mathcal{A})) \to \pi_1(M(\mathcal{B}_n))$, coincides with the abelianization map, $\mathrm{ab}\colon \pi \twoheadrightarrow \pi_{\mathrm{ab}} = \mathbb{Z}^n$.

COHOMOLOGY RING

- The logarithmic 1-form $\omega_H = \frac{1}{2\pi \mathrm{i}}\, \mathrm{d}\log \alpha_H \in \Omega_{\mathrm{dR}}(M)$ is a closed form, representing a class $e_H \in H^1(M, \mathbb{Z})$.
- Let E be the $\mathbb{Z}$-exterior algebra on $\{e_H \mid H \in \mathcal{A}\}$, and let $\partial\colon E^\bullet \to E^{\bullet-1}$ be the differential given by $\partial(e_H) = 1$.
- The ring $H^\bullet(M(\mathcal{A}), \mathbb{Z})$ is isomorphic to the OS-algebra E/I, where

$$I = \text{ideal}\left\{\partial\Big(\prod_{H\in\mathcal{B}} e_H\Big) \;\Big|\; \mathcal{B} \subseteq \mathcal{A} \text{ and } \operatorname{codim} \bigcap_{H\in\mathcal{B}} H < |\mathcal{B}| \right\}.$$

- Hence, the map $e_H \mapsto \omega_H$ extends to a cdga quasi-isomorphism, $\omega\colon (H^\bullet(M, \mathbb{R}), \mathrm{d} = 0) \longrightarrow \Omega^\bullet_{\mathrm{dR}}(M)$.
- Therefore, $M(\mathcal{A})$ is formal.
- $M(\mathcal{A})$ is minimally pure (i.e., $H^k(M(\mathcal{A}), \mathbb{Q})$ is pure of weight $2k$, for all k), which again implies formality (Dupont 2016).

- Let X be a connected, finite-type CW-complex, $\pi = \pi_1(X)$.
- Let G be a complex, linear algebraic group.
- The *representation variety* $\operatorname{Hom}(\pi, G)$ is an affine variety.
- Given a representation $\tau \colon \pi \to \mathrm{GL}(V)$, let V_τ be the left $\mathbb{C}[\pi]$-module V defined by $g \cdot v = \tau(g)v$.
- The *characteristic varieties* of X with respect to a rational representation $\iota \colon G \to \mathrm{GL}(V)$ are the algebraic subsets

$$\mathcal{V}^i_s(X, \iota) = \{\rho \in \operatorname{Hom}(\pi, G) \mid \dim H^i(X, V_{\iota\circ\rho}) \geqslant s\}.$$

- When $G = \mathbb{C}^*$ and $\iota \colon \mathbb{C}^* \xrightarrow{\simeq} \mathrm{GL}_1(\mathbb{C})$, we get the rank 1 characteristic varieties, $\mathcal{V}^i_s(X)$, sitting inside the character group $\operatorname{Char}(X) := \operatorname{Hom}(\pi, \mathbb{C}^*)$.

THEOREM (..., ARAPURA, ..., BUDUR–WANG)

If M is a quasi-projective manifold, the varieties $\mathcal{V}^i_s(M)$ are finite unions of torsion-translates of subtori of $\mathrm{Char}(M)$.

- A holomorphic map $f\colon M \to \Sigma$ is *admissible* if it surjective, its fibers are connected, and Σ is a smooth complex curve.
- The map $f_\sharp\colon \pi_1(M) \to \pi_1(\Sigma)$ is also surjective. Thus, the morphism $f^! := f_\sharp^*\colon \mathrm{Char}(\Sigma) \to \mathrm{Char}(M)$ is injective.
- Up to reparametrization at the target, there is a finite set $\mathcal{E}(M)$ of admissible maps with the property that $\chi(\Sigma) < 0$.

THEOREM (ARAPURA 1997)

The correspondence $f \rightsquigarrow f^!\,\mathrm{Char}(\Sigma)$ defines a bijection between $\mathcal{E}(M)$ and the set of positive-dimensional, irreducible components of $\mathcal{V}^1_1(M)$ passing through 1.

THEOREM (KAPOVICH–MILLSON UNIVERSALITY)

PSL_2*-representation varieties of Artin groups may have arbitrarily bad singularities away from the origin.*

THEOREM (KAPOVICH–MILLSON 1998)

Let M *be a quasi-projective manifold, and* G *be a reductive algebraic group. If* $\rho\colon \pi_1(M) \to G$ *is a representation with finite image, then the germ* $\mathrm{Hom}(\pi_1(M), G)_{(\rho)}$ *is analytically isomorphic to a quasi-homogeneous cone with generators of weight* 1 *and* 2 *and relations of weight* 2, 3, *and* 4.

THEOREM (CORLETTE-SIMPSON 08, LORAY-PEREIRA-TOUZET 16)

If $\rho\colon \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$ *is not virtually abelian, then there is an orbifold morphism* $f\colon M \to N$ *such that* $\tilde{\rho}\colon \pi_1(M) \to \mathrm{PSL}_2(\mathbb{C})$ *belongs to* $f^! \,\mathrm{Hom}(\pi_1(N), \mathrm{PSL}_2(\mathbb{C}))$, *where* N *is either a* 1*-dim complex orbifold, or a polydisk Shimura modular orbifold.*

- For an arrangement $\mathcal{A}$, all base curves Σ have genus 0, by purity of the MHS on $H^\bullet(M(\mathcal{A}), \mathbb{Q})$.
- Set $E(\mathcal{A}) = \mathcal{E}(M(\mathcal{A})) \cup \{\alpha\}$. Note that all maps $f \in E(\mathcal{A})$ are of the form $f\colon M(\mathcal{A}) \to M(\mathcal{A}_f)$, for some arrangement $\mathcal{A}_f$.
- Write $\pi = \pi_1(M(\mathcal{A}))$ and $\pi_f = \pi_1(M(\mathcal{A}_f))$

THEOREM (PAPADIMA–S. 2016)

Let $G = \mathrm{SL}_2(\mathbb{C})$ and let $\iota\colon G \to \mathrm{GL}(V)$ be a rational representation. Then,

$$\mathrm{Hom}(\pi, G)_{(1)} = \bigcup_{f \in E(\mathcal{A})} f^! \,\mathrm{Hom}(\pi_f, G)_{(1)}$$

$$\mathcal{V}^1_1(\pi, \iota)_{(1)} = \bigcup_{f \in E(\mathcal{A})} f^! \mathcal{V}^1_1(\pi_f, \iota)_{(1)}$$

THE TANGENT CONE THEOREM

- Let X be a connected, finite-type CW-complex, let $\Bbbk$ be a field ($\mathrm{char}(\Bbbk) \neq 2$), and set $A = H^{\bullet}(X, \Bbbk)$.
- For each $a \in A^1$, we get a cochain complex

$$(A, \cdot a)\colon \; A^0 \xrightarrow{\;a\;} A^1 \xrightarrow{\;a\;} A^2 \longrightarrow \cdots$$

- The *resonance varieties* of X are the homogeneous algebraic sets

$$\mathcal{R}^i_s(X, \Bbbk) = \{a \in H^1(X, \Bbbk) \mid \dim_{\Bbbk} H^i(A, a) \geqslant s\}.$$

THEOREM (DIMCA–PAPADIMA–S. 2010, DIMCA–PAPADIMA 2014)

Let X be a formal space. Then:

- *The homomorphism* $\exp\colon H^1(X, \mathbb{C}) \to H^1(X, \mathbb{C}^*)$ *induces isos of analytic germs,* $\mathcal{R}^i_s(X, \mathbb{C})_{(0)} \xrightarrow{\simeq} \mathcal{V}^i_s(X)_{(1)}$.
- *All irreducible components of* $\mathcal{R}^i_s(X, \mathbb{C})$ *are rationally defined linear subspaces.*

- X is an *abelian duality space* of dim n if $H^i(X, \mathbb{Z}\pi_{\mathrm{ab}}) = 0$ for $i \neq n$ and $B := H^n(X, \mathbb{Z}\pi_{\mathrm{ab}})$ is non-zero and torsion-free.
- $H^i(X, A) \cong H_{n-i}(X, B \otimes A)$, for any $\mathbb{Z}\pi_{\mathrm{ab}}$-module A.

THEOREM (DENHAM–S.–YUZVINSKY 2015/16)

Let X be an abelian duality space of dimension n. Then:

- $\mathcal{V}^1_1(X) \subseteq \cdots \subseteq \mathcal{V}^n_1(X)$.
- $b_1(X) \geqslant n - 1$.
- *If $n \geqslant 2$, then $b_i(X) \neq 0$, for all $0 \leqslant i \leqslant n$.*

- A cyclic, graded E-module $A = E/I$ has the *EPY property* if $A^*(n)$ is a Koszul module for some integer n.
- If $A = H^\bullet(X, \mathbb{k})$ has this property, we say that X has the EPY property over $\mathbb{k}$.

Propagation of resonance

Theorem (DSY)

Suppose X is a finite, connected CW-complex of dimension n with the EPY property over a field $\Bbbk$. Then the resonance varieties of X propagate:

$$\mathcal{R}^1(X,\Bbbk) \subseteq \cdots \subseteq \mathcal{R}^n(X,\Bbbk).$$

Theorem (DSY)

Let $\mathcal{A}$ be an essential arrangement in $\mathbb{C}^n$. Then $M(\mathcal{A})$ is an abelian duality space of dimension n (and also is formal and has the EPY property). Consequently, the characteristic and resonance varieties of $M(\mathcal{A})$ propagate.

- All irreducible components of $\mathcal{R}^i_s(M(\mathcal{A}),\mathbb{C})$ are linear.
- In general, $\mathcal{R}^1_1(M(\mathcal{A}),\Bbbk)$ may have non-linear components.

MULTINETS AND DEGREE 1 RESONANCE

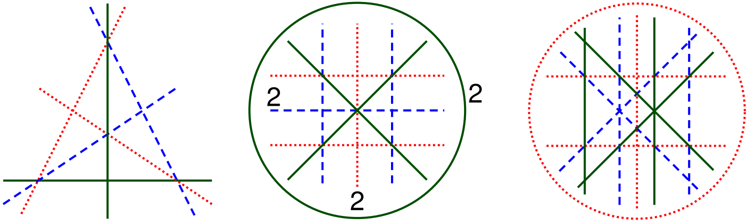


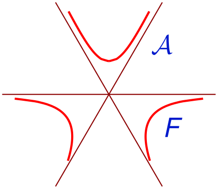
FIGURE: $(3, 2)$-net; $(3, 4)$-multinet; non-3-net, reduced $(3, 4)$-multinet

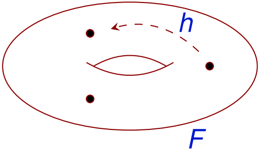
THEOREM (FALK, COHEN–S., LIBGOBER–YUZVINSKY, Falk–Yuz)

$$\mathcal{R}^1_s(M(\mathcal{A}), \mathbb{C}) = \bigcup_{\mathcal{B} \subseteq \mathcal{A}} \quad \bigcup_{\substack{\mathcal{N} \text{ a } k\text{-multinet on } \mathcal{B} \\ \text{with at least } s+2 \text{ parts}}} P_{\mathcal{N}}.$$

where $P_{\mathcal{N}}$ is the $(k-1)$-dimensional linear subspace spanned by the vectors $u_2 - u_1, \dots, u_k - u_1$, where $u_\alpha = \sum_{H \in \mathcal{B}_\alpha} m_H e_H$.

Milnor fibration





- Let $\mathcal{A}$ be an arrangement of n hyperplanes in $\mathbb{C}^{d+1}$. For each $H \in \mathcal{A}$ let α_H be a linear form with $\ker(\alpha_H) = H$, and let $Q = \prod_{H \in \mathcal{A}} \alpha_H$.
- $Q\colon \mathbb{C}^{d+1} \to \mathbb{C}$ restricts to a smooth fibration, $Q\colon M(\mathcal{A}) \to \mathbb{C}^*$. The *Milnor fiber* of the arrangement is $F(\mathcal{A}) := Q^{-1}(1)$.
- F is a Stein manifold. It has the homotopy type of a finite cell complex of dim d. In general, F is neither formal, nor minimal.
- $F = F(\mathcal{A})$ is the regular, $\mathbb{Z}_n$-cover of $U = U(\mathcal{A})$, classified by the morphism $\pi_1(U) \twoheadrightarrow \mathbb{Z}_n$ taking each loop x_H to 1.

Modular inequalities

- The monodromy diffeo, $h\colon F \to F$, is given by $h(z) = e^{2\pi \mathrm{i}/n} z$.
- Let $\Delta(t)$ be the characteristic polynomial of $h_*\colon H_1(F,\mathbb{C}) \circlearrowleft$. Since $h^n = \mathrm{id}$, we have

$$\Delta(t) = \prod_{r|n} \Phi_r(t)^{e_r(\mathcal{A})},$$

  where $\Phi_r(t)$ is the r-th cyclotomic polynomial, and $e_r(\mathcal{A}) \in \mathbb{Z}_{\geqslant 0}$.
- WLOG, we may assume $d = 2$, so that $\bar{\mathcal{A}} = \mathbb{P}(\mathcal{A})$ is an arrangement of lines in $\mathbb{CP}^2$.
- If there is no point of $\bar{\mathcal{A}}$ of multiplicity $q \geqslant 3$ such that $r \mid q$, then $e_r(\mathcal{A}) = 0$ (Libgober 2002).
- In particular, if $\bar{\mathcal{A}}$ has only points of multiplicity 2 and 3, then $\Delta(t) = (t-1)^{n-1}(t^2+t+1)^{e_3}$. If multiplicity 4 appears, then we also get factor of $(t+1)^{e_2} \cdot (t^2+1)^{e_4}$.

- Let $A = H^\bullet(M(\mathcal{A}), \Bbbk)$, and let $\sigma = \sum_{H \in \mathcal{A}} e_H \in A^1$.
- Assume $\Bbbk$ has characteristic $p > 0$, and define

$$\beta_p(\mathcal{A}) = \dim_{\Bbbk} H^1(A, \cdot\sigma).$$

That is, $\beta_p(\mathcal{A}) = \max\{s \mid \sigma \in \mathcal{R}^1_s(A, \Bbbk)\}$.

THEOREM (COHEN–ORLIK 2000, PAPADIMA–S. 2010)

$e_{p^m}(\mathcal{A}) \leqslant \beta_p(\mathcal{A})$, *for all* $m \geqslant 1$.

THEOREM (PAPADIMA–S. 2014)

- *Suppose* $\mathcal{A}$ *admits a* k*-net. Then* $\beta_p(\mathcal{A}) = 0$ *if* $p \nmid k$ *and* $\beta_p(\mathcal{A}) \geqslant k - 2$*, otherwise.*
- *If* $\mathcal{A}$ *admits a reduced* k*-multinet, then* $e_k(\mathcal{A}) \geqslant k - 2$*.*

THEOREM (PAPADIMA–S. 2014)

Suppose $\bar{\mathcal{A}}$ has no points of multiplicity $3r$ with $r > 1$. TFAE:

- *$\mathcal{A}$ admits a reduced 3-multinet.*
- *$\mathcal{A}$ admits a 3-net.*
- *$\beta_3(\mathcal{A}) \neq 0$.*

Moreover, the following hold:

- *$\beta_3(\mathcal{A}) \leqslant 2$.*
- *$e_3(\mathcal{A}) = \beta_3(\mathcal{A})$, and thus $e_3(\mathcal{A})$ is determined by $L_{\leqslant 2}(\mathcal{A})$.*

In particular, if $\bar{\mathcal{A}}$ has only double and triple points, then $\Delta(t)$ is combinatorially determined.

THEOREM (PS)

Suppose $\mathcal{A}$ supports a 4-net and $\beta_2(\mathcal{A}) \leqslant 2$. Then

$$e_2(\mathcal{A}) = e_4(\mathcal{A}) = \beta_2(\mathcal{A}) = 2.$$

Conjecture (PS)

The characteristic polynomial of the degree 1 algebraic monodromy for the Milnor fibration of an arrangement $\mathcal{A}$ of rank at least 3 is given by the combinatorial formula

$$\Delta_{\mathcal{A}}(t) = (t-1)^{|\mathcal{A}|-1}((t+1)(t^2+1))^{\beta_2(\mathcal{A})}(t^2+t+1)^{\beta_3(\mathcal{A})}.$$

The conjecture has been verified for several classes of arrangements, such as:

- All sub-arrangements of non-exceptional Coxeter arrangements (Măcinic, Papadima).
- All complex reflection arrangements (Măcinic, Papadima, Popescu, Dimca, Sticlaru).
- Certain types of complexified real arrangements (Yoshinaga, Bailet, Torielli, Settepanella).

THE BOUNDARY MANIFOLD

- Let $\mathcal{A}$ be a (central) arrangement of hyperplanes in $\mathbb{C}^{d+1}$.
- Let N be a (closed) regular neighborhood of the hypersurface $\bigcup_{H\in\mathcal{A}} \mathbb{P}(H) \subset \mathbb{CP}^d$.
- Let $\overline{U}(\mathcal{A}) = \mathbb{CP}^d \setminus \operatorname{int}(N)$. Clearly, $\overline{U} \simeq U$.
- The *boundary manifold* of $\mathcal{A}$ is $\partial\overline{U} = \partial N$. This is a compact, orientable, smooth manifold of dimension $2d-1$.

EXAMPLE

- Let $\mathcal{A}$ be a pencil of n hyperplanes in $\mathbb{C}^{d+1}$. If $n=1$, then $\partial\overline{U} = S^{2d-1}$. If $n>1$, then $\partial\overline{U} = \sharp^{n-1} S^1 \times S^{2(d-1)}$.
- Let $\mathcal{A}$ be a near-pencil of n planes in $\mathbb{C}^3$. Then $\partial\overline{U} = S^1 \times \Sigma_{n-2}$, where $\Sigma_g = \sharp^g S^1 \times S^1$.

- When $d = 2$, the boundary manifold $\partial \overline{U}$ is a 3-dimensional graph-manifold M_Γ, where
  - Γ is the incidence graph of $\mathcal{A}$, with $V(\Gamma) = L_1(\mathcal{A}) \cup L_2(\mathcal{A})$ and $E(\Gamma) = \{(L, P) \mid P \in L\}$.
  - Vertex manifolds $M_v = S^1 \times \left(S^2 \setminus \bigcup_{\{v,w\}\in E(\Gamma)} D^2_{v,w}\right)$ are glued along edge manifolds $M_e = S^1 \times S^1$ via flip maps.

THEOREM (JIANG–YAU 1993)

$U(\mathcal{A}) \cong U(\mathcal{A}') \Rightarrow M_\Gamma \cong M_{\Gamma'} \Rightarrow \Gamma \cong \Gamma' \Rightarrow L(\mathcal{A}) \cong L(\mathcal{A}')$.

THEOREM (COHEN–S. 2008)

$\mathcal{V}^1_1(M_\Gamma) = \bigcup_{v\in V(\Gamma)\,:\,\deg(v)\geqslant 3} \{\prod_{i\in v} t_i = 1\}$. *Moreover, TFAE:*

- M_Γ *is formal.*
- $\mathrm{TC}_1(\mathcal{V}^1_1(M_\Gamma)) = \mathcal{R}^1_1(M_\Gamma, \mathbb{C})$.
- $\mathcal{A}$ *is a pencil or a near-pencil.*

THE RFRp PROPERTY

DEFINITION (AGOL, KOBERDA–S.)

A finitely generated group G is *residually finite rationally* p for some prime p if there is a sequence of subgroups $G = G_0 > \cdots > G_i > G_{i+1} > \cdots$ such that $\bigcap_{i \geqslant 0} G_i = \{1\}$, and, for each i,

- $G_{i+1} \lhd G_i$;
- G_i / G_{i+1} is an elementary abelian p-group;
- $\ker(G_i \to H_1(G_i, \mathbb{Q}))$ is a subgroup of G_{i+1}.

- G RFRp $\Rightarrow$ residually p $\Rightarrow$ residually finite & residually nilpotent.
- G RFRp $\Rightarrow$ torsion-free.
- G finitely presented & RFRp $\Rightarrow$ has solvable word problem.
- The class of RFRp groups is closed under taking subgroups, finite direct products, and finite free products.

- Finitely generated free groups F_n, surface groups $\pi_1(\Sigma_g)$, and right-angled Artin groups A_Γ are RFRp, for all p.
- Finite groups and non-abelian nilpotent groups are *not* RFRp, for any p.

THEOREM (KOBERDA–S. 2016)

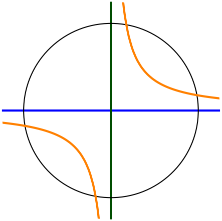
If G is a finitely presented group which is RFRp for infinitely many primes p, then either G is abelian or G is large (i.e., it virtually surjects onto a non-abelian free group).

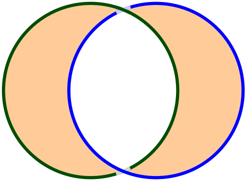
THEOREM (KS)

Let M_Γ be the boundary manifold of a line arrangement in $\mathbb{C}^2$. Then $\pi_1(M_\Gamma)$ is RFRp, for all primes p.

CONJECTURE (KS)

Let $\pi = \pi_1(M(\mathcal{A}))$ be an arrangement group. Then π is RFRp, for all p. (In particular, π is torsion-free and residually finite.)





- For an arrangement $\mathcal{A}$ in $\mathbb{C}^{d+1}$, let $\overline{F}(\mathcal{A}) = F(\mathcal{A}) \cap D^{2(d+1)}$ be the *closed Milnor fiber* of $\mathcal{A}$. Clearly, $F \simeq \overline{F}$.
- The *boundary of the Milnor fiber* of $\mathcal{A}$ is the compact, smooth, orientable, $(2d-1)$-manifold $\partial\overline{F} = F \cap S^{2d+1}$.
- The pair $(\overline{F}, \partial\overline{F})$ is $(d-1)$-connected. In particular, if $d \geqslant 2$, then $\partial\overline{F}$ is connected, and $\pi_1(\partial\overline{F}) \to \pi_1(\overline{F})$ is surjective.

Example

- Let $\mathcal{B}_n$ be the Boolean arrangement in $\mathbb{C}^n$. Recall $F = (\mathbb{C}^*)^{n-1}$. Hence, $\overline{F} = T^{n-1} \times D^{n-1}$ & and so $\partial\overline{F} = T^{n-1} \times S^{n-2}$.
- Let $\mathcal{A}$ be a near-pencil of n planes in $\mathbb{C}^3$. Then $\partial\overline{F} = S^1 \times \Sigma_{n-2}$.

The Hopf fibration $\pi\colon \mathbb{C}^{d+1}\setminus\{0\} \to \mathbb{CP}^d$ restricts to regular, cyclic n-fold covers, $\pi\colon \overline{F} \to \overline{U}$ and $\pi\colon \partial\overline{F} \to \partial\overline{U}$, which fit into

$$
\begin{array}{ccccccccc}
\mathbb{Z}_n & = & \mathbb{Z}_n & = & \mathbb{Z}_n & \longrightarrow & \mathbb{C}^* & = & \mathbb{C}^* \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\partial\overline{F} & \longrightarrow & \overline{F} & \xrightarrow{\simeq} & F & \longrightarrow & M & \longrightarrow & \mathbb{C}^{d+1}\setminus\{0\} \\
\downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\
\partial\overline{U} & \longrightarrow & \overline{U} & \xrightarrow{\simeq} & U & = & U & \longrightarrow & \mathbb{CP}^d
\end{array}
$$

Assume now that $d = 2$. The fundamental group of $\partial\overline{U} = M_\Gamma$ has generators $\overline{x}_H$ for $H \in \mathcal{A}$ and generators y_c corresponding to the cycles of Γ.

PROPOSITION (S. 2014)

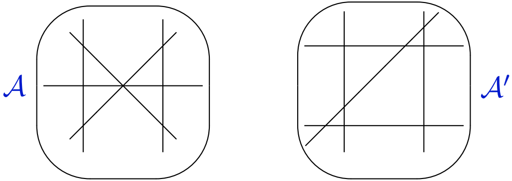
The $\mathbb{Z}_n$-cover $\pi\colon \partial\overline{F} \to \partial\overline{U}$ is classified by the homomorphism $\pi_1(\partial\overline{U}) \twoheadrightarrow \mathbb{Z}_n$ given by $x_H \mapsto 1$ and $y_c \mapsto 0$.

THEOREM (NÉMETHI–SZILARD 2012)

The characteristic polynomial of $h_*\colon H_1(\partial\overline{F}, \mathbb{C}) \circlearrowleft$ is given by

$$\delta(t) = \prod_{X \in L_2(\mathcal{A})} (t-1)(t^{\gcd(|\mathcal{A}_X|,|\mathcal{A}|)} - 1)^{|\mathcal{A}_X|-2}.$$

A pair of arrangements



- Let $\mathcal{A}$ and $\mathcal{A}'$ be the above pair of arrangements. Both have 2 triple points and 9 double points, yet $L(\mathcal{A}) \not\cong L(\mathcal{A}')$.
- As noted by Rose and Terao, the respective OS-algebras are isomorphic. In fact, as shown by Falk, $U(\mathcal{A}) \simeq U(\mathcal{A}')$.
- Since $L(\mathcal{A}) \not\cong L(\mathcal{A}')$, the corresponding boundary manifolds, $\partial\overline{U}$ and $\partial\overline{U}'$, are not homotopy equivalent.
- In fact, $\mathcal{V}^1_1(\partial\overline{U})$ consists of 7 codimension-1 subtori in $(\mathbb{C}^*)^{13}$, while $\mathcal{V}^1_1(\partial\overline{U}')$ consists of 8 such subtori.

- The corresponding Milnor fibers, F and F', have the same characteristic polynomial of the algebraic monodromy,

$$\Delta = \Delta' = (t-1)^5.$$

- Likewise for the boundaries of the Milnor fibers,

$$\delta = \delta' = (t-1)^{13}(t^2+t+1)^2.$$

- The characteristic varieties $\mathcal{V}^1_1(F)$ and $\mathcal{V}^1_1(F')$ consist of two 2-dimensional subtori of $(\mathbb{C}^*)^5$. On the other hand,

$$\mathcal{V}^1_2(F) = \{\mathbf{1}, (1,\omega,\omega,1,1), (1,\omega^2,\omega^2,1,1)\},$$
$$\mathcal{V}^1_2(F') = \{\mathbf{1}\}.$$

- Thus, $\pi_1(F) \not\cong \pi_1(F')$.

CONJECTURE

Let $\mathcal{A}$ and $\mathcal{A}'$ be two central arrangements in $\mathbb{C}^3$. Then

$$F(\mathcal{A}) \cong F(\mathcal{A}') \Rightarrow L(\mathcal{A}) \cong L(\mathcal{A}').$$

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