Hyperplane arrangements

at the crossroads of topology and combinatorics

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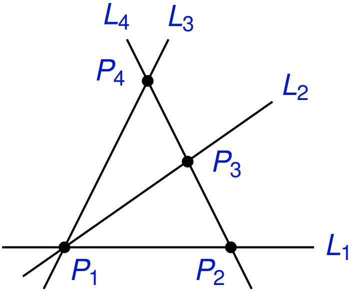
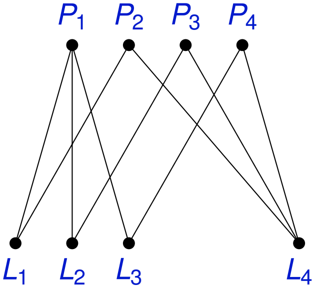
HYPERPLANE ARRANGEMENTS

- An *arrangement of hyperplanes* is a finite collection $\mathcal{A}$ of codimension 1 linear subspaces in $\mathbb{C}^{\ell}$.
- *Intersection lattice* $L(\mathcal{A})$: poset of all intersections of $\mathcal{A}$, ordered by reverse inclusion, and ranked by codimension.
- *Complement*: $M(\mathcal{A}) = \mathbb{C}^{\ell} \setminus \bigcup_{H\in\mathcal{A}} H$.





Example (The Boolean arrangement)

- $\mathcal{B}_n$: all coordinate hyperplanes $z_i = 0$ in $\mathbb{C}^n$.
- $L(\mathcal{B}_n)$: Boolean lattice of subsets of $\{0, 1\}^n$.
- $M(\mathcal{B}_n)$: complex algebraic torus $(\mathbb{C}^*)^n$.

Example (The braid arrangement)

- $\mathcal{A}_n$: all diagonal hyperplanes $z_i - z_j = 0$ in $\mathbb{C}^n$.
- $L(\mathcal{A}_n)$: lattice of partitions of $[n] := \{1, \dots, n\}$, ordered by refinement.
- $M(\mathcal{A}_n)$: configuration space of n ordered points in $\mathbb{C}$ (a classifying space for P_n, the pure braid group on n strings).

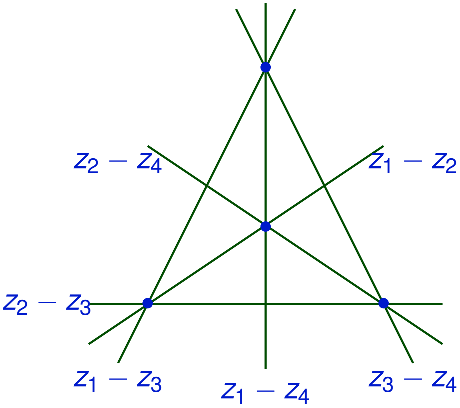


FIGURE : A planar slice of the braid arrangement $\mathcal{A}_4$

- We may assume that $\mathcal{A}$ is essential, i.e., $\bigcap_{H\in\mathcal{A}} H = \{0\}$.
- Fix an ordering $\mathcal{A} = \{H_1, \dots, H_n\}$, and choose linear forms $f_i\colon \mathbb{C}^\ell \to \mathbb{C}$ with $\ker(f_i) = H_i$. Define an injective linear map

$$\iota\colon \mathbb{C}^\ell \to \mathbb{C}^n, \quad z \mapsto (f_1(z), \dots, f_n(z)).$$

- This map restricts to an inclusion $\iota\colon M(\mathcal{A}) \hookrightarrow M(\mathcal{B}_n)$. Hence, $M(\mathcal{A}) = \iota(\mathbb{C}^\ell) \cap (\mathbb{C}^*)^n$ is a Stein manifold.
- Therefore, $M = M(\mathcal{A})$ has the homotopy type of a connected, finite cell complex of dimension ℓ.
- In fact, M has a minimal cell structure (Dimca–Papadima, Randell, Salvetti, Adiprasito,...). Consequently, $H_*(M, \mathbb{Z})$ is torsion-free.

COHOMOLOGY RING

- The Betti numbers $b_q(M) := \operatorname{rank} H_q(M, \mathbb{Z})$ are given by
$$\sum_{q=0}^{\ell} b_q(M) t^q = \sum_{X \in L(\mathcal{A})} \mu(X)(-t)^{\operatorname{rank}(X)},$$
where $\mu\colon L(\mathcal{A}) \to \mathbb{Z}$ is the Möbius function, defined recursively by $\mu(\mathbb{C}^\ell) = 1$ and $\mu(X) = -\sum_{Y \supsetneq X} \mu(Y)$.
- Let $E = \bigwedge(\mathcal{A})$ be the $\mathbb{Z}$-exterior algebra on degree 1 classes e_H dual to the meridians around the hyperplanes $H \in \mathcal{A}$.
- Let $\partial\colon E^\bullet \to E^{\bullet-1}$ be the differential given by $\partial(e_H) = 1$, and set $e_\mathcal{B} = \prod_{H \in \mathcal{B}} e_H$ for each $\mathcal{B} \subset \mathcal{A}$.
- The cohomology ring $H^*(M(\mathcal{A}), \mathbb{Z})$ is isomorphic to the Orlik–Solomon algebra $A(\mathcal{A}) = E/I$, where
$$I = \text{ideal}\left\langle \partial e_\mathcal{B} \,\middle|\, \operatorname{codim} \bigcap_{H \in \mathcal{B}} H < |\mathcal{B}| \right\rangle.$$

FUNDAMENTAL GROUP

- Given a generic projection of a generic slice of $\mathcal{A}$ in $\mathbb{C}^2$, the fundamental group $\pi = \pi_1(M(\mathcal{A}))$ can be computed from the resulting braid monodromy $\alpha = (\alpha_1, \dots, \alpha_s)$, where $\alpha_r \in P_n$.
- π has a (minimal) finite presentation with
  - Meridional generators $x_1, \dots, x_n$, where $n = |\mathcal{A}|$.
  - Commutator relators $x_i \alpha_j (x_i)^{-1}$, where each α_j acts on F_n via the Artin representation.
- Let $\pi/\gamma_k(\pi)$ be the $(k-1)^{\text{th}}$ nilpotent quotient of π. Then:
  - $\pi_{\text{ab}} = \pi/\gamma_2$ equals $\mathbb{Z}^n$.
  - π/γ_3 is determined by $A^{\leqslant 2}(\mathcal{A})$, and thus by $L_{\leqslant 2}(\mathcal{A})$.
  - π/γ_4 (and thus, π) is *not* determined by $L(\mathcal{A})$. (Rybnikov).

CHARACTERISTIC VARIETIES

- Let X be a connected, finite cell complex, and let $\pi = \pi_1(X, x_0)$.
- Let $\Bbbk$ be an algebraically closed field, and let $\operatorname{Hom}(\pi, \Bbbk^*)$ be the affine algebraic group of $\Bbbk$-valued, multiplicative characters on π.
- The *characteristic varieties* of X are the jump loci for homology with coefficients in rank-1 local systems on X:

  $$\mathcal{V}^q_s(X, \Bbbk) = \{\rho \in \operatorname{Hom}(\pi, \Bbbk^*) \mid \dim_{\Bbbk} H_q(X, \Bbbk_\rho) \geqslant s\}.$$

  Here, $\Bbbk_\rho$ is the local system defined by ρ, i.e, $\Bbbk$ viewed as a $\Bbbk\pi$-module, via $g \cdot x = \rho(g)x$, and $H_i(X, \Bbbk_\rho) = H_i(C_*(\widetilde{X}, \Bbbk) \otimes_{\Bbbk\pi} \Bbbk_\rho)$.
- These loci are Zariski closed subsets of the character group.
- The sets $\mathcal{V}^1_s(X, \Bbbk)$ depend only on π/π''.

EXAMPLE (CIRCLE)

We have $\widetilde{S^1} = \mathbb{R}$. Identify $\pi_1(S^1, *) = \mathbb{Z} = \langle t \rangle$ and $\Bbbk\mathbb{Z} = \Bbbk[t^{\pm 1}]$. Then:

$$C_*(\widetilde{S^1}, \Bbbk) : \ 0 \longrightarrow \Bbbk[t^{\pm 1}] \xrightarrow{t-1} \Bbbk[t^{\pm 1}] \longrightarrow 0 \ .$$

For $\rho \in \mathrm{Hom}(\mathbb{Z}, \Bbbk^*) = \Bbbk^*$, we get

$$C_*(\widetilde{S^1}, \Bbbk) \otimes_{\Bbbk\mathbb{Z}} \Bbbk_\rho : \ 0 \longrightarrow \Bbbk \xrightarrow{\rho-1} \Bbbk \longrightarrow 0 \ ,$$

which is exact, except for $\rho = 1$, when $H_0(S^1, \Bbbk) = H_1(S^1, \Bbbk) = \Bbbk$.
Hence: $\mathcal{V}^0_1(S^1, \Bbbk) = \mathcal{V}^1_1(S^1, \Bbbk) = \{1\}$ and $\mathcal{V}^i_s(S^1, \Bbbk) = \emptyset$, otherwise.

EXAMPLE (PUNCTURED COMPLEX LINE)

Identify $\pi_1(\mathbb{C}\backslash\{n \text{ points}\}) = F_n$, and $\widehat{F_n} = (\Bbbk^*)^n$. Then:

$$\mathcal{V}^1_s(\mathbb{C}\backslash\{n \text{ points}\}, \Bbbk) = \begin{cases} (\Bbbk^*)^n & \text{if } s < n, \\ \{1\} & \text{if } s = n, \\ \emptyset & \text{if } s > n. \end{cases}$$

RESONANCE VARIETIES

- Let $A = H^*(X, \Bbbk)$, where $\operatorname{char} \Bbbk \neq 2$. Then: $a \in A^1 \Rightarrow a^2 = 0$.
- We thus get a cochain complex

$$(A, \cdot a)\colon \ A^0 \xrightarrow{a} A^1 \xrightarrow{a} A^2 \longrightarrow \cdots .$$

- The *resonance varieties* of X are the jump loci for the cohomology of this complex

$$\mathcal{R}^q_s(X, \Bbbk) = \{a \in A^1 \mid \dim_{\Bbbk} H^q(A, \cdot a) \geqslant s\}$$

- E.g., $\mathcal{R}^1_1(X, \Bbbk) = \{a \in A^1 \mid \exists b \in A^1,\ b \neq \lambda a,\ ab = 0\}$.
- These loci are *homogeneous* subvarieties of $A^1 = H^1(X, \Bbbk)$.

EXAMPLE

- $\mathcal{R}^1_1(T^n, \Bbbk) = \{0\}$, for all $n > 0$.
- $\mathcal{R}^1_1(\mathbb{C} \backslash \{n \text{ points}\}, \Bbbk) = \Bbbk^n$, for all $n > 1$.

THE TANGENT CONE THEOREM

- Given a subvariety $W \subset (\mathbb{C}^*)^n$, let $\tau_1(W) = \{z \in \mathbb{C}^n \mid \exp(\lambda z) \in W, \ \forall \lambda \in \mathbb{C}\}$.
- (Dimca–Papadima–S. 2009) $\tau_1(W)$ is a finite union of rationally defined linear subspaces, and $\tau_1(W) \subseteq \mathrm{TC}_1(W)$.
- (Libgober 2002/DPS 2009)

$$\tau_1(\mathcal{V}^i_s(X)) \subseteq \mathrm{TC}_1(\mathcal{V}^i_s(X)) \subseteq \mathcal{R}^i_s(X).$$

- (DPS 2009/DP 2014): Suppose X is a k-formal space. Then, for each $i \leqslant k$ and $s > 0$,

$$\tau_1(\mathcal{V}^i_s(X)) = \mathrm{TC}_1(\mathcal{V}^i_s(X)) = \mathcal{R}^i_s(X).$$

- Consequently, $\mathcal{R}^i_s(X, \mathbb{C})$ is a union of rationally defined linear subspaces in $H^1(X, \mathbb{C})$.

JUMP LOCI OF ARRANGEMENTS

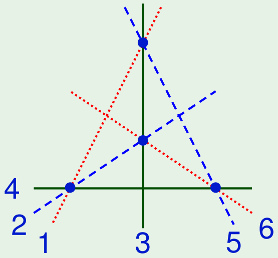
Work of Arapura, Falk, D.Cohen–A.S., Libgober, and Yuzvinsky, completely describes the varieties $\mathcal{R}_s(\mathcal{A}) := \mathcal{R}^1_s(M(\mathcal{A}), \mathbb{C})$:

- $\mathcal{R}_1(\mathcal{A})$ is a union of linear subspaces in $H^1(M(\mathcal{A}), \mathbb{C}) \cong \mathbb{C}^{|\mathcal{A}|}$.
- Each subspace has dimension at least 2, and each pair of subspaces meets transversely at 0.
- $\mathcal{R}_s(\mathcal{A})$ is the union of those linear subspaces that have dimension at least $s+1$.
- (Falk–Yuzvinsky 2007) Each k*-multinet* on a sub-arrangement $\mathcal{B} \subseteq \mathcal{A}$ gives rise to a component of $\mathcal{R}_1(\mathcal{A})$ of dimension $k-1$. Moreover, all components of $\mathcal{R}_1(\mathcal{A})$ arise in this way.

MULTINETS

- To compute $\mathcal{R}_1(\mathcal{A})$, we may assume $\mathcal{A}$ is an arrangement in $\mathbb{C}^3$. Its projectivization, $\bar{\mathcal{A}}$, is an arrangement of lines in $\mathbb{CP}^2$.
- $L_1(\mathcal{A}) \longleftrightarrow$ lines of $\bar{\mathcal{A}}$, $L_2(\mathcal{A}) \longleftrightarrow$ intersection points of $\bar{\mathcal{A}}$.
- A flat $X \in L_2(\mathcal{A})$ has multiplicity q if the point $\bar{X}$ has exactly q lines from $\bar{\mathcal{A}}$ passing through it.
- A (k, d)-*multinet* on $\mathcal{A}$ is a partition into $k \geqslant 3$ subsets, $\mathcal{A}_1, \dots, \mathcal{A}_k$, together with an assignment of multiplicities, $m\colon \mathcal{A} \to \mathbb{N}$, and a subset $\mathcal{X} \subseteq L_2(\mathcal{A})$, such that (basically):
  1. $\exists\ d \in \mathbb{N}$ such that $\sum_{H\in\mathcal{A}_\alpha} m_H = d$, for all $\alpha \in [k]$.
  2. If H and H' are in different classes, then $H \cap H' \in \mathcal{X}$.
  3. $\forall\ X \in \mathcal{X}$, the sum $n_X = \sum_{H\in\mathcal{A}_\alpha : H\supset X} m_H$ is independent of α.
- The multinet is *reduced* if $m_H = 1$, for all $H \in \mathcal{A}$.
- A *net* is a reduced multinet with $n_X = 1$, for all $X \in \mathcal{X}$.

EXAMPLE (BRAID ARRANGEMENT $\mathcal{A}_4$)



$\mathcal{R}_1(\mathcal{A}) \subset \mathbb{C}^6$ has 4 local components (from the triple points), and one essential component, from the above $(3,2)$-net:

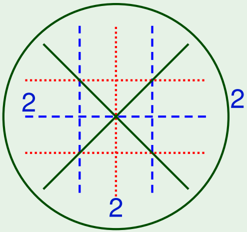
$$L_{124} = \{x_1 + x_2 + x_4 = x_3 = x_5 = x_6 = 0\},$$
$$L_{135} = \{x_1 + x_3 + x_5 = x_2 = x_4 = x_6 = 0\},$$
$$L_{236} = \{x_2 + x_3 + x_6 = x_1 = x_4 = x_5 = 0\},$$
$$L_{456} = \{x_4 + x_5 + x_6 = x_1 = x_2 = x_3 = 0\},$$
$$L = \{x_1 + x_2 + x_3 = x_1 - x_6 = x_2 - x_5 = x_3 - x_4 = 0\}.$$

- Let $\mathrm{Hom}(\pi_1(M(\mathcal{A})), \mathbb{C}^*) = (\mathbb{C}^*)^n$ be the character torus.
- The characteristic variety $\mathcal{V}_1(\mathcal{A}) := \mathcal{V}_1^1(M(\mathcal{A}), \mathbb{C})$ lies in the substorus $\{t \in (\mathbb{C}^*)^n \mid t_1 \cdots t_n = 1\}$.
- $\mathcal{V}_1(\mathcal{A})$ is a finite union of torsion-translates of algebraic subtori of $(\mathbb{C}^*)^n$.
- If a linear subspace $L \subset \mathbb{C}^n$ is a component of $\mathcal{R}_1(\mathcal{A})$, then the algebraic torus $T = \exp(L)$ is a component of $\mathcal{V}_1(\mathcal{A})$.
- All components of $\mathcal{V}_1(\mathcal{A})$ passing through the origin $1 \in (\mathbb{C}^*)^n$ arise in this way (and thus, are combinatorially determined).
- In general, though, there are translated subtori in $\mathcal{V}_1(\mathcal{A})$.

(Denham–S. 2014)

- Suppose there is a multinet $\mathcal{M}$ on $\mathcal{A}$, and there is a hyperplane H for which $m_H > 1$ and $m_H \mid n_X$ for each $X \in \mathcal{X}$ such that $X \subset H$.
- Then $\mathcal{V}_1(\mathcal{A} \setminus \{H\})$ has a component which is a 1-dimensional subtorus, translated by a character of order m_H.

EXAMPLE (THE DELETED B_3 ARRANGEMENT)



The B_3 arrangement supports a $(3, 4)$-multinet; $\mathcal{X}$ consists of 4 triple points ($n_X = 1$) and 3 quadruple points ($n_X = 2$). So pick H with $m_H = 2$ to get a translated torus in $\mathcal{V}_1(B_3 \setminus \{H\})$.

THE MILNOR FIBRATION(S) OF AN ARRANGEMENT

- Let $\mathcal{A}$ be a (central) hyperplane arrangement in $\mathbb{C}^\ell$.
- For each $H \in \mathcal{A}$, let $f_H \colon \mathbb{C}^\ell \to \mathbb{C}$ be a linear form with kernel H.
- For each choice of multiplicities $m = (m_H)_{H\in\mathcal{A}}$ with $m_H \in \mathbb{N}$, let

$$Q_m := Q_m(\mathcal{A}) = \prod_{H\in\mathcal{A}} f_H^{m_H},$$

  a homogeneous polynomial of degree $N = \sum_{H\in\mathcal{A}} m_H$.
- The map $Q_m \colon \mathbb{C}^\ell \to \mathbb{C}$ restricts to a map $Q_m \colon M(\mathcal{A}) \to \mathbb{C}^*$.
- This is the projection of a smooth, locally trivial bundle, known as the *Milnor fibration* of the multi-arrangement $(\mathcal{A}, m)$,

$$F_m(\mathcal{A}) \longrightarrow M(\mathcal{A}) \xrightarrow{Q_m} \mathbb{C}^*.$$

- The typical fiber, $F_m(\mathcal{A}) = Q_m^{-1}(1)$, is called the *Milnor fiber* of the multi-arrangement.
- $F_m(\mathcal{A})$ is a Stein manifold. It has the homotopy type of a finite cell complex, with $\gcd(m)$ connected components, of dim $\ell - 1$.
- The *(geometric) monodromy* is the diffeomorphism

$$h\colon F_m(\mathcal{A}) \to F_m(\mathcal{A}), \quad z \mapsto e^{2\pi \mathrm{i}/N} z.$$

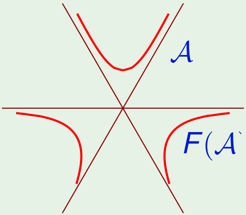
- If all $m_H = 1$, the polynomial $Q = Q(\mathcal{A})$ is the usual defining polynomial, and $F(\mathcal{A})$ is the usual Milnor fiber of $\mathcal{A}$.

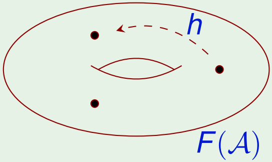
EXAMPLE

Let $\mathcal{A}$ be the single hyperplane $\{0\}$ inside $\mathbb{C}$. Then $M(\mathcal{A}) = \mathbb{C}^*$, $Q_m(\mathcal{A}) = z^m$, and $F_m(\mathcal{A}) = m$-roots of 1.

EXAMPLE

Let $\mathcal{A}$ be a pencil of 3 lines through the origin of $\mathbb{C}^2$. Then $F(\mathcal{A})$ is a thrice-punctured torus, and h is an automorphism of order 3:





More generally, if $\mathcal{A}$ is a pencil of n lines in $\mathbb{C}^2$, then $F(\mathcal{A})$ is a Riemann surface of genus $\binom{n-1}{2}$, with n punctures.

- Let $\mathcal{B}_n$ be the Boolean arrangement, with $Q_m(\mathcal{B}_n) = z_1^{m_1} \cdots z_n^{m_n}$. Then $M(\mathcal{B}_n) = (\mathbb{C}^*)^n$ and

$$F_m(\mathcal{B}_n) = \ker(\mathbb{Q}_m) \cong (\mathbb{C}^*)^{n-1} \times \mathbb{Z}_{\gcd(m)}$$

- Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an essential arrangement. The inclusion $\iota\colon M(\mathcal{A}) \to M(\mathcal{B}_n)$ restricts to a bundle map

$$\begin{array}{ccccc}
F_m(\mathcal{A}) & \longrightarrow & M(\mathcal{A}) & \xrightarrow{Q_m(\mathcal{A})} & \mathbb{C}^* \\
\downarrow & & \downarrow \iota & & \| \\
F_m(\mathcal{B}_n) & \longrightarrow & M(\mathcal{B}_n) & \xrightarrow{Q_m(\mathcal{B}_n)} & \mathbb{C}^*
\end{array}$$

- Thus,

$$F_m(\mathcal{A}) = M(\mathcal{A}) \cap F_m(\mathcal{B}_n)$$

THE HOMOLOGY OF THE MILNOR FIBER

- Let $(\mathcal{A}, m)$ be a multi-arrangement with $\gcd(m) = 1$. Set $N = \sum_{H \in \mathcal{A}} m_H$.
- The Milnor fiber $F_m(\mathcal{A})$ is a regular $\mathbb{Z}_N$-cover of the projectivized complement, $U(\mathcal{A}) = \mathbb{P}(M(\mathcal{A}))$, defined by the homomorphism

$$\delta_m \colon \pi_1(U(\mathcal{A})) \twoheadrightarrow \mathbb{Z}_N, \quad x_H \mapsto m_H \bmod N$$

- Let $\widehat{\delta_m} \colon \operatorname{Hom}(\mathbb{Z}_N, \Bbbk^*) \to \operatorname{Hom}(\pi_1(U(\mathcal{A})), \Bbbk^*)$ be the induced map between character groups.
- If $\operatorname{char}(\Bbbk) \nmid N$, the dimension of $H_q(F_m(\mathcal{A}), \Bbbk)$ may be computed by summing up the number of intersection points of $\operatorname{im}(\widehat{\delta_m})$ with the varieties $\mathcal{V}^q_s(U(\mathcal{A}), \Bbbk)$, for all $s \geqslant 1$.

- We now consider the simplest non-trivial case: that of an arrangement $\mathcal{A}$ of n planes in $\mathbb{C}^3$, and its Milnor fiber, $F(\mathcal{A})$.
- Let $\Delta_{\mathcal{A}}(t) = \det(t \cdot \mathrm{id} - h_*)$ be the characteristic polynomial of the algebraic monodromy, $h_* \colon H_1(F(\mathcal{A}), \mathbb{C}) \to H_1(F(\mathcal{A}), \mathbb{C})$.
- Since $h_*^n = \mathrm{id}$, we may write

$$\Delta_{\mathcal{A}}(t) = \prod_{d|n} \Phi_d(t)^{e_d(\mathcal{A})}, \qquad (\star)$$

where $\Phi_d(t)$ is the d-th cyclotomic polynomial, and $e_d(\mathcal{A}) \in \mathbb{Z}_{\geqslant 0}$.

PROBLEM

- *Is the polynomial $\Delta_{\mathcal{A}}$ (or, equivalently, the exponents $e_d(\mathcal{A})$) determined by the intersection lattice $L(\mathcal{A})$?*
- *In particular, is the first Betti number $b_1(F(\mathcal{A})) = \deg(\Delta_{\mathcal{A}})$ combinatorially determined?*

- By a transfer argument, $e_1(\mathcal{A}) = n - 1$.
- Not all divisors of n appear in ($\star$). E.g., if d does not divide at least one of the multiplicities of the intersection points, then $e_d(\mathcal{A}) = 0$.
- In particular, if $\mathcal{A}$ has only points of multiplicity 2 and 3, then $\Delta_{\mathcal{A}}(t) = (t-1)^{m-1}(t^2+t+1)^{e_3}$.
- If multiplicity 4 appears, then also get factor of $(t+1)^{e_2} \cdot (t^2+1)^{e_4}$.

EXAMPLE

Let $\mathcal{A} = \mathcal{A}_4$ be the braid arrangement. Then $\mathcal{V}_1(\mathcal{A})$ has a single 'essential' component,

$$T = \{t \in (\mathbb{C}^*)^6 \mid t_1 t_2 t_3 = t_1 t_6^{-1} = t_2 t_5^{-1} = t_3 t_4^{-1} = 1\}.$$

Clearly, $\delta^2 \in T$, yet $\delta \notin T$. Hence, $\Delta_{\mathcal{A}}(t) = (t-1)^5(t^2+t+1)$.

MODULAR INEQUALITIES

- Let $\sigma = \sum_{H\in\mathcal{A}} e_H \in A^1$ be the “diagonal" vector.
- Assume $\Bbbk$ has characteristic $p > 0$, and define

$$\beta_p(\mathcal{A}) = \dim_{\Bbbk} H^1(A, \cdot\sigma).$$

That is, $\beta_p(\mathcal{A}) = \max\{s \mid \sigma \in \mathcal{R}^1_s(A, \Bbbk)\}$.

THEOREM (COHEN–ORLIK 2000, PAPADIMA–S. 2010)

$e_{p^s}(\mathcal{A}) \leqslant \beta_p(\mathcal{A})$, *for all* $s \geqslant 1$.

THEOREM

1. *Suppose* $\mathcal{A}$ *admits a* k*-net. Then* $\beta_p(\mathcal{A}) = 0$ *if* $p \nmid k$ *and* $\beta_p(\mathcal{A}) \geqslant k-2$, *otherwise.*
2. *If* $\mathcal{A}$ *admits a reduced* k*-multinet, then* $e_k(\mathcal{A}) \geqslant k-2$.

THEOREM (PAPADIMA–S. 2014)

Suppose $\mathcal{A}$ has no points of multiplicity $3r$ with $r > 1$. Then $\mathcal{A}$ admits a reduced 3-multinet iff $\mathcal{A}$ admits a 3-net iff $\beta_3(\mathcal{A}) \neq 0$. Moreover,

- $\beta_3(\mathcal{A}) \leqslant 2$.
- $e_3(\mathcal{A}) = \beta_3(\mathcal{A})$, *and thus* $e_3(\mathcal{A})$ *is combinatorially determined.*

COROLLARY (PS)

Suppose all flats $X \in L_2(\mathcal{A})$ have multiplicity 2 or 3. Then $\Delta(t)$, and thus $b_1(F(\mathcal{A}))$, are combinatorially determined.

THEOREM (PS)

Suppose $\mathcal{A}$ supports a 4-net and $\beta_2(\mathcal{A}) \leqslant 2$. Then

$$e_2(\mathcal{A}) = e_4(\mathcal{A}) = \beta_2(\mathcal{A}) = 2.$$

CONJECTURE (PS)

Let $\mathcal{A}$ be an arrangement which is not a pencil. Then $e_{p^s}(\mathcal{A}) = 0$ for all primes p and integers $s \geqslant 1$, with two possible exceptions:

$$e_2(\mathcal{A}) = e_4(\mathcal{A}) = \beta_2(\mathcal{A}) \text{ and } e_3(\mathcal{A}) = \beta_3(\mathcal{A}).$$

If $e_d(\mathcal{A}) = 0$ for all divisors d of $|\mathcal{A}|$ which are not prime powers, this conjecture would give:

$$\Delta_{\mathcal{A}}(t) = (t-1)^{|\mathcal{A}|-1}((t+1)(t^2+1))^{\beta_2(\mathcal{A})}(t^2+t+1)^{\beta_3(\mathcal{A})}.$$

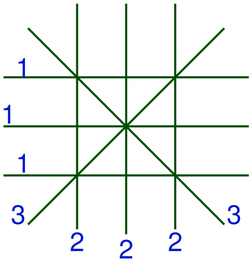
The conjecture has been verified for several classes of arrangements:

- Complex reflection arrangements (Măcinic–Papadima–Popescu).
- Certain types of real arrangements (Yoshinaga, Bailet, Torielli).

TORSION IN HOMOLOGY

THEOREM (COHEN–DENHAM–S. 2003)

For every prime $p \geqslant 2$, there is a multi-arrangement $(\mathcal{A}, m)$ such that $H_1(F_m(\mathcal{A}), \mathbb{Z})$ has non-zero p-torsion.



Simplest example: the arrangement of 8 hyperplanes in $\mathbb{C}^3$ with

$$Q_m(\mathcal{A}) = x^2y(x^2 - y^2)^3(x^2 - z^2)^2(y^2 - z^2)$$

Then $H_1(F_m(\mathcal{A}), \mathbb{Z}) = \mathbb{Z}^7 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

We now can generalize and reinterpret these examples, as follows.

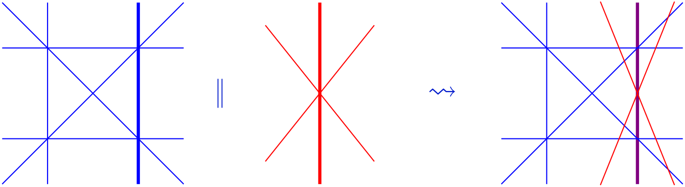
A *pointed multinet* on an arrangement $\mathcal{A}$ is a multinet structure, together with a distinguished hyperplane $H \in \mathcal{A}$ for which $m_H > 1$ and $m_H \mid n_X$ for each $X \in \mathcal{X}$ such that $X \subset H$.

THEOREM (DENHAM–S. 2014)

Suppose $\mathcal{A}$ admits a pointed multinet, with distinguished hyperplane H and multiplicity m. Let p be a prime dividing m_H. There is then a choice of multiplicities m' on the deletion $\mathcal{A}' = \mathcal{A} \setminus \{H\}$ such that $H_1(F_{m'}(\mathcal{A}'), \mathbb{Z})$ has non-zero p-torsion.

This torsion is explained by the fact that the geometry of $\mathcal{V}^1_1(M(\mathcal{A}'), \Bbbk)$ varies with $\operatorname{char}(\Bbbk)$.

To produce p-torsion in the homology of the usual Milnor fiber, we use a "polarization" construction:



$(\mathcal{A}, m) \rightsquigarrow \mathcal{A}\|m$, an arrangement of $N = \sum_{H\in\mathcal{A}} m_H$ hyperplanes, of rank equal to $\operatorname{rank}\mathcal{A} + |\{H \in \mathcal{A} \colon m_H \geqslant 2\}|$.

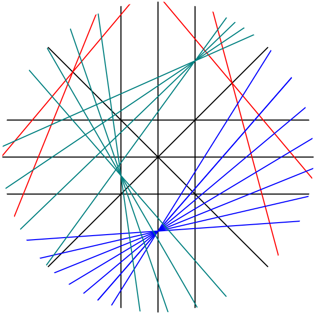
THEOREM (DS)

Suppose $\mathcal{A}$ admits a pointed multinet, with distinguished hyperplane H and multiplicity m. Let p be a prime dividing m_H.

There is then a choice of multiplicities m' on the deletion $\mathcal{A}' = \mathcal{A}\backslash\{H\}$ such that $H_q(F(\mathcal{B}), \mathbb{Z})$ has p-torsion, where $\mathcal{B} = \mathcal{A}'\|m'$ and $q = 1 + \left|\left\{K \in \mathcal{A}' : m'_K \geqslant 3\right\}\right|$.

COROLLARY (DS)

For every prime $p \geqslant 2$, there is an arrangement $\mathcal{A}$ such that $H_q(F(\mathcal{A}), \mathbb{Z})$ has non-zero p-torsion, for some $q > 1$.



Simplest example: the arrangement of 27 hyperplanes in $\mathbb{C}^8$ with

$$Q(\mathcal{A}) = xy(x^2-y^2)(x^2-z^2)(y^2-z^2)w_1w_2w_3w_4w_5(x^2-w_1^2)(x^2-2w_1^2)(x^2-3w_1^2)(x-4w_1)\cdot$$
$$((x-y)^2-w_2^2)((x+y)^2-w_3^2)((x-z)^2-w_4^2)((x-z)^2-2w_4^2)\cdot((x+z)^2-w_5^2)((x+z)^2-2w_5^2).$$

Then $H_6(F(\mathcal{A}), \mathbb{Z})$ has 2-torsion (of rank 108).

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