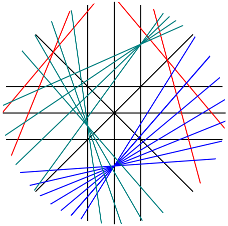
Topology and combinatorics of hyperplane arrangements

Alex Suciu

Northeastern University

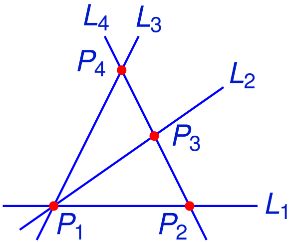
Mathematics Colloquium
University of Nevada at Reno

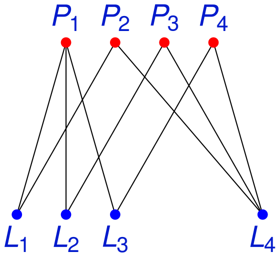
February 22, 2019



Hyperplane arrangements

- An *arrangement of hyperplanes* is a finite collection $\mathcal{A}$ of codimension 1 linear (or affine) subspaces in $\mathbb{C}^\ell$.
- *Intersection lattice* $L(\mathcal{A})$: poset of all intersections of $\mathcal{A}$, ordered by reverse inclusion, and ranked by codimension.





- *Complement*: $M(\mathcal{A}) = \mathbb{C}^\ell \setminus \bigcup_{H\in\mathcal{A}} H$.

Example (The Boolean arrangement)

- $\mathcal{B}_n$: all coordinate hyperplanes $z_i = 0$ in $\mathbb{C}^n$.
- $L(\mathcal{B}_n)$: Boolean lattice of subsets of $\{0, 1\}^n$.
- $M(\mathcal{B}_n)$: complex algebraic torus $(\mathbb{C}^*)^n$.

Example (The braid arrangement)

- $\mathcal{A}_n$: all diagonal hyperplanes $z_i - z_j = 0$ in $\mathbb{C}^n$.
- $L(\mathcal{A}_n)$: lattice of partitions of $[n] := \{1, \dots, n\}$, ordered by refinement.
- $M(\mathcal{A}_n)$: configuration space of n ordered points in $\mathbb{C}$ (a classifying space for P_n, the pure braid group on n strings).

- We may assume that $\mathcal{A}$ is essential, i.e., $\bigcap_{H\in\mathcal{A}} H = \{0\}$.
- Fix an ordering $\mathcal{A} = \{H_1, \dots, H_n\}$, and choose linear forms $f_i\colon \mathbb{C}^\ell \to \mathbb{C}$ with $\ker(f_i) = H_i$. Define an injective linear map

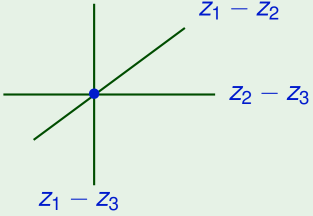
$$\iota\colon \mathbb{C}^\ell \to \mathbb{C}^n, \quad z \mapsto (f_1(z), \dots, f_n(z)).$$

- This map restricts to an inclusion $\iota\colon M(\mathcal{A}) \hookrightarrow M(\mathcal{B}_n)$. Hence, $M(\mathcal{A}) = \iota(\mathbb{C}^\ell) \cap (\mathbb{C}^*)^n$ is a Stein manifold.
- Therefore, $M = M(\mathcal{A})$ has the homotopy type of a connected, finite cell complex of dimension ℓ.
- In fact, M has a minimal cell structure. Consequently, $H_*(M, \mathbb{Z})$ is torsion-free.
- Let $U(\mathcal{A}) = \mathbb{P}(M(\mathcal{A})) = \mathbb{CP}^\ell \setminus \bigcup_{H\in\mathcal{A}} \mathbb{P}(H)$ be the projectivized complement. Then $M(\mathcal{A}) \cong U(\mathcal{A}) \times \mathbb{C}^*$.

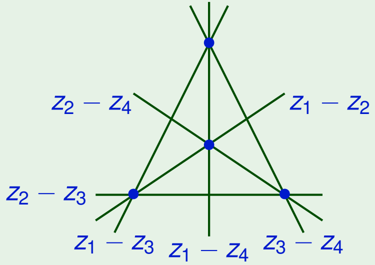
LINE ARRANGEMENTS

- Let $\mathcal{A}' = \{H \cap \mathbb{C}^2\}_{H \in \mathcal{A}}$ be a generic planar slice of $\mathcal{A}$. Then the arrangement group, $\pi = \pi_1(M(\mathcal{A}))$, is isomorphic to $\pi_1(M(\mathcal{A}'))$.
- So, for the purpose of studying π_1's, it is enough to consider arrangements of affine lines in $\mathbb{C}^2$, or projective lines in $\mathbb{CP}^2$.

EXAMPLE



$\pi = P_3 \cong F_2 \times \mathbb{Z}$



$\pi = P_4 \cong F_3 \rtimes P_3$

FUNDAMENTAL GROUP

- Let $\mathcal{A} = \{L_1, \dots, L_n\}$ be a line arrangement in $\mathbb{C}^2$, with multiple points $\mathcal{P} = \{P_1, \dots, P_s\}$.
- The incidence poset $L(\mathcal{A})$ is the corresponding point-line incidence diagram (a bipartite graph).
- Taking a generic projection $\mathbb{C}^2 \to \mathbb{C}$ yields the braid monodromy $\alpha = (\alpha_1, \dots, \alpha_s)$, where $\alpha_r \in P_n$.
- π has a (minimal) finite presentation with meridional generators $x_1, \dots, x_n$ and commutator relators $x_i \alpha_j(x_i)^{-1}$, where each α_j acts on F_n via the Artin representation.
- Let $\pi/\gamma_k(\pi)$ be the $(k-1)^{\text{th}}$ nilpotent quotient of π. Then:
  - $\pi_{\text{ab}} = \pi/\gamma_2$ equals $\mathbb{Z}^n$.
  - π/γ_3 is determined by $L(\mathcal{A})$.
  - π/γ_4 (and thus, π) is *not* determined by $L(\mathcal{A})$. (Rybnikov).

COHOMOLOGY RING

- The Betti numbers $b_q(M) := \operatorname{rank} H_q(M,\mathbb{Z})$ are given by
$$\sum_{q=0}^{\ell} b_q(M(\mathcal{A}))t^q = \sum_{X\in L(\mathcal{A})} \mu(X)(-t)^{\operatorname{rank}(X)},$$
with $\mu\colon L(\mathcal{A}) \to \mathbb{Z}$ given by $\mu(\mathbb{C}^\ell) = 1$ and $\mu(X) = -\sum_{Y\supsetneq X} \mu(Y)$.
- Let $E = \bigwedge(\mathcal{A})$ be the $\mathbb{Z}$-exterior algebra on degree-1 classes e_H dual to the meridians around the hyperplanes $H \in \mathcal{A}$.
- Let $\partial\colon E^{\bullet} \to E^{\bullet-1}$ be the differential given by $\partial(e_H) = 1$, and set $e_{\mathcal{B}} = \prod_{H\in\mathcal{B}} e_H$ for each $\mathcal{B} \subset \mathcal{A}$.
- Building on work of Arnold & Brieskorn, Orlik and Solomon described the cohomology ring of $M(\mathcal{A})$ solely in terms of $L(\mathcal{A})$:
$$H^*(M(\mathcal{A}),\mathbb{Z}) \cong E/\Big\langle \partial e_{\mathcal{B}} \;\Big|\; \operatorname{codim} \bigcap_{H\in\mathcal{B}} H < |\mathcal{B}| \Big\rangle.$$
- The space $M(\mathcal{A})$ is $\mathbb{Q}$-formal but not $\mathbb{F}_p$-formal in general.

Resonance varieties

- Let X be a connected, finite cell complex,
- Let $A = H^*(X, \Bbbk)$, where $\operatorname{char} \Bbbk \neq 2$. Then: $a \in A^1 \Rightarrow a^2 = 0$.
- We thus get a cochain complex

$$(A, \cdot a)\colon \; A^0 \xrightarrow{\;a\;} A^1 \xrightarrow{\;a\;} A^2 \longrightarrow \cdots .$$

- The *resonance varieties* of X are the jump loci for the cohomology of this complex

$$\mathcal{R}^q_s(X, \Bbbk) = \{a \in A^1 \mid \dim_{\Bbbk} H^q(A, \cdot a) \geqslant s\}$$

- E.g., $\mathcal{R}^1_1(X, \Bbbk) = \{a \in A^1 \mid \exists b \in A^1,\; b \neq \lambda a,\; ab = 0\}$.
- These loci are homogeneous subvarieties of $A^1 = H^1(X, \Bbbk)$. In general, they can be arbitrarily complicated.

Resonance varieties of arrangements

Work of Arapura, Falk, D.Cohen, A.S., Libgober, and Yuzvinsky, completely describes the varieties $\mathcal{R}_s(\mathcal{A}) = \mathcal{R}^1_s(M(\mathcal{A}), \mathbb{C})$.

- $\mathcal{R}_1(\mathcal{A})$ is a union of linear subspaces in $H^1(M(\mathcal{A}), \mathbb{C}) \cong \mathbb{C}^{|\mathcal{A}|}$.
- Each subspace has dimension at least 2, and each pair of subspaces meets transversely at 0.
- $\mathcal{R}_s(\mathcal{A})$ is the union of those linear subspaces that have dimension at least $s+1$.
- Each k*-multinet* on a sub-arrangement $\mathcal{B} \subseteq \mathcal{A}$ gives rise to a component of $\mathcal{R}_1(\mathcal{A})$ of dimension $k-1$. Moreover, all components of $\mathcal{R}_1(\mathcal{A})$ arise in this way.

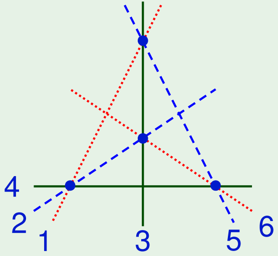
Definition (Falk and Yuzvinsky)

A *multinet* on $\mathcal{A}$ is a partition of the set $\mathcal{A}$ into $k \geqslant 3$ subsets $\mathcal{A}_1, \dots, \mathcal{A}_k$, together with an assignment of multiplicities, $m\colon \mathcal{A} \to \mathbb{N}$, and a subset $\mathcal{X} \subseteq L_2(\mathcal{A})$, such that:

1. $\exists\, d \in \mathbb{N}$ such that $\sum_{H \in \mathcal{A}_\alpha} m_H = d$, for all $\alpha \in [k]$.
2. If H and H' are in different classes, then $H \cap H' \in \mathcal{X}$.
3. $\forall\, X \in \mathcal{X}$, the sum $n_X = \sum_{H \in \mathcal{A}_\alpha : H \supset X} m_H$ is independent of α.
4. $\left(\bigcup_{H \in \mathcal{A}_\alpha} H\right) \backslash \mathcal{X}$ is connected, for each α.

- Such a multinet is also called a (k, d)-multinet, or k-multinet.
- It is *reduced* if $m_H = 1$, for all $H \in \mathcal{A}$.
- A *net* is a reduced multinet with $n_X = 1$, for all $X \in \mathcal{X}$.

EXAMPLE (BRAID ARRANGEMENT $\mathcal{A}_4$)



$\mathcal{R}_1(\mathcal{A}) \subset \mathbb{C}^6$ has 4 local components (from the triple points), and one essential component, from the above $(3,2)$-net:

$$L_{124} = \{x_1 + x_2 + x_4 = x_3 = x_5 = x_6 = 0\},$$
$$L_{135} = \{x_1 + x_3 + x_5 = x_2 = x_4 = x_6 = 0\},$$
$$L_{236} = \{x_2 + x_3 + x_6 = x_1 = x_4 = x_5 = 0\},$$
$$L_{456} = \{x_4 + x_5 + x_6 = x_1 = x_2 = x_3 = 0\},$$
$$L = \{x_1 + x_2 + x_3 = x_1 - x_6 = x_2 - x_5 = x_3 - x_4 = 0\}.$$

Characteristic varieties

- Let X be a connected, finite cell complex, let $\pi = \pi_1(X, x_0)$, and let $\operatorname{Hom}(\pi, \mathbb{C}^*)$ be the affine algebraic group of $\mathbb{C}$-valued, multiplicative characters on π.
- The *characteristic varieties* of X are the jump loci for homology with coefficients in rank-1 local systems on X:

$$\mathcal{V}^q_s(X) = \{\rho \in \operatorname{Hom}(\pi, \mathbb{C}^*) \mid \dim H_q(X, \mathbb{C}_\rho) \geqslant s\}.$$

  Here, $\mathbb{C}_\rho$ is the local system defined by ρ, i.e, $\mathbb{C}$ viewed as a $\mathbb{C}[\pi]$-module, via $g \cdot x = \rho(g)x$, and $H_i(X, \mathbb{C}_\rho) = H_i(C_*(\widetilde{X}, \mathbb{C}) \otimes_{\mathbb{C}[\pi]} \mathbb{C}_\rho)$.
- These loci are Zariski closed subsets of the character group. In general, they can be arbitrarily complicated.
- The sets $\mathcal{V}^1_s(X)$ depend only on π/π''.

CHARACTERISTIC VARIETIES OF ARRANGEMENTS

- Let $\mathcal{A}$ be an arrangement of n hyperplanes, and let $\operatorname{Hom}(\pi_1(M(\mathcal{A})), \mathbb{C}^*) = (\mathbb{C}^*)^n$ be the character torus.
- The characteristic variety $\mathcal{V}_1(\mathcal{A}) := \mathcal{V}_1^1(M(\mathcal{A}))$ lies in the subtorus $\{t \in (\mathbb{C}^*)^n \mid t_1 \cdots t_n = 1\}$; it is a finite union of torsion-translates of algebraic subtori of $(\mathbb{C}^*)^n$.
- If a linear subspace $L \subset \mathbb{C}^n$ is a component of $\mathcal{R}_1(\mathcal{A})$, then the algebraic torus $T = \exp(L)$ is a component of $\mathcal{V}_1(\mathcal{A})$.
- All components of $\mathcal{V}_1(\mathcal{A})$ passing through the origin $1 \in (\mathbb{C}^*)^n$ arise in this way (and thus, are combinatorially determined).
- In general, though, there are translated subtori in $\mathcal{V}_1(\mathcal{A})$, which are not *a priori* determined by $L(\mathcal{A})$.

ABELIAN DUALITY

DEFINITION (BIERI–ECKMANN 1978)

X is a *duality space* of dimension n if $H^i(X, \mathbb{Z}\pi) = 0$ for $i \neq n$ and $H^n(X, \mathbb{Z}\pi) \neq 0$ and torsion-free.

DEFINITION (DENHAM–S.–YUZVINSKY 2016/17)

X is an *abelian duality space* of dimension n if $H^i(X, \mathbb{Z}\pi_{\mathrm{ab}}) = 0$ for $i \neq n$ and $H^n(X, \mathbb{Z}\pi_{\mathrm{ab}}) \neq 0$ and torsion-free.

THEOREM (DSY)

Let X be an abelian duality space of dimension n. Then:

- $b_1(X) \geqslant n - 1$.
- $b_i(X) \neq 0$, *for* $0 \leqslant i \leqslant n$ *and* $b_i(X) = 0$ *for* $i > n$.
- $(-1)^n \chi(X) \geqslant 0$.
- *The characteristic varieties "propagate":* $\mathcal{V}^1_1(X) \subseteq \cdots \subseteq \mathcal{V}^n_1(X)$.

Theorem (Denham–S. 2018)

Let X be a connected, smooth, complex quasi-projective variety of dimension n. Suppose X has a smooth compactification Y for which

- *Components of $Y \setminus X$ form an arrangement of hypersurfaces $\mathcal{A}$;*
- *For each submanifold X in the intersection poset $L(\mathcal{A})$, the complement of the restriction of $\mathcal{A}$ to X is a Stein manifold.*

Then X is both a duality and an abelian duality space of dimension n.

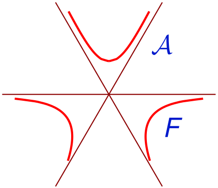
Theorem (DS18)

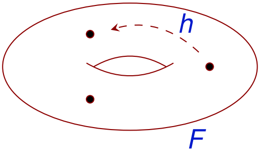
Suppose that $\mathcal{A}$ is one of the following:

- *A hyperplane arrangement in $\mathbb{C}^n$ or $\mathbb{CP}^n$;*
- *A non-empty elliptic arrangement in E^n;*
- *A toric arrangement in $(\mathbb{C}^*)^n$.*

Then $M(\mathcal{A})$ is both a duality and an abelian duality space of dimension $n - r$, $n + r$, and n, respectively, where $r = \operatorname{corank}(\mathcal{A})$.

MILNOR FIBRATION





- Let $\mathcal{A}$ be a central arrangement in $\mathbb{C}^\ell$. For each $H \in \mathcal{A}$ let α_H be a linear form with $\ker(\alpha_H) = H$, and let $Q = \prod_{H\in\mathcal{A}} \alpha_H$.
- $Q\colon \mathbb{C}^\ell \to \mathbb{C}$ restricts to a smooth fibration, $Q\colon M(\mathcal{A}) \to \mathbb{C}^*$. The *Milnor fiber* of the arrangement is $F(\mathcal{A}) := Q^{-1}(1)$.
- F is a Stein manifold. It has the homotopy type of a finite CW-complex of dimension $\ell - 1$.
- In general, F is not $\mathbb{Q}$-formal, and $H_*(F,\mathbb{Z})$ may have torsion.
- $F = F(\mathcal{A})$ is the regular, $\mathbb{Z}_n$-cover of $U = U(\mathcal{A})$, classified by the morphism $\pi_1(U) \twoheadrightarrow \mathbb{Z}_n$ taking each loop x_H to 1 (where $n = |\mathcal{A}|$).

MODULAR INEQUALITIES

- The monodromy diffeo, $h\colon F \to F$, is given by $h(z) = e^{2\pi \mathrm{i}/n} z$.
- Let $\Delta(t)$ be the characteristic polynomial of $h_*\colon H_1(F,\mathbb{C}) \circlearrowleft$. Since $h^n = \mathrm{id}$, we have
$$\Delta(t) = \prod_{r|n} \Phi_r(t)^{e_r(\mathcal{A})},$$
where $\Phi_r(t)$ is the r-th cyclotomic polynomial, and $e_r(\mathcal{A}) \in \mathbb{Z}_{\geqslant 0}$.
- To compute h_*, we may assume $\ell = 3$, so that $\bar{\mathcal{A}} = \mathbb{P}(\mathcal{A})$ is an arrangement of lines in $\mathbb{CP}^2$.
- If there is no point of $\bar{\mathcal{A}}$ of multiplicity $q \geqslant 3$ such that $r \mid q$, then $e_r(\mathcal{A}) = 0$ (Libgober 2002).
- In particular, if $\bar{\mathcal{A}}$ has only points of multiplicity 2 and 3, then $\Delta(t) = (t-1)^{n-1}(t^2+t+1)^{e_3}$. If multiplicity 4 appears, then we also get factor of $(t+1)^{e_2} \cdot (t^2+1)^{e_4}$.

- Let $A = H^{\bullet}(M(\mathcal{A}), \Bbbk)$, and let $\sigma = \sum_{H \in \mathcal{A}} e_H \in A^1$.
- Assume $\Bbbk$ has characteristic $p > 0$, and define

$$\beta_p(\mathcal{A}) = \dim_{\Bbbk} H^1(A, \cdot\sigma).$$

That is, $\beta_p(\mathcal{A}) = \max\{s \mid \sigma \in \mathcal{R}^1_s(A, \Bbbk)\}$.

Theorem (Cohen–Orlik 2000, Papadima–S. 2010)

$e_{p^m}(\mathcal{A}) \leqslant \beta_p(\mathcal{A})$, for all $m \geqslant 1$.

Theorem (Papadima–S. 2017)

- *Suppose $\mathcal{A}$ admits a k-net. Then $\beta_p(\mathcal{A}) = 0$ if $p \nmid k$ and $\beta_p(\mathcal{A}) \geqslant k - 2$, otherwise.*
- *If $\mathcal{A}$ admits a reduced k-multinet, then $e_k(\mathcal{A}) \geqslant k - 2$.*

Combinatorics and monodromy

Theorem (PS)

Suppose $\mathcal{A}$ has no points of multiplicity $3r$ with $r > 1$. Then $\mathcal{A}$ admits a reduced 3-multinet iff $\mathcal{A}$ admits a 3-net iff $\beta_3(\mathcal{A}) \neq 0$. Moreover,

- $\beta_3(\mathcal{A}) \leqslant 2$.
- $e_3(\mathcal{A}) = \beta_3(\mathcal{A})$, *and thus* $e_3(\mathcal{A})$ *is combinatorially determined.*

Corollary

Suppose all flats $X \in L_2(\mathcal{A})$ have multiplicity 2 or 3. Then $\Delta(t)$, and thus $b_1(F(\mathcal{A}))$, are combinatorially determined.

Theorem (PS)

Suppose $\mathcal{A}$ supports a 4-net and $\beta_2(\mathcal{A}) \leqslant 2$. Then

$$e_2(\mathcal{A}) = e_4(\mathcal{A}) = \beta_2(\mathcal{A}) = 2.$$

Conjecture (PS)

The characteristic polynomial of the degree 1 algebraic monodromy for the Milnor fibration of an arrangement $\mathcal{A}$ of rank at least 3 is given by the combinatorial formula

$$\Delta_{\mathcal{A}}(t) = (t-1)^{|\mathcal{A}|-1}((t+1)(t^2+1))^{\beta_2(\mathcal{A})}(t^2+t+1)^{\beta_3(\mathcal{A})}.$$

- The conjecture has been verified for
  - All sub-arrangements of non-exceptional Coxeter arrangements (Măcinic, Papadima).
  - All complex reflection arrangements (Măcinic, Papadima, Popescu, Dimca, Sticlaru).
  - Certain types of complexified real arrangements (Yoshinaga, Bailet, Torielli, Settepanella).
- A counterexample has been announced by Yoshinaga (2019): there is an arrangement of 16 planes in $\mathbb{C}^3$ with $e_2 = 0$ but $\beta_2 = 1$.

The boundary manifold

- Let $\mathcal{A}$ be a (central) arrangement of hyperplanes in $\mathbb{C}^\ell$.
- Let N be a (closed) regular neighborhood of the hypersurface $V(\mathcal{A}) = \bigcup_{H\in\mathcal{A}} \mathbb{P}(H)$ inside $\mathbb{CP}^{\ell-1}$.
- Let $\overline{U}(\mathcal{A}) = \mathbb{CP}^{\ell-1} \setminus \operatorname{int}(N)$. Clearly, $\overline{U} \simeq U$.
- The *boundary manifold* of $\mathcal{A}$ is $\partial\overline{U} = \partial N$. This is a compact, orientable, smooth manifold of dimension $2\ell-3$.

Example

- Let $\mathcal{A}$ be a pencil of n hyperplanes in $\mathbb{C}^\ell$, defined by $Q = z_1^n - z_2^n$. If $n = 1$, then $\partial\overline{U} = S^{2\ell-3}$. If $n > 1$, then $\partial\overline{U} = \sharp^{n-1} S^1 \times S^{2(\ell-2)}$.
- Let $\mathcal{A}$ be a near-pencil of n planes in $\mathbb{C}^3$, defined by $Q = z_1(z_2^{n-1} - z_3^{n-1})$. Then $\partial\overline{U} = S^1 \times \Sigma_{n-2}$.

- When $\ell = 3$, the boundary manifold $\partial \overline{U}$ is a 3-dimensional graph-manifold M_Γ, where
  - Γ is the incidence graph of $\mathcal{A}$, with $V(\Gamma) = L_1(\mathcal{A}) \cup L_2(\mathcal{A})$ and $E(\Gamma) = \{(L, P) \mid P \in L\}$.
  - Vertex manifolds $M_v = S^1 \times \left(S^2 \setminus \bigcup_{\{v,w\} \in E(\Gamma)} D^2_{v,w}\right)$ are glued along edge manifolds $M_e = S^1 \times S^1$ via flip maps.
- $b_1(M_\Gamma) = |\mathcal{A}| + b_1(\Gamma) - 1$.

THEOREM (JIANG–YAU 1993)

$U(\mathcal{A}) \cong U(\mathcal{A}') \;\Rightarrow\; M_\Gamma \cong M_{\Gamma'} \;\Rightarrow\; \Gamma \cong \Gamma' \;\Rightarrow\; L(\mathcal{A}) \cong L(\mathcal{A}')$.

THEOREM (COHEN–S. 2008)

$$\mathcal{V}^1_1(M_\Gamma) = \bigcup_{v \in V(\Gamma)\,:\, \deg(v) \geqslant 3} \left\{ t \in (\mathbb{C}^*)^{b_1(M_\Gamma)} \mid \prod_{i \in v} t_i = 1 \right\}.$$

THE RFRp PROPERTY

DEFINITION (AGOL, KOBERDA–S.)

A finitely generated group G is *residually finite rationally* p for some prime p if there is a sequence of subgroups,

$$G = G_0 > \cdots > G_i > G_{i+1} > \cdots$$

such that $\bigcap_{i \geqslant 0} G_i = \{1\}$, and, for each i,

- $G_{i+1} \lhd G_i$;
- G_i/G_{i+1} is an elementary abelian p-group;
- $\ker(G_i \to H_1(G_i, \mathbb{Q}))$ is a subgroup of G_{i+1}.

- G RFRp $\Rightarrow$ residually p $\Rightarrow$ residually finite & residually nilpotent.
- G RFRp $\Rightarrow$ torsion-free.
- G finitely presented & RFRp $\Rightarrow$ has solvable word problem.
- The class of RFRp groups is closed under taking subgroups, finite direct products, and finite free products.

- Finitely generated free groups F_n, surface groups $\pi_1(\Sigma_g)$, and right-angled Artin groups A_Γ are RFRp, for all p.
- Finite groups and non-abelian nilpotent groups are *not* RFRp, for any p.

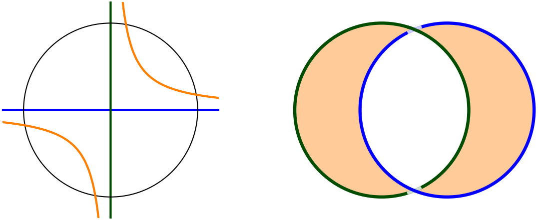
THEOREM (KOBERDA–S. 2016)

If G is a finitely presented group which is RFRp for infinitely many primes p, then either G is abelian or G is large (i.e., it virtually surjects onto a non-abelian free group).

THEOREM (KS)

Let M_Γ be the boundary manifold of a line arrangement in $\mathbb{C}^2$. Then $\pi_1(M_\Gamma)$ is RFRp, for all primes p.

THE BOUNDARY OF THE MILNOR FIBER



- For an arrangement $\mathcal{A}$ in $\mathbb{C}^\ell$, let $\overline{F}(\mathcal{A}) = F(\mathcal{A}) \cap D^{2\ell}$ be the *closed Milnor fiber* of $\mathcal{A}$. Clearly, $F \simeq \overline{F}$.
- The *boundary of the Milnor fiber* of $\mathcal{A}$ is the compact, smooth, orientable, $(2\ell - 3)$-manifold $\partial\overline{F} = F \cap S^{2\ell-1}$.
- The pair $(\overline{F}, \partial\overline{F})$ is $(\ell - 2)$-connected. In particular, if $\ell \geqslant 3$, then $\partial\overline{F}$ is connected, and $\pi_1(\partial\overline{F}) \to \pi_1(\overline{F})$ is surjective.

EXAMPLE

- Let $\mathcal{B}_n$ be the Boolean arrangement in $\mathbb{C}^n$. Then $F = (\mathbb{C}^*)^{n-1}$. Hence, $\overline{F} = T^{n-1} \times D^{n-1}$ and so $\partial\overline{F} = T^{n-1} \times S^{n-2}$.
- Let $\mathcal{A}$ be a near-pencil of n planes in $\mathbb{C}^3$. Then $\partial\overline{F} = S^1 \times \Sigma_{n-2}$.

Set $n = |\mathcal{A}|$. The Hopf fibration $\pi\colon \mathbb{C}^\ell\backslash\{0\} \to \mathbb{CP}^{\ell-1}$ restricts to regular, cyclic n-fold covers, $\pi\colon \overline{F} \to \overline{U}$ and $\pi\colon \partial\overline{F} \to \partial\overline{U}$, which fit into

$$
\begin{array}{ccccccccc}
\mathbb{Z}_n & = & \mathbb{Z}_n & = & \mathbb{Z}_n & \longrightarrow & \mathbb{C}^* & = & \mathbb{C}^* \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\partial\overline{F} & \longrightarrow & \overline{F} & \xrightarrow{\simeq} & F & \longrightarrow & M & \longrightarrow & \mathbb{C}^\ell\backslash\{0\} \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\
\partial\overline{U} & \longrightarrow & \overline{U} & \xrightarrow{\simeq} & U & = & U & \longrightarrow & \mathbb{CP}^{\ell-1}
\end{array}
$$

Assume now that $\ell = 3$. The fundamental group of $\partial\overline{U} = M_\Gamma$ has generators $\{\overline{x}_H \mid H \in \mathcal{A}\}$ and $\{y_c \mid c \text{ a cycle in } \Gamma\}$.

PROPOSITION (S. 2014)

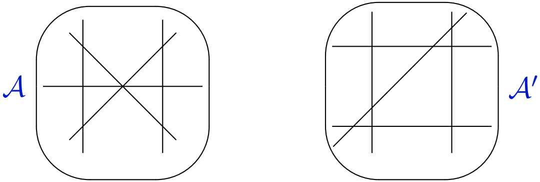
The $\mathbb{Z}_n$-cover $\pi\colon \partial\overline{F} \to \partial\overline{U}$ is classified by the homomorphism $\pi_1(\partial\overline{U}) \twoheadrightarrow \mathbb{Z}_n$ given by $x_H \mapsto 1$ and $y_c \mapsto 0$.

THEOREM (NÉMETHI–SZILARD 2012)

The characteristic polynomial of $h_*\colon H_1(\partial\overline{F}, \mathbb{C}) \circlearrowleft$ is given by

$$\delta(t) = \prod_{P \in L_2(\mathcal{A})} (t-1)(t^{\gcd(n_P, n)} - 1)^{n_P - 2}.$$

A pair of arrangements



- Let $\mathcal{A}$ and $\mathcal{A}'$ be the above pair of arrangements. Both have 2 triple points and 9 double points, yet $L(\mathcal{A}) \not\cong L(\mathcal{A}')$.
- Nevertheless, $U(\mathcal{A}) \simeq U(\mathcal{A}')$.
- Since $L(\mathcal{A}) \not\cong L(\mathcal{A}')$, the corresponding boundary manifolds, $\partial\overline{U}$ and $\partial\overline{U}'$, are not homotopy equivalent.
- In fact, $\mathcal{V}_1^1(\partial\overline{U})$ consists of 7 codimension-1 subtori in $(\mathbb{C}^*)^{13}$, while $\mathcal{V}_1^1(\partial\overline{U}')$ consists of 8 such subtori.

- The corresponding Milnor fibers, F and F', have the same characteristic polynomial of the algebraic monodromy,

$$\Delta = \Delta' = (t-1)^5.$$

- Likewise for the boundaries of the Milnor fibers,

$$\delta = \delta' = (t-1)^{13}(t^2+t+1)^2.$$

- The varieties $\mathcal{V}_1^1(F)$ and $\mathcal{V}_1^1(F')$ consist of two 2-dimensional subtori of $(\mathbb{C}^*)^5$. On the other hand, $\mathcal{V}_2^1(F) \not\cong \mathcal{V}_2^1(F')$.
- Thus, $\pi_1(F) \not\cong \pi_1(F')$.

CONJECTURE

Let $\mathcal{A}$ and $\mathcal{A}'$ be two central arrangements in $\mathbb{C}^3$. Then

$$F(\mathcal{A}) \cong F(\mathcal{A}') \Rightarrow L(\mathcal{A}) \cong L(\mathcal{A}').$$

References

- G. Denham, A. Suciu, *Local systems on arrangements of smooth, complex algebraic hypersurfaces*, Forum Math. Sigma **6** (2018), e6, 20 pp.
- G. Denham, A. Suciu, S. Yuzvinsky, *Combinatorial covers and vanishing of cohomology*, Selecta Math. **22** (2016), no. 2, 561–594.
- G. Denham, A. Suciu, S. Yuzvinsky, *Abelian duality and propagation of resonance*, Selecta Math. **23** (2017), no. 4, 2331–2367.
- T. Koberda, A. Suciu, *Residually finite rationally p groups*, Commun. Contemp. Math. (2019).
- S. Papadima, A. Suciu, *The Milnor fibration of a hyperplane arrangement: from modular resonance to algebraic monodromy*, Proc. London Math. Soc. **114** (2017), no. 6, 961–1004.
- A. Suciu, *Hyperplane arrangements and Milnor fibrations*, Ann. Fac. Sci. Toulouse Math. **23** (2014), no. 2, 417–481.
- A. Suciu, *On the topology of Milnor fibrations of hyperplane arrangements*, Rev. Roumaine Math. Pures Appl. **62** (2017), no. 1, 191–215.