Combinatorial covers, abelian duality, and propagation of resonance

Alex Suciu

Northeastern University

Joint work with Graham Denham and Sergey Yuzvinsky

Algebra, Topology and Combinatorics Seminar
University of Pisa
May 29, 2014

COHOMOLOGY JUMP LOCI

- Let $\Bbbk$ be an algebraically closed field.
- Let S be a commutative, finitely-generated $\Bbbk$-algebra.
- Let $\mathrm{Spec}(S) = \mathrm{Hom}_{\Bbbk\text{-alg}}(S, \Bbbk)$ be the maximal spectrum of S.
- Let

$$C:\ 0 \longrightarrow C^0 \longrightarrow \cdots \longrightarrow C^i \xrightarrow{d_i} C^{i+1} \longrightarrow \cdots \longrightarrow C^n \longrightarrow 0$$

  be a (bounded) cochain complex over S.
- The *cohomology jump loci* of C are defined as

$$\mathcal{V}^i(C) := \{\mathfrak{m} \in \mathrm{Spec}(S) \mid H^i(C \otimes_S S/\mathfrak{m}) \neq 0\}.$$

PROPAGATION

- The sets $V^i(C)$ depend only on the chain-homotopy equivalence class of C.
- Assume C is a cochain complex of free, finitely-generated S-modules. Then $\mathcal{V}^i(C)$ are Zariski closed subsets of $\operatorname{Spec}(S)$.
- We say the jump loci of C *propagate* if

$$\mathcal{V}^{i-1}(P) \subseteq \mathcal{V}^i(P) \qquad \text{for } 0 < i \leqslant n.$$

THE BGG CORRESPONDENCE

- Let V be a finite-dimensional $\Bbbk$-vector space.
- Fix basis $e_1, \dots, e_n$ for V, and dual basis $x_1, \dots, x_n$ for $V^\vee$.
- Let $E = \bigwedge V$ and $S = \operatorname{Sym} V^\vee$.
- Let P be a finitely-generated, graded E-module.
  - E.g., a graded, graded-commutative $\Bbbk$-algebra A ($\operatorname{char} \Bbbk \neq 2$).
- BGG yields a cochain complex of free, finitely-generated S-modules,

$$\mathbf{L}(P)\colon \quad \cdots \longrightarrow P^i \otimes_\Bbbk S \xrightarrow{d_i} P^{i+1} \otimes_\Bbbk S \longrightarrow \cdots,$$

with differentials $d_i(p \otimes s) = \sum_{j=1}^n e_j p \otimes x_j s$.

RESONANCE VARIETIES

- Evaluating $\mathbf{L}(P)$ at $a \in V$ gives the (Aomoto) cochain complex

$$(P, a) := \mathbf{L}(P) \otimes_S S/\mathfrak{m}_a \colon \quad \cdots \longrightarrow P^i \xrightarrow{\cdot a} P^{i+1} \longrightarrow \cdots$$

- The *resonance varieties* of P are the cohomology jump loci of $\mathbf{L}(P)$:

$$\mathcal{R}^i(P) := \mathcal{V}^i(\mathbf{L}(P)) = \{a \in V \mid H^i(P, a) \neq 0\}.$$

  They are closed cones inside the affine space $V = \mathrm{Spec}(S)$.

PROPAGATION OF RESONANCE

Motivating result:

THEOREM (EISENBUD–POPESCU–YUZVINSKY 2003)

Let A be the Orlik–Solomon algebra of an arrangement. Then the resonance varieties of A propagate.

Using similar techniques, we obtain the following generalization.

THEOREM (DSY)

Suppose the $\Bbbk$-dual module, $\widehat{P}$, has a linear free resolution over E. Then the resonance varieties of P propagate.

JUMP LOCI OF SPACES

- Let X be a connected, finite CW-complex.
- Fundamental group $\pi = \pi_1(X, x_0)$: a finitely generated, discrete group, with $\pi_{ab} \cong H_1(X, \mathbb{Z})$.
- Let $S = \Bbbk[\pi_{ab}]$ and identify $\operatorname{Spec}(S)$ with the character group $\operatorname{Hom}(\pi, \Bbbk^*) = H^1(X, \Bbbk^*)$.
- The *characteristic varieties* of X are the cohomology jump loci of the free S-cochain complex $C = C^*(X^{ab}, \Bbbk)$:

$$\mathcal{V}^i(X, \Bbbk) = \{\rho \in H^1(X, \Bbbk^*) \mid H^i(X, \Bbbk_\rho) \neq 0\}.$$

- The *resonance varieties* of X are the jump loci associated to the cohomology algebra $A = H^*(X, \Bbbk)$:

$$\mathcal{R}^i(X, \Bbbk) = \{a \in H^1(X, \Bbbk) \mid H^i(A, a) \neq 0\}.$$

THEOREM (PAPADIMA–S. 2010)

Let X be a minimal CW-complex. Then the linearization of the cellular cochain complex $C^*(X^{\mathrm{ab}}, \Bbbk)$ coincides with the complex $\mathbf{L}(A)$, where $A = H^*(X, \Bbbk)$.

DUALITY SPACES

In order to study propagation of jump loci in a topological setting, we start by recalling a notion due to Bieri and Eckmann (1978).

- X is a *duality space* of dimension n if $H^i(X, \mathbb{Z}\pi) = 0$ for $i \neq n$ and $H^n(X, \mathbb{Z}\pi) \neq 0$ and torsion-free.
- Let $D = H^n(X, \mathbb{Z}\pi)$ be the dualizing $\mathbb{Z}\pi$-module. Given any $\mathbb{Z}\pi$-module A, we have $H^i(X, A) \cong H_{n-i}(X, D \otimes A)$.
- If $D = \mathbb{Z}$, with trivial $\mathbb{Z}\pi$-action, then X is a Poincaré duality space.
- If $X = K(\pi, 1)$ is a duality space, then π is a *duality group*.

ABELIAN DUALITY SPACES

We introduce an analogous notion, by replacing $\pi \rightsquigarrow \pi_{ab}$.

- X is an *abelian duality space* of dimension n if $H^i(X, \mathbb{Z}\pi_{ab}) = 0$ for $i \neq n$ and $H^n(X, \mathbb{Z}\pi_{ab}) \neq 0$ and torsion-free.
- Let $B = H^n(X, \mathbb{Z}\pi_{ab})$ be the dualizing $\mathbb{Z}\pi_{ab}$-module. Given any $\mathbb{Z}\pi_{ab}$-module A, we have $H^i(X, A) \cong H_{n-i}(X, B \otimes A)$.
- The two notions of duality are independent. E.g.:
    - Every orientable surface of genus $g > 1$ is a PD space, but not an abelian duality space.
    - Let $H = \langle x_1, \dots, x_4 \mid x_1^{-2}x_2x_1x_2^{-1}, \dots, x_4^{-2}x_1x_4x_1^{-1} \rangle$. Then $\pi = \mathbb{Z}^2 * H$ is a 2-dim abelian duality group, but not a duality group.

PROPAGATION OF JUMP LOCI

THEOREM

Let X be an abelian duality space of dimension n. If $\rho\colon \pi_1(X) \to \Bbbk^*$ satisfies $H^i(X, \Bbbk_\rho) \neq 0$, then $H^j(X, \Bbbk_\rho) \neq 0$, for all $i \leqslant j \leqslant n$.

Consequences:

- The characteristic varieties propagate: $\mathcal{V}^1(X, \Bbbk) \subseteq \cdots \subseteq \mathcal{V}^n(X, \Bbbk)$.
- $\dim_{\Bbbk} H^1(X, \Bbbk) \geqslant n - 1$.
- If $n \geqslant 2$, then $H^i(X, \Bbbk) \neq 0$, for all $0 \leqslant i \leqslant n$.

THEOREM

If, moreover, X admits a minimal cell structure, then the resonance varieties also propagate: $\mathcal{R}^1(X, \Bbbk) \subseteq \cdots \subseteq \mathcal{R}^n(X, \Bbbk)$.

Corollary

Let M be a compact, connected, orientable manifold of dimension n. Suppose M admits a minimal cell structure, and $\mathcal{R}^1(M, \Bbbk) \neq 0$. Then M is not *an abelian duality space.*

Proof.

Let $\omega \in H^n(M, \Bbbk) \cong \Bbbk$ be the orientation class. By Poincaré duality, for any $a \in H^1(M, \Bbbk)$, there is $b \in H^{n-1}(M, \Bbbk)$ such that $a \cup b = \omega$. Hence, $\mathcal{R}^n(M, \Bbbk) = \{0\}$, thus contradicting propagation of resonance. □

Example

- Let M be the 3-dimensional Heisenberg nilmanifold.
- M admits a perfect Morse function.
- Characteristic varieties propagate: $\mathcal{V}^i(M) = \{1\}$ for $i \leqslant 3$;.
- Resonance does not propagate: $\mathcal{R}^1(M, \Bbbk) = \Bbbk^2$, $\mathcal{R}^3(M, \Bbbk) = 0$.

COMBINATORIAL COVERS

A *combinatorial cover* for a space X is a triple $(\mathscr{C}, \phi, \rho)$, where

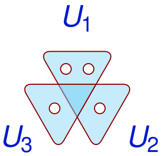
1. $\mathscr{C}$ is a countable cover which is either open, or closed and locally finite.
2. $\phi\colon N(\mathscr{C}) \to P$ is an order-preserving, surjective map from the nerve of the cover to a finite poset P, such that, if $S \leqslant T$ and $\phi(S) = \phi(T)$, then $\cap T \hookrightarrow \cap S$ admits a homotopy inverse.
3. If $S \leqslant T$ and $\bigcap S = \bigcap T$, then $\phi(S) = \phi(T)$.
4. $\rho\colon P \to \mathbb{Z}$ is an order-preserving map whose fibers are antichains.

We say that $\mathscr{C}$ is a *strong* combinatorial cover if, moreover, ϕ induces a homotopy equivalence, $\phi\colon |N(\mathscr{C})| \to |P|$.

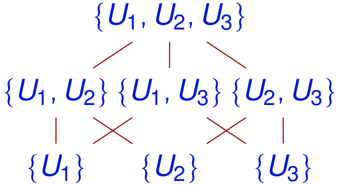
 COMBINATORIAL COVERS

Example: $X = D^2 \setminus \{4 \text{ points}\}$.

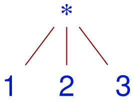
$\mathscr{C}$:



$N(\mathscr{C})$:



P :



- $\phi \colon N(\mathscr{C}) \to P$: $\phi(\{U_i\}) = i$ and $\phi(S) = *$ if $|S| \neq 1$.
- $\rho \colon P \to \mathbb{Z}$: $\rho(*) = 1$ and $\rho(i) = 0$.
- $\cap S = \cap T$ for any $S, T \in \phi^{-1}(*)$.
- Both $|N(\mathscr{C})|$ and $|P|$ are contractible.
- Thus, $\mathscr{C}$ is a strong combinatorial cover.

A SPECTRAL SEQUENCE

- Suppose X has a combinatorial cover $(\mathscr{C}, \phi, \rho)$. For each $x \in P$, let $P_{\leqslant x} = \{y \in P \mid y \leqslant x\}$; then $\phi^{-1}(P_{\leqslant x})$ is a sub-poset of $N(\mathscr{C})$.
- Choose a set $S \in N(\mathscr{C})$ with $\phi(S) = x$, and write $U_x = \cap S$; then U_x is well-defined up to homotopy.

THEOREM

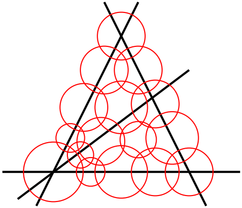
For every locally constant sheaf $\mathcal{F}$ on X, there is a spectral sequence with

$$E_2^{pq} = \prod_{x \in P} H^{p-\rho(x)}(\phi^{-1}(P_{\leqslant x}), \phi^{-1}(P_{<x}); \; H^{q+\rho(x)}(X, \mathcal{F}|_{U_x})),$$

converging to $H^{p+q}(X, \mathcal{F})$. Moreover, if $\mathscr{C}$ is a strong combinatorial cover, then

$$E_2^{pq} = \prod_{x \in P} \widetilde{H}^{p-\rho(x)-1}(\mathrm{lk}_{|P|}(x); \; H^{q+\rho(x)}(X, \mathcal{F}|_{U_x})).$$

ARRANGEMENTS OF SUBMANIFOLDS



Let $\mathcal{A}$ be an arrangement of submanifolds in a smooth, connected manifold. Assume that the intersection of any subset of $\mathcal{A}$ is also smooth, and has only finitely many connected components.

THEOREM

1. *If each submanifold is either compact or open, then the complement $M(\mathcal{A})$ has a combinatorial cover $(\mathscr{C}, \phi, \rho)$ over the (ranked) intersection poset $L(\mathcal{A})$.*
2. *If, moreover, each submanifold in $L(\mathcal{A})$ is contractible, then $\mathscr{C}$ is a strong combinatorial cover.*

THEOREM

Let $\mathcal{F}$ be a locally constant sheaf on $M(\mathcal{A})$. There is then a spectral sequence with

$$E_2^{pq} = \prod_{X \in L(\mathcal{A})} H^{p-\rho(X)}(X, D_X; H^{q+\rho(X)}(M(\mathcal{A}), \mathcal{F}_{U_X})),$$

converging to $H^{p+q}(M(\mathcal{A}), \mathcal{F})$.

Here,

- $D_X = \bigcup_{Z \in L(\mathcal{A})_{<X}} Z$.
- $U_X \in \mathscr{C}$ is such that $\min\{X \in L(\mathcal{A}) : X \cap \overline{U} \neq \emptyset\} = X$.

HYPERPLANE ARRANGEMENTS

- Let $\mathcal{A}$ be a central, essential hyperplane arrangement in $\mathbb{C}^n$.
- Its complement, $M(\mathcal{A})$, has the homotopy type of a minimal CW-complex of dimension n.

THEOREM

Suppose $A = \mathbb{Z}[\pi]$ or $A = \mathbb{Z}[\pi_{\text{ab}}]$. Then $H^p(M(\mathcal{A}), A) = 0$ for all $p \neq n$, and $H^n(M(\mathcal{A}), A)$ is a free abelian group.

COROLLARY

1. *$M(\mathcal{A})$ is a duality space of dimension n (due to [DJO 2011]).*
2. *$M(\mathcal{A})$ is an abelian duality space of dimension n.*
3. *The characteristic and resonance varieties of $M(\mathcal{A})$ propagate.*

Toric arrangements

- A *toric arrangement* is a finite collection of codimension-1 subtori (possibly translated) in a complex algebraic torus.
- Studied by DeConcini–Procesi, Moci, Moci–Settepanella, d'Antonio–Delucchi, Davis–Settepanella, Callegaro–Delucchi, ...
- The complement is again a minimal space (Adiprasito–Delucchi).

Using some of this work and our machinery, we obtain:

Theorem

Let $\mathcal{A}$ be a toric arrangement in $(\mathbb{C}^*)^n$. Then:

1. *$M(\mathcal{A})$ is a duality space of dimension n (due to [DS 2013]).*
2. *$M(\mathcal{A})$ is an abelian duality space of dimension n.*
3. *The characteristic and resonance varieties of $M(\mathcal{A})$ propagate.*

ELLIPTIC ARRANGEMENTS

- An *elliptic arrangement* is a finite collection $\mathcal{A}$ of subvarieties in a product of elliptic curves E^n, each subvariety being a fiber of a group homomorphism $E^n \to E$.
- If $\mathcal{A}$ is essential, the complement $M(\mathcal{A})$ is a Stein manifold.
- $M(\mathcal{A})$ is minimal, but it's not formal, in general.

THEOREM

The complement of an essential, unimodular elliptic arrangement in E^n is both a duality space and an abelian duality space of dimension n.

Our approach recovers and generalizes a result Levin and Varchenko.

THEOREM (LV 2012)

Let $\mathcal{A}$ be an elliptic arrangement in E^n, and let $\Bbbk_\rho$ be a 'convenient' rank-1 local system on its complement. Then $H^i(M(\mathcal{A}), \Bbbk_\rho) = 0$ for $i < n$ and $H^n(M(\mathcal{A}), \Bbbk_\rho) = H^n(M(\mathcal{A}), \Bbbk)$.

TORIC COMPLEXES

- Let L be simplicial complex on n vertices.
- The *toric complex* T_L is the subcomplex of the n-torus obtained by deleting the cells corresponding to the missing simplices of L.
- By construction, T_L is a minimal CW-complex, of dimension $\dim L + 1$.
- $\pi_\Gamma := \pi_1(T_L)$ is the *right-angled Artin group* associated to the graph $\Gamma = L^{(1)}$.
- $K(\pi_\Gamma, 1) = T_{\Delta_\Gamma}$, where Δ_Γ is the *flag complex* of Γ.
- $H^*(T_L, \Bbbk) = E/J_L$ is the *exterior Stanley–Reisner ring* of L.

- L is *Cohen–Macaulay* if for each simplex $\sigma \in L$, the reduced cohomology of $\operatorname{lk}(\sigma)$ is concentrated in degree $\dim(L) - |\sigma|$ and is torsion-free.

Theorem (N. Brady–Meier 2001, Jensen–Meier 2005)

A right-angled Artin group π_Γ is a duality group if and only if Δ_Γ is Cohen–Macaulay. Moreover, π_Γ is a Poincaré duality group if and only if Γ is a complete graph.

Theorem

T_L is an abelian duality space (of dimension $\dim(L)+1$) if and only if L is Cohen-Macaulay.

In this case, the resonance varieties of T_L propagate. In general, though, they don't.

Given a (finite, simplicial) graph Γ, the corresponding Bestvina–Brady group is defined as

$$N_\Gamma = \ker(\nu \colon G_\Gamma \to \mathbb{Z}),$$

where $\nu(v) = 1$, for each vertex v of Γ.

PROPOSITION (DAVIS–OKUN 2012)

Suppose Δ_Γ is acyclic. Then N_Γ is a duality group if and only if Δ_Γ is Cohen–Macaulay.

PROPOSITION

N_Γ is an abelian duality group if and only if Δ_Γ is acyclic and Cohen-Macaulay.

REFERENCES

Graham Denham, Alexander I. Suciu, and Sergey Yuzvinsky, *Combinatorial covers and vanishing cohomology*, preprint, 2014.

Graham Denham, Alexander I. Suciu, and Sergey Yuzvinsky, *Abelian duality and propagation of resonance*, preprint, 2014.