TOPOLOGY OF HYPERPLANE ARRANGEMENTS

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HYPERPLANE ARRANGEMENTS

- An *arrangement of hyperplanes* is a finite set $\mathcal{A}$ of codimension-1 linear subspaces in $\mathbb{C}^\ell$.
- *Intersection lattice* $L(\mathcal{A})$: poset of all intersections of $\mathcal{A}$, ordered by reverse inclusion, and ranked by codimension.
- *Complement*: $M(\mathcal{A}) = \mathbb{C}^\ell \setminus \bigcup_{H\in\mathcal{A}} H$.
- The Boolean arrangement $\mathcal{B}_n$
  - $\mathcal{B}_n$: all coordinate hyperplanes $z_i = 0$ in $\mathbb{C}^n$.
  - $L(\mathcal{B}_n)$: Boolean lattice of subsets of $\{0,1\}^n$.
  - $M(\mathcal{B}_n)$: complex algebraic torus $(\mathbb{C}^*)^n$.
- The braid arrangement $\mathcal{A}_n$ (or, reflection arr. of type A_{n-1})
  - $\mathcal{A}_n$: all diagonal hyperplanes $z_i - z_j = 0$ in $\mathbb{C}^n$.
  - $L(\mathcal{A}_n)$: lattice of partitions of $[n] = \{1,\dots,n\}$.
  - $M(\mathcal{A}_n)$: configuration space of n ordered points in $\mathbb{C}$ (a classifying space for the pure braid group on n strings).

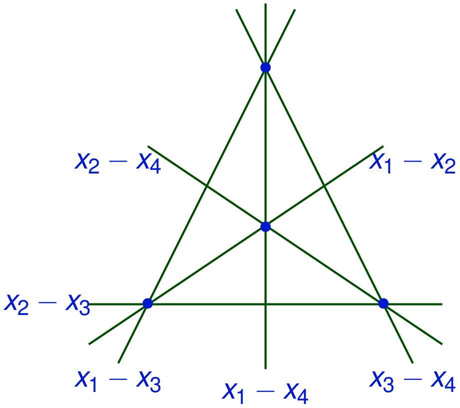


FIGURE : A planar slice of the braid arrangement $\mathcal{A}_4$

- We may assume that $\mathcal{A}$ is essential, i.e., $\bigcap_{H\in\mathcal{A}} H = \{0\}$.
- Fix an ordering $\mathcal{A} = \{H_1, \dots, H_n\}$, and choose linear forms $f_i \colon \mathbb{C}^\ell \to \mathbb{C}$ with $\ker(f_i) = H_i$.
- Define an injective linear map

$$\iota \colon \mathbb{C}^\ell \to \mathbb{C}^n, \quad z \mapsto (f_1(z), \dots, f_n(z)).$$

- This map restricts to an inclusion $\iota \colon M(\mathcal{A}) \hookrightarrow M(\mathcal{B}_n)$. Hence,

$$M(\mathcal{A}) = \iota(\mathbb{C}^\ell) \cap (\mathbb{C}^*)^n,$$

  a “very affine" subvariety of $(\mathbb{C}^*)^n$, and thus, a Stein manifold.
- Therefore, $M(\mathcal{A})$ has the homotopy type of a connected, finite cell complex of dimension ℓ.

- In fact, $M = M(\mathcal{A})$ admits a minimal cell structure. Consequently, $H_*(M, \mathbb{Z})$ is torsion-free.

- The Betti numbers $b_q(M) := \operatorname{rank} H_q(M, \mathbb{Z})$ are given by

$$\sum_{q=0}^{\ell} b_q(M) t^q = \sum_{X \in L(\mathcal{A})} \mu(X)(-t)^{\operatorname{rank}(X)},$$

  where $\mu \colon L(\mathcal{A}) \to \mathbb{Z}$ is the Möbius function, defined recursively by $\mu(\mathbb{C}^\ell) = 1$ and $\mu(X) = -\sum_{Y \supsetneq X} \mu(Y)$.

- The Orlik–Solomon algebra $H^*(M, \mathbb{Z})$ is the quotient of the exterior algebra on generators $\{e_H \mid H \in \mathcal{A}\}$ by an ideal determined by the circuits in the matroid of $\mathcal{A}$.

- Thus, the ring $H^*(M, \Bbbk)$ is determined by $L(\mathcal{A})$, for every field $\Bbbk$.

- To compute the fundamental group of the complement, we may assume $\mathcal{A}$ is an arrangement of planes in $\mathbb{C}^3$.
- Its projectivization, $\bar{\mathcal{A}}$, is an arrangement of lines in $\mathbb{CP}^2$.
- $L_1(\mathcal{A}) \longleftrightarrow$ lines of $\bar{\mathcal{A}}$, while $L_2(\mathcal{A}) \longleftrightarrow$ intersection points of $\bar{\mathcal{A}}$.
- Poset structure of $L_{\leqslant 2}(\mathcal{A}) \longleftrightarrow$ incidence structure of $\bar{\mathcal{A}}$.
- A flat $X \in L_2(\mathcal{A})$ has multiplicity q if the point $\bar{X}$ has exactly q lines from $\bar{\mathcal{A}}$ passing through it.

MULTINETS

DEFINITION (FALK AND YUZVINSKY 2007)

A *multinet* on a hyperplane arrangement $\mathcal{A}$ consists of

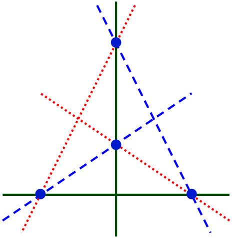
1. a partition of $\mathcal{A}$ into $k \geqslant 3$ subsets $\mathcal{A}_1, \dots, \mathcal{A}_k$;
2. an assignment of multiplicities, $m \colon \mathcal{A} \to \mathbb{N}$;
3. a subset $\mathcal{X} \subseteq L_2(\mathcal{A})$, called the base locus,

such that

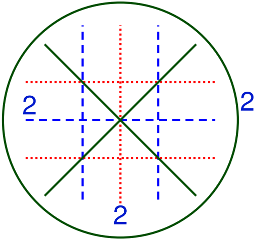
1. there is $d \in \mathbb{N}$ such that $\sum_{H\in\mathcal{A}_\alpha} m_H = d$, for all $\alpha \in [k]$;
2. if H and H' are in different classes, then $H \cap H' \in \mathcal{X}$;
3. for each $X \in \mathcal{X}$, the sum $n_X = \sum_{H\in\mathcal{A}_\alpha : H\supset X} m_H$ is independent of α;
4. each space $\big(\bigcup_{H\in\mathcal{A}_\alpha} H\big) \setminus \mathcal{X}$ is connected.

- A multinet as above is also called a (k, d)-multinet, or a k-multinet.
- If $m_H = 1$, for all $H \in \mathcal{A}$, the multinet is *reduced*.

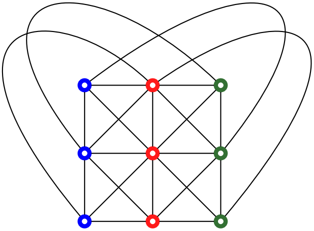
- If, furthermore, $n_X = 1$, for all $X \in \mathcal{X}$, this is a *net*. In this case, $|\mathcal{A}_\alpha| = |\mathcal{A}|/k = d$, for all α. Moreover, $\bar{\mathcal{X}}$ has size d^2, and is encoded by a $(k-2)$-tuple of orthogonal Latin squares.
- If $\mathcal{A}$ has no flats of multiplicity kr, for some $r > 1$, then every reduced k-multinet is a k-net.

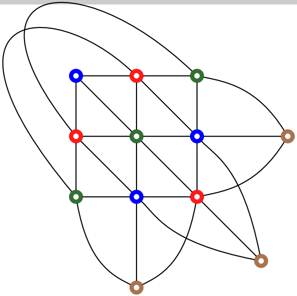


A $(3,2)$-net on the A_3 arrangement
$\bar{\mathcal{X}}$ consists of 4 triple points ($n_X = 1$)



A $(3,4)$-multinet on the B_3 arrangement
$\bar{\mathcal{X}}$ consists of 4 triple points ($n_X = 1$)
and 3 triple points ($n_X = 2$)





A $(3,3)$-net on the Ceva matroid. A $(4,3)$-net on the Hessian matroid.

- (Yuzvinsky 2004 & 2009, Pereira–Yuzvinsky 2008): If $\mathcal{A}$ supports a k-multinet with $|\mathcal{X}| > 1$, then $k = 3$ or 4; moreover, if the multinet is not reduced, then $k = 3$.
- Conjecture (Yuz): The only 4-multinet is the Hessian $(4,3)$-net.
- (Cordovil–Forge 2003, Torielli–Yoshinaga 2014): Conjecture is true if $\mathcal{A}$ is defined by real equations.

COHOMOLOGY JUMP LOCI

- Let X be a connected, finite cell complex, and let $\pi = \pi_1(X, x_0)$.
- Let $\Bbbk$ be an algebraically closed field, and let $\mathrm{Hom}(\pi, \Bbbk^*)$ be the affine algebraic group of $\Bbbk$-valued, multiplicative characters on π.
- The *characteristic varieties* of X are the jump loci for homology with coefficients in rank-1 local systems on X:

$$\mathcal{V}^q_s(X, \Bbbk) = \{\rho \in \mathrm{Hom}(\pi, \Bbbk^*) \mid \dim_{\Bbbk} H_q(X, \Bbbk_\rho) \geqslant s\}.$$

  Here, $\Bbbk_\rho$ is the local system defined by ρ, i.e, $\Bbbk$ viewed as a $\Bbbk\pi$-module, via $g \cdot x = \rho(g)x$, and $H_i(X, \Bbbk_\rho) = H_i(C_*(\widetilde{X}, \Bbbk) \otimes_{\Bbbk\pi} \Bbbk_\rho)$.
- These loci are Zariski closed subsets of the character group.
- The sets $\mathcal{V}^1_s(X, \Bbbk)$ depend only on π/π''.

EXAMPLE (CIRCLE)

We have $\widetilde{S^1} = \mathbb{R}$. Identify $\pi_1(S^1, *) = \mathbb{Z} = \langle t \rangle$ and $\Bbbk\mathbb{Z} = \Bbbk[t^{\pm 1}]$. Then:

$$C_*(\widetilde{S^1}, \Bbbk):\ 0 \longrightarrow \Bbbk[t^{\pm 1}] \xrightarrow{t-1} \Bbbk[t^{\pm 1}] \longrightarrow 0\ .$$

For $\rho \in \mathrm{Hom}(\mathbb{Z}, \Bbbk^*) = \Bbbk^*$, we get

$$C_*(\widetilde{S^1}, \Bbbk) \otimes_{\Bbbk\mathbb{Z}} \Bbbk_\rho:\ 0 \longrightarrow \Bbbk \xrightarrow{\rho-1} \Bbbk \longrightarrow 0\ ,$$

which is exact, except for $\rho = 1$, when $H_0(S^1, \Bbbk) = H_1(S^1, \Bbbk) = \Bbbk$.
Hence: $\mathcal{V}^0_1(S^1, \Bbbk) = \mathcal{V}^1_1(S^1, \Bbbk) = \{1\}$ and $\mathcal{V}^i_s(S^1, \Bbbk) = \emptyset$, otherwise.

EXAMPLE (PUNCTURED COMPLEX LINE)

Identify $\pi_1(\mathbb{C}\backslash\{n \text{ points}\}) = F_n$, and $\widehat{F_n} = (\Bbbk^*)^n$. Then:

$$\mathcal{V}^1_s(\mathbb{C}\backslash\{n \text{ points}\}, \Bbbk) = \begin{cases} (\Bbbk^*)^n & \text{if } s < n, \\ \{1\} & \text{if } s = n, \\ \emptyset & \text{if } s > n. \end{cases}$$

- Let $A = H^*(X, \Bbbk)$. If $\operatorname{char} \Bbbk = 2$, assume that $H_1(X, \mathbb{Z})$ has no 2-torsion. Then: $a \in A^1 \Rightarrow a^2 = 0$.
- Thus, we get a cochain complex

$$(A, \cdot a)\colon \; A^0 \xrightarrow{\;a\;} A^1 \xrightarrow{\;a\;} A^2 \longrightarrow \cdots,$$

known as the *Aomoto complex* of A.

- The *resonance varieties* of X are the jump loci for the Aomoto-Betti numbers

$$\mathcal{R}^q_s(X, \Bbbk) = \{a \in A^1 \mid \dim_{\Bbbk} H^q(A, \cdot a) \geqslant s\},$$

- These loci are *homogeneous* subvarieties of $A^1 = H^1(X, \Bbbk)$.

EXAMPLE

- $\mathcal{R}^1_1(T^n, \Bbbk) = \{0\}$, for all $n > 0$.
- $\mathcal{R}^1_1(\mathbb{C} \backslash \{n \text{ points}\}, \Bbbk) = \Bbbk^n$, for all $n > 1$.
- $\mathcal{R}^1_1(\Sigma_g, \Bbbk) = \Bbbk^{2g}$, for all $g > 1$.

JUMP LOCI OF ARRANGEMENTS

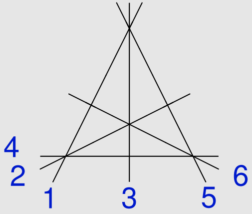
- Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an arrangement in $\mathbb{C}^3$, and identify $H^1(M(\mathcal{A}), \Bbbk) = \Bbbk^n$, with basis dual to the meridians.
- The resonance varieties $\mathcal{R}^1_s(\mathcal{A}, \Bbbk) := \mathcal{R}^1_s(M(\mathcal{A}), \Bbbk) \subset \Bbbk^n$ lie in the hyperplane $\{x \in \Bbbk^n \mid x_1 + \cdots + x_n = 0\}$.
- $\mathcal{R}(\mathcal{A}) = \mathcal{R}^1_1(\mathcal{A}, \mathbb{C})$ is a union of linear subspaces in $\mathbb{C}^n$.
- Each subspace has dimension at least 2, and each pair of subspaces meets transversely at 0.
- $\mathcal{R}^1_s(\mathcal{A}, \mathbb{C})$ is the union of those linear subspaces that have dimension at least $s + 1$.

- Each flat $X \in L_2(\mathcal{A})$ of multiplicity $k \geqslant 3$ gives rise to a *local* component of $\mathcal{R}(\mathcal{A})$, of dimension $k-1$.

- More generally, every k-multinet of a sub-arrangement $\mathcal{B} \subseteq \mathcal{A}$ gives rise to a component of dimension $k-1$, and all components of $\mathcal{R}(\mathcal{A})$ arise in this way.

- The resonance varieties $\mathcal{R}^1(\mathcal{A}, \Bbbk)$ can be more complicated, e.g., they may have non-linear components.

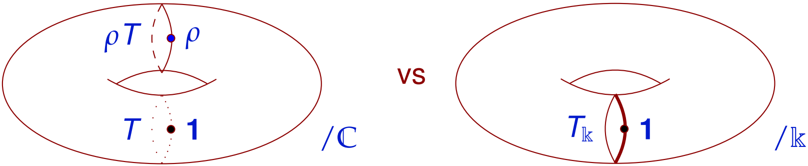
EXAMPLE (BRAID ARRANGEMENT $\mathcal{A}_4$)



$\mathcal{R}^1(\mathcal{A}, \mathbb{C}) \subset \mathbb{C}^6$ has 4 local components (from triple points), and one non-local component, from the $(3, 2)$-net:

$$L_{124} = \{x_1 + x_2 + x_4 = x_3 = x_5 = x_6 = 0\},$$
$$L_{135} = \{x_1 + x_3 + x_5 = x_2 = x_4 = x_6 = 0\},$$
$$L_{236} = \{x_2 + x_3 + x_6 = x_1 = x_4 = x_5 = 0\},$$
$$L_{456} = \{x_4 + x_5 + x_6 = x_1 = x_2 = x_3 = 0\},$$
$$L = \{x_1 + x_2 + x_3 = x_1 - x_6 = x_2 - x_5 = x_3 - x_4 = 0\}.$$

- Let $\operatorname{Hom}(\pi_1(M), \mathbb{k}^*) = (\mathbb{k}^*)^n$ be the character torus.
- The characteristic variety $\mathcal{V}^1(\mathcal{A}, \mathbb{k}) := \mathcal{V}^1_1(M(\mathcal{A}), \mathbb{k}) \subset (\mathbb{k}^*)^n$ lies in the substorus $\{t \in (\mathbb{k}^*)^n \mid t_1 \cdots t_n = 1\}$.
- $\mathcal{V}^1(\mathcal{A}, \mathbb{C})$ is a finite union of torsion-translates of algebraic subtori of $(\mathbb{C}^*)^n$.
- If a linear subspace $L \subset \mathbb{C}^n$ is a component of $\mathcal{R}^1(\mathcal{A}, \mathbb{C})$, then the algebraic torus $T = \exp(L)$ is a component of $\mathcal{V}^1(\mathcal{A}, \mathbb{C})$.
- All components of $\mathcal{V}^1(\mathcal{A}, \mathbb{C})$ passing through the origin $1 \in (\mathbb{C}^*)^n$ arise in this way (and thus, are combinatorially determined).



- In general, though, there are translated subtori in $\mathcal{V}^1(\mathcal{A}, \Bbbk)$.
- When this happens, the characteristic varieties $\mathcal{V}^1(\mathcal{A}, \Bbbk)$ may depend (qualitatively) on $\operatorname{char}(\Bbbk)$.

THE MILNOR FIBRATION(S) OF AN ARRANGEMENT

- For each $H \in \mathcal{A}$, let $f_H \colon \mathbb{C}^\ell \to \mathbb{C}$ be a linear form with kernel H.
- For each choice of multiplicities $m = (m_H)_{H\in\mathcal{A}}$ with $m_H \in \mathbb{N}$, let

$$Q_m := Q_m(\mathcal{A}) = \prod_{H\in\mathcal{A}} f_H^{m_H},$$

a homogeneous polynomial of degree $N = \sum_{H\in\mathcal{A}} m_H$.
- The map $Q_m \colon \mathbb{C}^\ell \to \mathbb{C}$ restricts to a map $Q_m \colon M(\mathcal{A}) \to \mathbb{C}^*$.
- This is the projection of a smooth, locally trivial bundle, known as the *Milnor fibration* of the multi-arrangement $(\mathcal{A}, m)$,

$$F_m(\mathcal{A}) \longrightarrow M(\mathcal{A}) \xrightarrow{Q_m} \mathbb{C}^*.$$

- The typical fiber, $F_m(\mathcal{A}) = Q_m^{-1}(1)$, is called the *Milnor fiber* of the multi-arrangement.
- $F_m(\mathcal{A})$ has the homotopy type of a finite cell complex, with $\gcd(m)$ connected components, and of dimension $\ell - 1$.
- The *(geometric) monodromy* is the diffeomorphism

$$h\colon F_m(\mathcal{A}) \to F_m(\mathcal{A}), \quad z \mapsto e^{2\pi \mathrm{i}/N} z.$$

- If all $m_H = 1$, the polynomial $Q = Q_m(\mathcal{A})$ is the usual defining polynomial, and $F(\mathcal{A}) = F_m(\mathcal{A})$ is the usual Milnor fiber of $\mathcal{A}$.

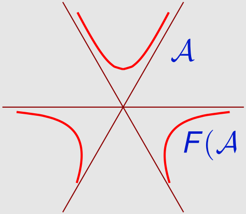
EXAMPLE

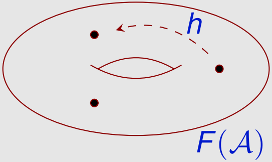
Let $\mathcal{A}$ be the single hyperplane $\{0\}$ inside $\mathbb{C}$. Then:

- $M(\mathcal{A}) = \mathbb{C}^*$.
- $Q_m(\mathcal{A}) = z^m$.
- $F_m(\mathcal{A}) = m$-roots of 1.

EXAMPLE

Let $\mathcal{A}$ be a pencil of 3 lines through the origin of $\mathbb{C}^2$. Then $F(\mathcal{A})$ is a thrice-punctured torus, and h is an automorphism of order 3:





More generally, if $\mathcal{A}$ is a pencil of n lines in $\mathbb{C}^2$, then $F(\mathcal{A})$ is a Riemann surface of genus $\binom{n-1}{2}$, with n punctures.

- Let $\mathcal{B}_n$ be the Boolean arrangement, with $Q_m(\mathcal{B}_n) = z_1^{m_1} \cdots z_n^{m_n}$. Then $M(\mathcal{B}_n) = (\mathbb{C}^*)^n$ and

$$F_m(\mathcal{B}_n) = \ker(Q_m) \cong (\mathbb{C}^*)^{n-1} \times \mathbb{Z}_{\gcd(m)}$$

- Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an essential arrangement. The inclusion $\iota\colon M(\mathcal{A}) \to M(\mathcal{B}_n)$ restricts to a bundle map

$$\begin{array}{ccccc} F_m(\mathcal{A}) & \longrightarrow & M(\mathcal{A}) & \xrightarrow{Q_m(\mathcal{A})} & \mathbb{C}^* \\ \downarrow & & \downarrow{\scriptstyle \iota} & & \| \\ F_m(\mathcal{B}_n) & \longrightarrow & M(\mathcal{B}_n) & \xrightarrow{Q_m(\mathcal{B}_n)} & \mathbb{C}^* \end{array}$$

- Thus,

$$F_m(\mathcal{A}) = M(\mathcal{A}) \cap F_m(\mathcal{B}_n)$$

THE HOMOLOGY OF THE MILNOR FIBER

Two basic questions about the topology of the Milnor fibration:

(Q1) Are the homology groups $H_q(F(\mathcal{A}), \Bbbk)$ determined by $L(\mathcal{A})$?
If so, is the characteristic polynomial

$$\Delta_q(t) = \det(t \cdot \mathrm{id} - h_*),$$

of the algebraic monodromy, $h_* \colon H_q(F(\mathcal{A}), \Bbbk) \to H_q(F(\mathcal{A}), \Bbbk)$, also determined by $L(\mathcal{A})$?

(Q2) Are the homology groups $H_q(F(\mathcal{A}), \mathbb{Z})$ torsion-free?
If so, does $F(\mathcal{A})$ admit a minimal cell structure?

- Let $(\mathcal{A}, m)$ be a multi-arrangement with $\gcd\{m_H \mid H \in \mathcal{A}\} = 1$. Set $N = \sum_{H\in\mathcal{A}} m_H$.
- The Milnor fiber $F_m(\mathcal{A})$ is a regular $\mathbb{Z}_N$-cover of $U(\mathcal{A}) = \mathbb{P}(M(\mathcal{A}))$ defined by the homomorphism

$$\delta_m\colon \pi_1(U(\mathcal{A})) \twoheadrightarrow \mathbb{Z}_N, \quad x_H \mapsto m_H \bmod N$$

- Let $\widehat{\delta_m}\colon \operatorname{Hom}(\mathbb{Z}_N, \Bbbk^*) \to \operatorname{Hom}(\pi_1(U(\mathcal{A})), \Bbbk^*)$. If $\operatorname{char}(\Bbbk) \nmid N$, then

$$\dim_{\Bbbk} H_q(F_m(\mathcal{A}), \Bbbk) = \sum_{s\geqslant 1} \left| \mathcal{V}^q_s(U(\mathcal{A}), \Bbbk) \cap \operatorname{im}(\widehat{\delta_m}) \right|.$$

- This gives a formula for $\Delta_q(t)$ in terms of the characteristic varieties of $U(\mathcal{A})$.

- The characteristic polynomial of h_* acting on $H_1(F(\mathcal{A}), \mathbb{C})$ can be written as
$$\Delta(t) = \prod_{d|n} \Phi_d(t)^{e_d(\mathcal{A})},$$
where $\Phi_d(t)$ is the d-th cyclotomic polynomial, and $e_d(\mathcal{A}) \in \mathbb{Z}_{\geqslant 0}$.

EXAMPLE

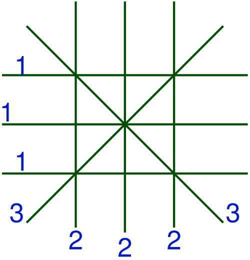
Let $\mathcal{A}$ be the braid arrangement. $\mathcal{V}_1(\mathcal{A})$ has a single essential component, $T = \{t \in (\mathbb{C}^*)^6 \mid t_1 t_2 t_3 = t_1 t_6^{-1} = t_2 t_5^{-1} = t_3 t_4^{-1} = 1\}$. Clearly, $\delta^2 \in T$, yet $\delta \notin T$; hence,
$$\Delta(t) = (t-1)^5(t^2+t+1).$$

- Thus, for $q = 1$ and $\mathbb{k} = \mathbb{C}$, question (Q1) is equivalent to: are the integers $e_d(\mathcal{A})$ determined by $L_{\leqslant 2}(\mathcal{A})$?
- A partial (positive) answer is given in joint work with Stefan Papadima (2014).

TORSION IN HOMOLOGY

THEOREM (COHEN–DENHAM–S. 2003)

For every prime $p \geqslant 2$, there is a multi-arrangement $(\mathcal{A}, m)$ such that $H_1(F_m(\mathcal{A}), \mathbb{Z})$ has non-zero p-torsion.



Simplest example: the arrangement of 8 hyperplanes in $\mathbb{C}^3$ with

$$Q_m(\mathcal{A}) = x^2y(x^2 - y^2)^3(x^2 - z^2)^2(y^2 - z^2)$$

Then $H_1(F_m(\mathcal{A}), \mathbb{Z}) = \mathbb{Z}^7 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

We now can generalize and reinterpret these examples, as follows.

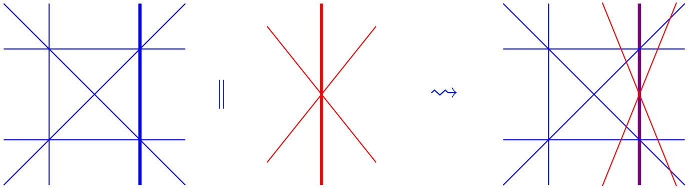
A *pointed multinet* on an arrangement $\mathcal{A}$ is a multinet structure, together with a distinguished hyperplane $H \in \mathcal{A}$ for which $m_H > 1$ and $m_H \mid n_X$ for each $X \in \mathcal{X}$ such that $X \subset H$.

THEOREM (DENHAM–S. 2014)

Suppose $\mathcal{A}$ admits a pointed multinet, with distinguished hyperplane H and multiplicity m. Let p be a prime dividing m_H. There is then a choice of multiplicities m' on the deletion $\mathcal{A}' = \mathcal{A} \setminus \{H\}$ such that $H_1(F_{m'}(\mathcal{A}'), \mathbb{Z})$ has non-zero p-torsion.

This torsion is explained by the fact that the geometry of $\mathcal{V}(\mathcal{A}', \Bbbk)$ varies with $\operatorname{char}(\Bbbk)$.

To produce p-torsion in the homology of the usual Milnor fiber, we use a "polarization" construction:



$(\mathcal{A}, m) \rightsquigarrow \mathcal{A} \| m$, an arrangement of $N = \sum_{H \in \mathcal{A}} m_H$ hyperplanes, of rank equal to $\operatorname{rank} \mathcal{A} + |\{H \in \mathcal{A} \colon m_H \geqslant 2\}|$.

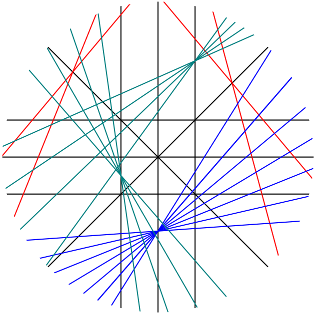
THEOREM (DS)

Suppose $\mathcal{A}$ admits a pointed multinet, with distinguished hyperplane H and multiplicity m. Let p be a prime dividing m_H.

There is then a choice of multiplicities m' on the deletion $\mathcal{A}' = \mathcal{A} \backslash \{H\}$ such that $H_q(F(\mathcal{B}), \mathbb{Z})$ has p-torsion, where $\mathcal{B} = \mathcal{A}' \| m'$ and $q = 1 + \left|\{K \in \mathcal{A}' : m'_K \geqslant 3\}\right|$.

COROLLARY (DS)

For every prime $p \geqslant 2$, there is an arrangement $\mathcal{A}$ such that $H_q(F(\mathcal{A}), \mathbb{Z})$ has non-zero p-torsion, for some $q > 1$.



Simplest example: the arrangement of 27 hyperplanes in $\mathbb{C}^8$ with

$$Q(\mathcal{A}) = xy(x^2-y^2)(x^2-z^2)(y^2-z^2)w_1w_2w_3w_4w_5(x^2-w_1^2)(x^2-2w_1^2)(x^2-3w_1^2)(x-4w_1)\cdot$$

$$((x-y)^2-w_2^2)((x+y)^2-w_3^2)((x-z)^2-w_4^2)((x-z)^2-2w_4^2)\cdot((x+z)^2-w_5^2)((x+z)^2-2w_5^2).$$

Then $H_6(F(\mathcal{A}), \mathbb{Z})$ has 2-torsion (of rank 108).

THE BOUNDARY MANIFOLD

- Let $\mathcal{A}$ be a (central) arrangement of hyperplanes in $\mathbb{C}^{d+1}$ ($d \geqslant 1$).
- Let $\mathbb{P}(\mathcal{A}) = \{\mathbb{P}(H)\}_{H\in\mathcal{A}}$, and let $\nu(W)$ be a regular neighborhood of the algebraic hypersurface $W = \bigcup_{H\in\mathcal{A}} \mathbb{P}(H)$ inside $\mathbb{CP}^d$.
- Let $\overline{U} = \mathbb{CP}^d \setminus \operatorname{int}(\nu(W))$ be the *exterior* of $\mathbb{P}(\mathcal{A})$.
- The *boundary manifold* of $\mathcal{A}$ is $\partial\overline{U} = \partial\nu(W)$: a compact, orientable, smooth manifold of dimension $2d-1$.

EXAMPLE

Let $\mathcal{A}$ be a pencil of n hyperplanes in $\mathbb{C}^{d+1}$, defined by $Q = z_1^n - z_2^n$. If $n = 1$, then $\partial\overline{U} = S^{2d-1}$. If $n > 1$, then $\partial\overline{U} = \sharp^{n-1} S^1 \times S^{2(d-1)}$.

EXAMPLE

Let $\mathcal{A}$ be a near-pencil of n planes in $\mathbb{C}^3$, defined by $Q = z_1(z_2^{n-1} - z_3^{n-1})$. Then $\partial\overline{U} = S^1 \times \Sigma_{n-2}$, where $\Sigma_g = \sharp^g S^1 \times S^1$.

- By Lefschetz duality: $H_q(\partial\overline{U},\mathbb{Z}) \cong H_q(U,\mathbb{Z}) \oplus H_{2d-q-1}(U,\mathbb{Z})$
- Let $A = H^*(U,\mathbb{Z})$; then $\check{A} = \operatorname{Hom}_{\mathbb{Z}}(A,\mathbb{Z})$ is an A-bimodule, with $(a\cdot f)(b) = f(ba)$ and $(f\cdot a)(b) = f(ab)$.

THEOREM (COHEN–S. 2006)

The ring $\widehat{A} = H^*(\partial\overline{U},\mathbb{Z})$ is the "double" of A, that is: $\widehat{A} = A \oplus \check{A}$, with multiplication given by $(a,f)\cdot(b,g) = (ab, ag+fb)$, and grading $\widehat{A}^q = A^q \oplus \check{A}^{2d-q-1}$.

- Now assume $d = 2$. Then $\partial\overline{U}$ is a graph-manifold of dimension 3, modeled on a graph Γ based on the poset $L_{\leqslant 2}(\mathcal{A})$.

THEOREM (COHEN–S. 2008)

The manifold $\partial\overline{U}$ admits a minimal cell structure. Moreover,

$$\mathcal{V}_1^1(\partial\overline{U}) = \bigcup_{v\in \mathrm{V}(\Gamma)\,:\, d_v \geqslant 3} \{t_v - 1 = 0\},$$

where d_v denotes the degree of the vertex v, and $t_v = \prod_{i\in v} t_i$.

THE BOUNDARY OF THE MILNOR FIBER

- Let $(\mathcal{A}, m)$ be a multi-arrangement in $\mathbb{C}^{d+1}$.
- Define $\overline{F}_m(\mathcal{A}) = F_m(\mathcal{A}) \cap D^{2(d+1)}$ to be the *closed Milnor fiber* of $(\mathcal{A}, m)$. Clearly, $F_m(\mathcal{A})$ deform-retracts onto $\overline{F}_m(\mathcal{A})$.
- The *boundary of the Milnor fiber* of $(\mathcal{A}, m)$ is the compact, smooth, orientable, $(2d-1)$-manifold $\partial\overline{F}_m(\mathcal{A}) = F_m(\mathcal{A}) \cap S^{2d+1}$.
- The pair $(\overline{F}_m, \partial\overline{F}_m)$ is $(d-1)$-connected. In particular, if $d \geqslant 2$, then $\partial\overline{F}_m$ is connected, and $\pi_1(\partial\overline{F}_m) \to \pi_1(\overline{F}_m)$ is surjective.

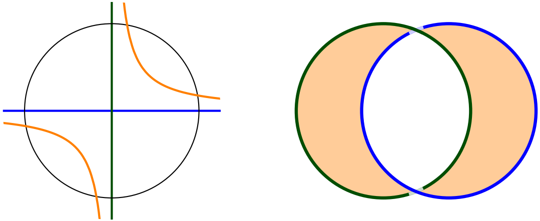


FIGURE : Closed Milnor fiber for $Q(\mathcal{A}) = xy$

EXAMPLE

- Let $\mathcal{B}_n$ be the Boolean arrangement in $\mathbb{C}^n$. Recall $F = (\mathbb{C}^*)^{n-1}$. Hence, $\overline{F} = T^{n-1} \times D^{n-1}$, and so $\partial\overline{F} = T^{n-1} \times S^{n-2}$.
- Let $\mathcal{A}$ be a near-pencil of n planes in $\mathbb{C}^3$. Then $\partial\overline{F} = S^1 \times \Sigma_{n-2}$.

The Hopf fibration $\pi\colon \mathbb{C}^{d+1}\backslash\{0\} \to \mathbb{CP}^d$ restricts to regular, cyclic n-fold covers, $\pi\colon \overline{F} \to \overline{U}$ and $\pi\colon \partial\overline{F} \to \partial\overline{U}$, which fit into the ladder

$$
\begin{array}{ccccccccc}
\mathbb{Z}_n & = & \mathbb{Z}_n & = & \mathbb{Z}_n & \longrightarrow & \mathbb{C}^* & = & \mathbb{C}^* \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\partial\overline{F} & \longrightarrow & \overline{F} & \xrightarrow{\simeq} & F & \longrightarrow & M & \longrightarrow & \mathbb{C}^{d+1}\backslash\{0\} \\
\downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\
\partial\overline{U} & \longrightarrow & \overline{U} & \xrightarrow{\simeq} & U & = & U & \longrightarrow & \mathbb{CP}^d
\end{array}
$$

Assume now that $d = 2$. The group $\pi_1(\partial\overline{U})$ has generators $x_1, \dots, x_{n-1}$ corresponding to the meridians around the first $n-1$ lines in $\mathbb{P}(\mathcal{A})$, and generators $y_1, \dots, y_s$ corresponding to the cycles in the associated graph Γ.

PROPOSITION (S. 2014)

The $\mathbb{Z}_n$-cover $\pi\colon \partial\overline{F} \to \partial\overline{U}$ is classified by the homomorphism $\pi_1(\partial\overline{U}) \twoheadrightarrow \mathbb{Z}_n$ given by $x_i \mapsto 1$ and $y_i \mapsto 0$.

EXAMPLE

Let $\mathcal{A}$ be a pencil of $n+1$ planes in $\mathbb{C}^3$. Since $\partial\overline{U} = \sharp^n S^1 \times S^2$, and $\partial\overline{F} \to \partial\overline{U}$ is a cover with $n+1$ sheets, we see that $\partial\overline{F} = \sharp^{n^2} S^1 \times S^2$.

THEOREM (NÉMETHI–SZILARD 2012)

Let $\mathcal{A}$ be an arrangement of n planes in $\mathbb{C}^3$. The characteristic polynomial of the algebraic monodromy acting on $H_1(\partial\overline{F}, \mathbb{C})$ is given by

$$\Delta(t) = \prod_{X \in L_2(\mathcal{A})} (t-1)(t^{\gcd(\mu(X)+1,n)} - 1)^{\mu(X)-1}.$$

- This shows that $b_1(\partial\overline{F})$ is a much less subtle invariant than $b_1(F)$: it depends only on the number and type of multiple points of $\mathbb{P}(\mathcal{A})$, but not on their relative position.
- On the other hand, the torsion in $H_1(\partial\overline{F}, \mathbb{Z})$ is still not understood.
- For a generic arrangement of n planes in $\mathbb{C}^3$, I expect that $H_1(\partial\overline{F}, \mathbb{Z}) = \mathbb{Z}^{n(n-1)/2} \oplus \mathbb{Z}_n^{(n-2)(n-3)/2}$.
- In general, it would be interesting to see whether all the torsion in $H_1(\partial\overline{F}(\mathcal{A}), \mathbb{Z})$ consists of $\mathbb{Z}_n$-summands, where $n = |\mathcal{A}|$.

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