COUNTING HOMOMORPHISMS TO FINITE GROUPS: PRESENTED BY ALEXANDER I. SUCIU

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1. Motivation

The study of infinite groups is an exceedingly hard problem. Even if we restrict ourselves to the case of finitely generated groups, the problem of showing two groups to be the same is very nontrivial. One of the many tools at our disposal is studying the quotient groups of a give group. It can even occur that the same group can appear as a quotient in many different ways. The invariants we will discuss involve counting the number of homomorphisms and epimorphisms from a given finitely generated group to a given quotient group. I would like to than Alex Suciu for his help in making these notes.

2. Introduction

2.1. **Assumptions and Notation.** Though out this document we will make the assumption that all groups are finitely generated. Given a f.g. group G we let $|Hom(G, \Gamma)|$ denote the number of homomorphisms from G to Γ. This is easily seen to be an invariant by the functorial properties of Hom. We also set $\delta_\Gamma(G) = |Epi(G, \Gamma)|/|Aut(\Gamma)|$, the number of epimorphism from G to Γ divided by the order of the automorphism group of Γ. Our main goal is to find formulas for these invariants of G.

2.2. **Beginnings.** Our first lemma will aid us in our goal of explicitly determining these invariants.

Lemma 2.1.

$$|Hom(G, \Gamma)| = \sum_{H \leq \Gamma} |Epi(G, H)|.$$

The proof of this lemma is seen in that the image of a homomorphism is a subgroup of the codomain, and thus will be omitted.

We are now in a position to do a few examples.

Example 2.1. $G = \mathbb{Z}$ $\quad \Gamma = \mathbb{Z}_k$ *For this case we have*

$$|Hom(\mathbb{Z}, \mathbb{Z}_k)| = k$$

since the generator of $\mathbb{Z}$ *can be mapped to any of the elements. The number of epimorphisms is a different case. For a homomorphism to be an epimorphism we require that* 1 *get mapped to a generator. Thus the number of* *epimorphisms is the same as the number of generators of our target group so*

$$|Epi(\mathbb{Z}, \mathbb{Z}_k)| = \phi(k).$$

Example 2.2. *We now consider the case of the free group of rank* n, F_n *and* $\Gamma = \mathbb{Z}_p$ *(*p *- prime). To count the number of homomorphisms we simply need to say where each generator of* F_n *should be sent. Since* $\mathbb{Z}_p$ *is a group of order* p *we have* p *choices for each of* n *generators, whence*

$$|Hom(F_n, \mathbb{Z}_p)| = p^n.$$

To count the number of epimorphisms we note that $\mathbb{Z}_p$ *is a simple abelian group, hence has no nontrivial subgroups, and our lemma says*

$$|Hom(F_n, \mathbb{Z}_p)| = |Epi(F_n, \mathbb{Z}_p)| + |Epi(F_n, \{0\})|,$$

thus

$$|Epi(F_n, \mathbb{Z}_p)| = |Hom(F_n, \mathbb{Z}_p)| - |Epi(F_n, \{0\})| = p^n - 1.$$

This can be generalized to the case when Γ *is a finite group of order* $|\Gamma| = \gamma$. *We then have*

$$|Hom(F_n, \Gamma)| = \gamma^n$$

and

$$|Epi(F_n, \Gamma)| = \phi_\Gamma(n) = |\{(\alpha_1, \ldots, \alpha_n) | \langle \alpha_1, \ldots, \alpha_n \rangle = \Gamma\}|.$$

We finish this section with a second lemma.

Lemma 2.2. *If* $\Gamma = \Gamma_1 \times \Gamma_2$ *with* $(|\Gamma_1|, |\Gamma_2|) = 1$ *then*

$$|Hom(G, \Gamma)| = |Hom(G, \Gamma_1)| \cdot |Hom(G, \Gamma_2)|$$

and

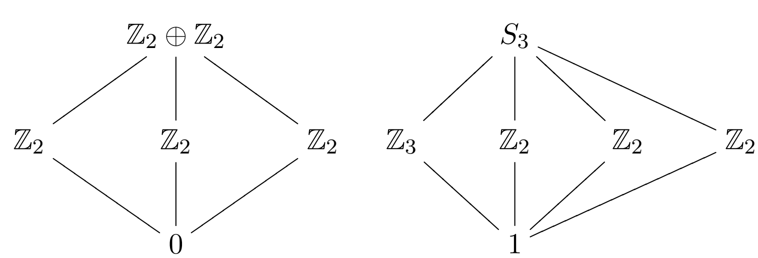
$$|Epi(G, \Gamma)| = |Epi(G, \Gamma_1)| \cdot |Epi(G, \Gamma_2)|.$$

Exercise 1. *Prove this lemma.*

3. Möbius Inversion

A classical idea from number theory is that if we are given a function defined as the series of another function can we invert the formula. Namely our goal in this section is to find a formula for $|Epi(G, \Gamma)|$ in terms of $|Hom(G, \Lambda)|$ where $\Lambda < \Gamma$. We begin by considering the lattice of subgroups of Γ ordered by inclusion.

Example 3.1.



Definition 3.1. *We define the* Möbius *function to be the function*

$$\mu\colon L(\Gamma) \to \mathbb{Z} \qquad \textit{with} \qquad \begin{cases} \mu(\Gamma) = 1 \\ \sum\limits_{K \leq \Lambda \leq \Gamma} \mu(\Lambda) = 0. \end{cases}$$

Exercise 2. *Show that for the group $\mathbb{Z}_n$ the möbius function for the group is exactly the möbius function from number theory,*

$$\mu(n) = \begin{cases} \mu(1) = 1 \\ \mu(n) = 0 & \textit{if } p^2 \mid n \textit{ for some prime } p \\ \mu(p_1 \cdot p_k) = (-1)^k & \textit{for } p_1, \dots, p_k \textit{ distinct primes.} \end{cases}$$

We now proceed to the main result due to Rota,

Theorem 3.1. *(Rota)*

$$|Epi(G, \Gamma)| = \sum_{\Lambda \leq \Gamma} \mu(\Lambda) |Hom(G, \Lambda)|.$$

4. Finite Index Subgroups

We begin by defining some invariants of a group.

Definition 4.1.

$$a_k(G) = \#\{H \leq G \mid [G : H] = k\}$$

These invariants were introduced and studied by Philip Hall. He proved the following recursion formula.

Theorem 4.1.

$$a_k(G) = \frac{|Hom(G, S_k)|}{(k-1)!} - \sum_{l=1}^{k-1} \frac{|Hom(G, S_{k-l})|}{(k-l)!} a_l(G),$$

where S_k is the symmetric group on k elements.

These invariants contain a good amount of information about a group but the following "Hall Invariants" contain more information and give us $a_k(G)$.

Definition 4.2.

$$\delta_\Gamma(G) = \frac{|Epi(G, \Gamma)|}{|Aut(\Gamma)|}.$$

We then find the following

$$\begin{aligned} a_1 &= 1 \\ a_2 &= \delta_{\mathbb{Z}_2} \\ a_3 &= \delta_{\mathbb{Z}_3} + 3\delta_{S_3} \\ a_4 &= -\binom{\delta_{\mathbb{Z}_2}}{2} + \delta_{\mathbb{Z}_4} + 4\delta_{\mathbb{Z}_2 \oplus \mathbb{Z}_2} + 4\delta_{D_8} + 4\delta_{A_4} + 4\delta_{S_4}. \end{aligned}$$

As an example to show that the invariants δ_Γ contain more information than a_k we consider the following.

Example 4.1. *Let M_g be the orientable surface of genus g, and N_g the non-orientable surface of genus g. We now consider the fundamental groups of these two classes of spaces*

$$\pi_1(M_g) = \langle x_1, y_1, \ldots, x_g, y_g \mid [x_1, y_1] \cdots [x_g, y_g] = 1 \rangle$$
$$\pi_1(N_g) = \langle x_1, \ldots, x_g \mid x_1^2 \cdots x_g^2 = 1 \rangle.$$

We then have

$$a_k(\pi_1(M_g)) = a_k(\pi_1(N_{2g})) \qquad \forall g \geq 1, k \geq 1$$

but

$$\delta_{\mathbb{Z}_3}(\pi_1(M_g)) = \frac{3^{2g}-1}{2} \qquad \delta_{\mathbb{Z}_3}(\pi_1(N_{2g})) = \frac{3^{2g-1}-1}{2}.$$

This is due to the $\mathbb{Z}_2$ factor in the abelianization of $\pi_1(N_g)$.

5. Cohomology

5.1. **Setup.** Let G be a group. We consider A as a $\mathbb{Z}G$ module, where $\mathbb{Z}G$ is the group ring of G. The action of G on A is defined by

$$g \cdot a = \alpha(g)(a)$$

where $\alpha \colon G \to Aut(A)$. We are now in a position to build a cochain complex on A

$$\begin{array}{ccccccccc} C^0 & \xrightarrow{\delta_0} & C^1 & \xrightarrow{\delta_1} & C^2 & \xrightarrow{\delta^2} & \cdots \longrightarrow & C^k & \xrightarrow{\delta^k} \cdots \\ \| & & \| & & \| & & & \| & \\ A & & Map(G, A) & & Map(G \times G, A) & & & Map(G^{\times k}, A) & \end{array}$$

The boundary maps are defined by

$$\delta^0(a)(x) = x \cdot a - a$$
$$\delta^1(f)(x, y) = x \cdot f(y) - f(xy) + f(x)$$
$$\vdots$$
$$\delta^k(f)(x_0, \ldots, x_k) = x_0 \cdot f(x_1, \ldots, x_n) - f(x_0 x_1, \ldots, x_n) + f(x_0, x_1 x_2, \ldots, x_n) + \cdots + (-1)^{k-1} f(x_0, \ldots, x_{k-1}).$$

Exercise 3. *Show that this is a chain complex, i.e.*

$$\delta^k \delta^{k-1} = 0.$$

Once you have a chain complex the most natural thing to consider is the homology groups,

$$H^k_\alpha(G, A) = Z^k_\alpha(G, A)/B^k_\alpha(G, A) = ker(\delta^k)/im(\delta^{k-1}).$$

We want to find a practical method of computing these cohomology groups, the next section we will see that the group H^1 can be computed.

5.2. **A Practical Approach to H^1.** Let G be a finitely presented group,

$$G = \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle,$$

alternatively we mean there is a short exact sequnce

$$1 \longrightarrow R \longrightarrow F_n \overset{\phi}{\twoheadrightarrow} G \longrightarrow 1.$$

We now need to introduce the free derivatives.

Definition 5.1. *The* Free *Derivatives on F_n are defined to be the following:*

$$\frac{\partial}{\partial x_i} \colon F_n \to \mathbb{Z}F_n$$

satisfying
1)

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij}$$

2)

$$\frac{\partial uv}{\partial x_j} = \frac{\partial u}{\partial x_j} + u \frac{\partial v}{\partial x_j}.$$

Now consider a free resolution of $\mathbb{Z}$ by $\mathbb{Z}G$ modules

$$\cdots \longrightarrow \mathbb{Z}G^m \xrightarrow{J_G} \mathbb{Z}G^n \xrightarrow{d_1} \mathbb{Z}G \longrightarrow \mathbb{Z} \longrightarrow 0.$$

We know two of the maps in this sequence,

$$d_1 = \begin{pmatrix} x_1 - 1 \\ x_2 - 1 \\ \vdots \\ x_n - 1 \end{pmatrix} \qquad J_G = \left(\frac{\partial r_i}{\partial x_j} \right)^{\phi}.$$

Now apply the functor $Hom_{\mathbb{Z}G}(-, A)$ to obtain the cochain complex

$$A \xrightarrow{d_1^\alpha} A^n \xrightarrow{J_G^\alpha} A^m \longrightarrow \cdots.$$

From this we find the first cohomology group

$$H^1_\alpha(G, A) = ker(J_G^\alpha)/im(d_1^\alpha).$$