DUALITY AND RESONANCE

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POINCARÉ DUALITY ALGEBRAS

- Let A be a graded, graded-commutative algebra over a field $\Bbbk$.
  - $A = \bigoplus_{i \geqslant 0} A^i$, where A^i are $\Bbbk$-vector spaces.
  - $\cdot\colon A^i \otimes A^j \to A^{i+j}$.
  - $ab = (-1)^{ij} ba$ for all $a \in A^i$, $b \in B^j$.
- We will assume that A is connected ($A^0 = \Bbbk \cdot 1$), and locally finite (all the Betti numbers $b_i(A) := \dim_{\Bbbk} A^i$ are finite).
- A is a *Poincaré duality* $\Bbbk$*-algebra* of dimension m if there is a $\Bbbk$-linear map $\varepsilon\colon A^m \to \Bbbk$ (called an *orientation*) such that all the bilinear forms $A^i \otimes_{\Bbbk} A^{m-i} \to \Bbbk$, $a \otimes b \mapsto \varepsilon(ab)$ are non-singular.
- Consequently,
  - $b_i(A) = b_{m-i}(A)$, and $A^i = 0$ for $i > m$.
  - ε is an isomorphism.
  - The maps $\mathrm{PD}\colon A^i \to (A^{m-i})^*$, $\mathrm{PD}(a)(b) = \varepsilon(ab)$ are isomorphisms.
  - Each $a \in A^i$ has a *Poincaré dual*, $a^\vee \in A^{m-i}$, such that $\varepsilon(aa^\vee) = 1$.
  - The *orientation class* is defined as $\omega_A = 1^\vee$, so that $\varepsilon(\omega_A) = 1$.

THE ASSOCIATED ALTERNATING FORM

- Associated to a $\Bbbk$-PD$_m$ algebra there is an alternating m-form,

$$\mu_A\colon \bigwedge^m A^1 \to \Bbbk, \quad \mu_A(a_1 \wedge \cdots \wedge a_m) = \varepsilon(a_1 \cdots a_m).$$

- Assume now that $m = 3$, and set $n = b_1(A)$. Fix a basis $\{e_1, \dots, e_n\}$ for A^1, and let $\{e_1^\vee, \dots, e_n^\vee\}$ be the PD basis for A^2.
- The multiplication in A, then, is given on basis elements by

$$e_i e_j = \sum_{k=1}^{n} \mu_{ijk}\, e_k^\vee, \quad e_i e_j^\vee = \delta_{ij}\omega,$$

  where $\mu_{ijk} = \mu(e_i \wedge e_j \wedge e_k)$.
- Alternatively, let $A_i = (A^i)^*$, and let $e^i \in A_1$ be the (Kronecker) dual of e_i. We may then view μ dually as a trivector,

$$\mu = \sum \mu_{ijk}\, e^i \wedge e^j \wedge e^k \in \bigwedge^3 A_1,$$

  which encodes the algebra structure of A.

POINCARÉ DUALITY IN ORIENTABLE MANIFOLDS

- If M is a compact, connected, orientable, m-dimensional manifold, then the cohomology ring $A = H^{\cdot}(M, \Bbbk)$ is a PD_m algebra over $\Bbbk$.
- Sullivan (1975): for every finite-dimensional $\mathbb{Q}$-vector space V and every alternating 3-form $\mu \in \bigwedge^3 V^*$, there is a closed 3-manifold M with $H^1(M, \mathbb{Q}) = V$ and cup-product form $\mu_M = \mu$.
- Such a 3-manifold can be constructed via “Borromean surgery."

- If M bounds an oriented 4-manifold W such that the cup-product pairing on $H^2(W, M)$ is non-degenerate (e.g., if M is the link of an isolated surface singularity), then $\mu_M = 0$.

RESONANCE VARIETIES OF GRADED ALGEBRAS

- Let A be a connected, finite-type cga over $\Bbbk = \mathbb{C}$.
- For each $a \in A^1$, there is a cochain complex of $\Bbbk$-vector spaces,

$$(A, \delta_a)\colon \ A^0 \xrightarrow{\delta_a^0} A^1 \xrightarrow{\delta_a^1} A^2 \xrightarrow{\delta_a^2} \cdots ,$$

  with differentials $\delta_a(b) = a \cdot b$, for $b \in A^i$.
- The *resonance varieties* of A are the sets

$$\mathcal{R}^i_s(A) = \{a \in A^1 \mid \dim_{\Bbbk} H^i(A, \delta_a) \geqslant s\}.$$

- An element $a \in A^1$ belongs to $\mathcal{R}^i_s(A)$ if and only if

$$\operatorname{rank} \delta_a^{i+1} + \operatorname{rank} \delta_a^i \leqslant b_i(A) - s.$$

- Fix a $\Bbbk$-basis $\{e_1, \dots, e_n\}$ for A^1, and let $\{x_1, \dots, x_n\}$ be the dual basis for $A_1 = (A^1)^*$.
- Identify $\mathrm{Sym}(A_1)$ with $S = \Bbbk[x_1, \dots, x_n]$, the coordinate ring of the affine space A^1.
- Define a cochain complex of free S-modules, $\mathbf{L}(A) := (A^\bullet \otimes S, \delta)$,

$$\cdots \longrightarrow A^i \otimes S \xrightarrow{\delta^i} A^{i+1} \otimes S \xrightarrow{\delta^{i+1}} A^{i+2} \otimes S \longrightarrow \cdots,$$

  where $\delta^i(u \otimes s) = \sum_{j=1}^n e_j u \otimes s x_j$.
- The specialization of $(A \otimes S, \delta)$ at $a \in A^1$ coincides with (A, δ_a).
- Hence, $\mathcal{R}^i_s(A)$ is the zero-set of the ideal generated by all minors of size $b_i - s + 1$ of the block-matrix $\delta^{i+1} \oplus \delta^i$.
- In particular, $\mathcal{R}^1_s(A) = V(I_{n-s}(\delta^1))$, the zero-set of the ideal of codimension s minors of δ^1.

RESONANCE VARIETIES OF PD-ALGEBRAS

- Let A be a PD_m algebra.
- For all $0 \leqslant i \leqslant m$ and all $a \in A^1$, the square

$$\begin{array}{ccc} (A^{m-i})^* & \xrightarrow{(\delta_a^{m-i-1})^*} & (A^{m-i-1})^* \\ \mathrm{PD}\uparrow\cong & & \mathrm{PD}\uparrow\cong \\ A^i & \xrightarrow{\delta_a^i} & A^{i+1} \end{array}$$

  commutes up to a sign of $(-1)^i$.
- Consequently,

$$\left(H^i(A,\delta_a)\right)^* \cong H^{m-i}(A,\delta_{-a}).$$

- Hence, for all i and s,

$$\mathcal{R}^i_s(A) = \mathcal{R}^{m-i}_s(A).$$

- In particular, $\mathcal{R}^m_1(A) = \{0\}$.

3-DIMENSIONAL POINCARÉ DUALITY ALGEBRAS

- Let A be a PD_3-algebra with $b_1(A) = n > 0$. Then
  - $\mathcal{R}^3_1(A) = \mathcal{R}^0_1(A) = \{0\}$.
  - $\mathcal{R}^2_s(A) = \mathcal{R}^1_s(A)$ for $1 \leqslant s \leqslant n$.
  - $\mathcal{R}^i_s(A) = \emptyset$, otherwise.
- Write $\mathcal{R}_s(A) = \mathcal{R}^1_s(A)$. Work of Buchsbaum and Eisenbud on Pfaffians of skew-symmetric matrices implies that
  - $\mathcal{R}_{2k}(A) = \mathcal{R}_{2k+1}(A)$ if n is even.
  - $\mathcal{R}_{2k-1}(A) = \mathcal{R}_{2k}(A)$ if n is odd.
- If μ_A has rank $n \geqslant 3$, then $\mathcal{R}_{n-2}(A) = \mathcal{R}_{n-1}(A) = \mathcal{R}_n(A) = \{0\}$.
  - Here, the *rank* of a form $\mu\colon \bigwedge^3 V \to \Bbbk$ is the minimum dimension of a linear subspace $W \subset V$ such that μ factors through $\bigwedge^3 W$.
  - The *nullity* of μ is the maximum dimension of a subspace $U \subset V$ such that $\mu(a \wedge b \wedge c) = 0$ for all $a, b \in U$ and $c \in V$.

- If $n \geqslant 4$, then $\dim \mathcal{R}_1(A) \geqslant \operatorname{null}(\mu_A) \geqslant 2$.
- If n is even, then $\mathcal{R}_1(A) = \mathcal{R}_0(A) = A^1$.
- If $n = 2g+1 > 1$, then $\mathcal{R}_1(A) \neq A^1$ if and only if μ_A is 'generic' in the sense of Berceanu and Papadima (1994).
- That is, $\exists\ c \in A^1$ such that the 2-form $\gamma_c \in \bigwedge^2 A_1$ given by $\gamma_c(a \wedge b) = \mu_A(a \wedge b \wedge c)$ has rank $2g$, i.e., $\gamma_c^g \neq 0$ in $\bigwedge^{2g} A_1$.
- In that case, $\mathcal{R}_1(A)$ is the hypersurface $\operatorname{Pf}(\mu_A) = 0$, where $\operatorname{pf}(\delta^1(i;i)) = (-1)^{i+1} x_i \operatorname{Pf}(\mu_A)$.

Example

Let $M = S^1 \times \Sigma_g$, where $g \geqslant 2$. Then $\mu_M = \sum_{i=1}^{g} a_i b_i c$ is generic, and $\operatorname{Pf}(\mu_M) = x_{2g+1}^{g-1}$. Hence, $\mathcal{R}_1 = \cdots = \mathcal{R}_{2g-2} = \{x_{2g+1} = 0\}$ and $\mathcal{R}_{2g-1} = \mathcal{R}_{2g} = \mathcal{R}_{2g+1} = \{0\}$.

Resonance varieties of 3-forms of low rank

n	μ	$\mathcal{R}_1$
3	123	0

n	μ	$\mathcal{R}_1 = \mathcal{R}_2$	$\mathcal{R}_3$
5	125+345	$\{x_5 = 0\}$	0

n	μ	$\mathcal{R}_1$	$\mathcal{R}_2 = \mathcal{R}_3$	$\mathcal{R}_4$
6	123+456	$\mathbb{C}^6$	$\{x_1 = x_2 = x_3 = 0\} \cup \{x_4 = x_5 = x_6 = 0\}$	0
	123+236+456	$\mathbb{C}^6$	$\{x_3 = x_5 = x_6 = 0\}$	0

n	μ	$\mathcal{R}_1 = \mathcal{R}_2$	$\mathcal{R}_3 = \mathcal{R}_4$	$\mathcal{R}_5$
7	147+257+367	$\{x_7 = 0\}$	$\{x_7 = 0\}$	0
	456+147+257+367	$\{x_7 = 0\}$	$\{x_4 = x_5 = x_6 = x_7 = 0\}$	0
	123+456+147	$\{x_1 = 0\} \cup \{x_4 = 0\}$	$\{x_1 = x_2 = x_3 = x_4 = 0\} \cup \{x_1 = x_4 = x_5 = x_6 = 0\}$	0
	123+456+147+257	$\{x_1 x_4 + x_2 x_5 = 0\}$	$\{x_1 = x_2 = x_4 = x_5 = x_7^2 - x_3 x_6 = 0\}$	0
	123+456+147+257+367	$\{x_1 x_4 + x_2 x_5 + x_3 x_6 = x_7^2\}$	0	0

n	μ	$\mathcal{R}_1$	$\mathcal{R}_2 = \mathcal{R}_3$	$\mathcal{R}_4 = \mathcal{R}_5$
8	147+257+367+358	$\mathbb{C}^8$	$\{x_7 = 0\}$	$\{x_3 = x_5 = x_7 = x_8 = 0\} \cup \{x_1 = x_3 = x_4 = x_5 = x_7 = 0\}$
	456+147+257+367+358	$\mathbb{C}^8$	$\{x_5 = x_7 = 0\}$	$\{x_3 = x_4 = x_5 = x_7 = x_1 x_8 + x_6^2 = 0\}$
	123+456+147+358	$\mathbb{C}^8$	$\{x_1 = x_5 = 0\} \cup \{x_3 = x_4 = 0\}$	$\{x_1 = x_3 = x_4 = x_5 = x_2 x_6 + x_7 x_8 = 0\}$
	123+456+147+257+358	$\mathbb{C}^8$	$\{x_1 = x_5 = 0\} \cup \{x_3 = x_4 = x_5 = 0\}$	$\{x_1 = x_2 = x_3 = x_4 = x_5 = x_7 = 0\}$
	123+456+147+257+367+358	$\mathbb{C}^8$	$\{x_3 = x_5 = x_1 x_4 - x_7^2 = 0\}$	$\{x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = x_7 = 0\}$
	147+268+358	$\mathbb{C}^8$	$\{x_1 = x_4 = x_7 = 0\} \cup \{x_8 = 0\}$	$\{x_1 = x_4 = x_7 = x_8 = 0\} \cup \{x_2 = x_3 = x_5 = x_6 = x_8 = 0\}$
	147+257+268+358	$\mathbb{C}^8$	$L_1 \cup L_2 \cup L_3$	$L_1 \cup L_2$
	456+147+257+268+358	$\mathbb{C}^8$	$C_1 \cup C_2$	$L_1 \cup L_2$
	147+257+367+268+358	$\mathbb{C}^8$	$L_1 \cup L_2 \cup L_3 \cup L_4$	$L'_1 \cup L'_2 \cup L'_3$
	456+147+257+367+268+358	$\mathbb{C}^8$	$C_1 \cup C_2 \cup C_3$	$L_1 \cup L_2 \cup L_3$
	123+456+147+268+358	$\mathbb{C}^8$	$C_1 \cup C_2$	L
	123+456+147+257+268+358	$\mathbb{C}^8$	$\{f_1 = \cdots = f_{20} = 0\}$	0
	123+456+147+257+367+268+358	$\mathbb{C}^8$	$\{g_1 = \cdots = g_{20} = 0\}$	0

CHARACTERISTIC VARIETIES

- Let X be a connected, finite-type CW-complex.
- The fundamental group $\pi = \pi_1(X, x_0)$ is a finitely presented group, with abelianization $\pi_{ab} \cong H_1(X, \mathbb{Z})$.
- The group-algebra $R = \mathbb{C}[\pi_{ab}]$ is the coordinate ring of the character group, $\operatorname{Char}(X) = \operatorname{Hom}(\pi, \mathbb{C}^*) \cong (\mathbb{C}^*)^n \times \operatorname{Tors}(\pi_{ab})$, where $n = b_1(X)$.
- The *characteristic varieties* of X are the homology jump loci

$$\mathcal{V}^i_s(X) = \{\rho \in \operatorname{Char}(X) \mid \dim_{\mathbb{C}} H_i(X, \mathbb{C}_\rho) \geqslant s\}.$$

- Away from 1, we have that $\mathcal{V}^1_s(X) = V(E_s(A_\pi))$, the zero-set of the ideal of codimension s minors of the Alexander matrix of abelianized Fox derivatives of the relators of π.

THE ALEXANDER POLYNOMIAL

- The group-algebra $\mathbb{C}[\pi_{\text{ab}}/\operatorname{Tors}(\pi_{\text{ab}})]$ is isomorphic to $\Lambda = \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$, the coordinate ring of $\operatorname{Char}^0(X) \cong (\mathbb{C}^*)^n$.
- The *Alexander polynomial* Δ_X is the gcd of $E_1(A_\pi \otimes_R \Lambda)$.
- Dimca–Papadima–S. (2011): The zero-set $V(\Delta_X)$ coincides (away from 1) with the union of all codimension 1 irreducible components of $\mathcal{V}_1^1(X) \cap \operatorname{Char}^0(X)$.

EXAMPLE

Let K be a knot in S^3. Its complement, X, is a homology circle. The Alexander polynomial, $\Delta = \Delta_X$, satisfies $\Delta(1) = \pm 1$, and so $1 \notin V(\Delta)$. On the other hand, $\mathcal{V}_1^1(X) = V(\Delta) \cup \{1\}$.

TANGENT CONES AND EXPONENTIAL MAPS

- The map $\exp\colon \mathbb{C}^n \to (\mathbb{C}^*)^n$, $(z_1,\dots,z_n) \mapsto (e^{z_1},\dots,e^{z_n})$ is a homomorphism taking 0 to 1.
- For a Zariski-closed subset $W = V(I)$ inside $(\mathbb{C}^*)^n$, define:
  - The *tangent cone* at 1 to W as $\mathrm{TC}_1(W) = V(\mathrm{in}(I))$.
  - The *exponential tangent cone* at 1 to W as

$$\tau_1(W) = \{z \in \mathbb{C}^n \mid \exp(\lambda z) \in W,\ \forall \lambda \in \mathbb{C}\}$$

- These sets are homogeneous subvarieties of $\mathbb{C}^n$, which depend only on the analytic germ of W at 1.
- Both commute with finite unions and arbitrary intersections.
- $\tau_1(W) \subseteq \mathrm{TC}_1(W)$.
  - $=$ if all irred components of W are subtori.
  - $\neq$ in general.
- $\tau_1(W)$ is a finite union of rationally defined subspaces.

THE TANGENT CONE THEOREM

- The *resonance varieties* of a space X are the jump loci $\mathcal{R}^i_s(X) \subset H^1(X, \mathbb{C}) = \mathbb{C}^n$ associated to the algebra $A = H^*(X, \mathbb{C})$.
- We also have the characteristic varieties $\mathcal{V}^i_s(X) \subset \mathrm{Char}(X)$.
- (Libgober 2002)

$$\mathrm{TC}_1(\mathcal{V}^i_s(X)) \subseteq \mathcal{R}^i_s(X).$$

- Thus,

$$\tau_1(\mathcal{V}^i_s(X)) \subseteq \mathrm{TC}_1(\mathcal{V}^i_s(X)) \subseteq \mathcal{R}^i_s(X).$$

- (DPS 2009/DP 2014) If X is formal, then

$$\tau_1(\mathcal{V}^i_s(X)) = \mathrm{TC}_1(\mathcal{V}^i_s(X)) = \mathcal{R}^i_s(X).$$

A TANGENT CONE THEOREM FOR 3-MANIFOLDS

- Let M be a closed, orientable, 3-dimensional manifold.
- C. McMullen (2000): Let I be the augmentation ideal of Λ. Then

$$E_1(M) = \begin{cases} (\Delta_M) & \text{if } b_1(M) \leqslant 1, \\ I^2 \cdot (\Delta_M) & \text{if } b_1(M) \geqslant 2. \end{cases}$$

- It follows that $\mathcal{V}^1_1(M) \cap \mathrm{Char}^0(M) = V(\Delta_M)$, at least away from 1.
- Using the previous discussion, as well as work of Turaev (2002), we obtain:

THEOREM

Suppose $b_1(M)$ is odd and μ_M is generic. Then

$$\mathrm{TC}_1(\mathcal{V}^1_1(M)) = \mathcal{R}^1_1(M).$$

- If $b_1(M)$ is even, the conclusion of the theorem may or may not hold:
  - Let $M = S^1 \times S^2 \# S^1 \times S^2$; then $\mathcal{V}^1_1(M) = \mathrm{Char}(M) = (\mathbb{C}^*)^2$, and so $\mathrm{TC}_1(\mathcal{V}^1_1(M)) = \mathcal{R}^1_1(M) = \mathbb{C}^2$.
  - Let M be the Heisenberg nilmanifold; then $\mathrm{TC}_1(\mathcal{V}^1_1(M)) = \{0\}$, whereas $\mathcal{R}^1_1(M) = \mathbb{C}^2$.
- If M is not formal, the first half of the Tangent Cone theorem may fail to hold, i.e., $\tau_1(\mathcal{V}^1_1(M)) \nsubseteq \mathrm{TC}_1(\mathcal{V}^1_1(M))$.
  - Let M be a closed, orientable 3-manifold with $b_1 = 7$ and $\mu = e_1e_3e_5 + e_1e_4e_7 + e_2e_5e_7 + e_3e_6e_7 + e_4e_5e_6$. Then μ is generic and $\mathrm{Pf}(\mu) = (x_5^2 + x_7^2)^2$. Hence, $\mathcal{R}^1_1(M) = \{x_5^2 + x_7^2 = 0\}$ splits as a union of two hyperplanes over $\mathbb{C}$, but not over $\mathbb{Q}$.

Propagation of jump loci

- We say that the resonance varieties of a graded algebra $A = \bigoplus_{i=0}^{n} A^i$ *propagate* if
$$\mathcal{R}_1^1(A) \subseteq \cdots \subseteq \mathcal{R}_1^n(A).$$
- Likewise, the characteristic varieties of an n-dimensional CW-complex X propagate if
$$\mathcal{V}^1(X) \subseteq \cdots \subseteq \mathcal{V}^n(X).$$
- (Eisenbud–Popescu–Yuzvinsky 2003) If X is the complement of a hyperplane arrangement, then its resonance varieties propagate.

Theorem (Denham–S.–Yuzvinsky 2016/17, generalizing EPY)

Suppose the $\Bbbk$-dual of a graded algebra A has a linear free resolution over $E = \bigwedge A^1$. Then the resonance varieties of A propagate.

DUALITY SPACES

In order to study propagation of jump loci in a topological setting, we turn to a notion due to Bieri and Eckmann (1978).

- X is a *duality space* of dimension n if $H^i(X, \mathbb{Z}\pi) = 0$ for $i \neq n$ and $H^n(X, \mathbb{Z}\pi) \neq 0$ and torsion-free.
- Let $D = H^n(X, \mathbb{Z}\pi)$ be the dualizing $\mathbb{Z}\pi$-module. Given any $\mathbb{Z}\pi$-module A, we have $H^i(X, A) \cong H_{n-i}(X, D \otimes A)$.
- If $D = \mathbb{Z}$, with trivial $\mathbb{Z}\pi$-action, then X is a Poincaré duality space.
- If $X = K(\pi, 1)$ is a duality space, then π is a *duality group*.

ABELIAN DUALITY SPACES

We introduce in (DSY17) an analogous notion, by replacing $\pi \rightsquigarrow \pi_{ab}$.

- X is an *abelian duality space* of dimension n if $H^i(X, \mathbb{Z}\pi_{ab}) = 0$ for $i \neq n$ and $H^n(X, \mathbb{Z}\pi_{ab}) \neq 0$ and torsion-free.
- Let $B = H^n(X, \mathbb{Z}\pi_{ab})$ be the dualizing $\mathbb{Z}\pi_{ab}$-module. Given any $\mathbb{Z}\pi_{ab}$-module A, we have $H^i(X, A) \cong H_{n-i}(X, B \otimes A)$.
- The two notions of duality are independent.

THEOREM (DSY)

Let X be an abelian duality space of dimension n. If $\rho\colon \pi_1(X) \to \mathbb{C}^*$ satisfies $H^i(X, \mathbb{C}_\rho) \neq 0$, then $H^j(X, \mathbb{C}_\rho) \neq 0$, for all $i \leqslant j \leqslant n$.

COROLLARY (DSY)

Let X be an abelian duality space of dimension n. Then:

- *The characteristic varieties propagate: $\mathcal{V}_1^1(X) \subseteq \cdots \subseteq \mathcal{V}_1^n(X)$.*
- $b_1(X) \geqslant n-1$.
- *If $n \geqslant 2$, then $b_i(X) \neq 0$, for all $0 \leqslant i \leqslant n$.*

PROPOSITION (DSY)

Let M be a closed, orientable 3-manifold. If $b_1(M)$ is even and non-zero, then the resonance varieties of M do not propagate.

EXAMPLE

- Let M be the 3-dimensional Heisenberg nilmanifold.
- Characteristic varieties propagate: $\mathcal{V}_1^i(M) = \{1\}$ for $i \leqslant 3$.
- Resonance does not propagate: $\mathcal{R}_1^1(M) = \Bbbk^2$, $\mathcal{R}_1^3(M) = 0$.

Toric complexes

- Let L be a simplicial complex on vertex set $\mathsf{V} = \{v_1, \dots, v_m\}$.
- Define $T_L = \mathcal{Z}_L(S^1, *)$ to be the subcomplex of T^m obtained by deleting the cells corresponding to the missing simplices of L.
- T_L is a finite, connected CW-complex, and $\dim T_L = \dim L + 1$.
- T_L is formal. (Notbohm–Ray 2005).
- (Kim–Roush 1980, Charney–Davis 1995) The cohomology algebra $H^*(T_L, \Bbbk)$ is the exterior Stanley–Reisner ring

$$\Bbbk\langle L\rangle = \bigwedge V^* / (v^*_\sigma \mid \sigma \notin L),$$

  where $\Bbbk = \mathbb{Z}$ or a field, V is the free $\Bbbk$-module on V, and $V^* = \operatorname{Hom}_{\Bbbk}(V, \Bbbk)$, while $v^*_\sigma = v^*_{i_1} \cdots v^*_{i_s}$ for $\sigma = \{i_1, \dots, i_s\}$.
- If $H^*(T_K, \mathbb{Z}) \cong H^*(T_L, \mathbb{Z})$, then $K \cong L$. (Stretch 2017)

RIGHT ANGLED ARTIN GROUPS

- The fundamental group $\pi_\Gamma := \pi_1(T_L, *)$ is the RAAG associated to the graph $\Gamma := L^{(1)} = (\mathsf{V}, \mathsf{E})$,

$$\pi_\Gamma = \langle v \in V \mid [v, w] = 1 \text{ if } \{v, w\} \in E \rangle.$$

- If $\Gamma = \overline{K}_n$ then $G_\Gamma = F_n$, while if $\Gamma = K_n$, then $G_\Gamma = \mathbb{Z}^n$.
- If $\Gamma = \Gamma' \coprod \Gamma''$, then $G_\Gamma = G_{\Gamma'} * G_{\Gamma''}$.
- If $\Gamma = \Gamma' * \Gamma''$, then $G_\Gamma = G_{\Gamma'} \times G_{\Gamma''}$.
- $K(\pi_\Gamma, 1) = T_{\Delta_\Gamma}$, where Δ_Γ is the flag complex of Γ.
- (Kim–Makar-Limanov–Neggers–Roush 1980, Droms 1987)

$$\Gamma \cong \Gamma' \iff \pi_\Gamma \cong \pi_{\Gamma'}.$$

Identify $H^1(T_L, \mathbb{C})$ with $\mathbb{C}^{\mathsf{V}}$, the $\mathbb{C}$-vector space with basis $\{v \mid v \in \mathsf{V}\}$.

THEOREM (PAPADIMA–S. 2010)

$$\mathcal{R}^i_s(T_L) = \bigcup_{\substack{\mathsf{W}\subseteq \mathsf{V}\\ \sum_{\sigma\in L_{\mathsf{V}\setminus \mathsf{W}}} \dim \widetilde{H}_{i-1-|\sigma|}(\mathrm{lk}_{L_{\mathsf{W}}}(\sigma),\mathbb{C})\geqslant s}} \mathbb{C}^{\mathsf{W}},$$

where L_{W} is the subcomplex induced by L on W, and $\mathrm{lk}_K(\sigma)$ is the link of a simplex σ in a subcomplex $K \subseteq L$.

In particular (PS06):

$$\mathcal{R}^1_1(G_\Gamma) = \bigcup_{\substack{\mathsf{W}\subseteq \mathsf{V}\\ \Gamma_{\mathsf{W}} \text{ disconnected}}} \mathbb{C}^{\mathsf{W}}.$$

Similar formula holds for $\mathcal{V}^i_s(T_L)$, with $\mathbb{C}^{\mathsf{W}}$ replaced by $(\mathbb{C}^*)^{\mathsf{W}}$.

The Cohen–Macaulay property

A simplicial complex L is *Cohen–Macaulay* if for each simplex $\sigma \in L$, the reduced cohomology of $\mathrm{lk}(\sigma)$ is concentrated in degree $\dim L - |\sigma|$ and is torsion-free.

Theorem (N. Brady–Meier 2001, Jensen–Meier 2005)

A RAAG π_Γ is a duality group if and only if Δ_Γ is Cohen–Macaulay. Moreover, π_Γ is a Poincaré duality group if and only if Γ is a complete graph.

Theorem (DSY17)

A toric complex T_L is an abelian duality space (of dimension $\dim L + 1$) if and only if L is Cohen-Macaulay, in which case both the resonance and characteristic varieties of T_L propagate.

BESTVINA–BRADY GROUPS

- The *Bestvina–Brady group* associated to a graph Γ is defined as $N_\Gamma = \ker(\varphi\colon \pi_\Gamma \to \mathbb{Z})$, where $\varphi(v) = 1$, for each $v \in V(\Gamma)$.
- A counterexample to either the Eilenberg–Ganea conjecture or the Whitehead conjecture can be constructed from these groups.
- The cohomology ring $H^*(N_\Gamma, \mathbb{Z})$ was computed by Papadima–S. (2007) and Leary–Saadetoğlu (2011).
- The jump loci $\mathcal{R}^1_1(N_\Gamma)$ and $\mathcal{V}^1_1(N_\Gamma)$ were computed in PS07.

THEOREM (DAVIS–OKUN 2012)

Suppose Δ_Γ is acyclic. Then N_Γ is a duality group if and only if Δ_Γ is Cohen–Macaulay.

THEOREM (DSY17)

N_Γ is an abelian duality group if and only if Δ_Γ is acyclic and Cohen–Macaulay.

ARRANGEMENTS OF SMOOTH HYPERSURFACES

THEOREM (DENHAM–S. 2017)

Let U be a connected, smooth, complex quasi-projective variety of dimension n. Suppose U has a smooth compactification Y for which

1. *Components of $Y \backslash U$ form an arrangement of hypersurfaces $\mathcal{A}$;*
2. *For each submanifold X in the intersection poset $L(\mathcal{A})$, the complement of the restriction of $\mathcal{A}$ to X is a Stein manifold.*

Then:

1. *U is both a duality space and an abelian duality space of dimension n.*
2. *If A is a finite-dimensional representation of $\pi = \pi_1(U)$, and if $A^{\gamma_g} = 0$ for all g in a building set $\mathcal{G}_X$, for some $X \in L(\mathcal{A})$, then $H^i(U, A) = 0$ for all $i \neq n$.*
3. *The ℓ_2-Betti numbers of U vanish for all $i \neq n$.*

LINEAR, ELLIPTIC, AND TORIC ARRANGEMENTS

THEOREM (DS17)

Suppose that $\mathcal{A}$ is one of the following:

1. *An affine-linear arrangement in $\mathbb{C}^n$, or a hyperplane arrangement in $\mathbb{CP}^n$;*
2. *A non-empty elliptic arrangement in E^n;*
3. *A toric arrangement in $(\mathbb{C}^*)^n$.*

Then the complement $M(\mathcal{A})$ is both a duality space and an abelian duality space of dimension $n - r$, $n + r$, and n, respectively, where r is the corank of the arrangement.

This theorem extends several previous results:

1. Davis, Januszkiewicz, Leary, and Okun (2011);
2. Levin and Varchenko (2012);
3. Davis and Settepanella (2013), Esterov and Takeuchi (2014).

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