Algebraic models and cohomology jump loci

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1. COHOMOLOGY JUMP LOCI
   - Resonance varieties of a cdga
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   - Gysin models and cohomology jump loci
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RESONANCE VARIETIES OF A CDGA

- Let $A = (A^{\bullet}, \mathrm{d})$ be a commutative, differential graded $\mathbb{C}$-algebra.
  - Multiplication $\cdot\colon A^i \otimes A^j \to A^{i+j}$ is graded-commutative.
  - Differential $\mathrm{d}\colon A^i \to A^{i+1}$ satisfies the graded Leibniz rule.
- Assume
  - A is connected, i.e., $A^0 = \mathbb{C}$.
  - A is of finite-type, i.e., $\dim A^i < \infty$ for all $i \geqslant 0$.
- For each $a \in Z^1(A) \cong H^1(A)$, we get a cochain complex,

$$(A^{\bullet}, \delta_a)\colon \; A^0 \xrightarrow{\delta_a^0} A^1 \xrightarrow{\delta_a^1} A^2 \xrightarrow{\delta_a^2} \cdots,$$

  with differentials $\delta_a^i(u) = a \cdot u + \mathrm{d}\, u$, for all $u \in A^i$.
- *Resonance varieties*:

$$\mathcal{R}^i(A) = \{a \in H^1(A) \mid H^i(A^{\bullet}, \delta_a) \neq 0\}.$$

- Fix $\mathbb{C}$-basis $\{e_1, \dots, e_n\}$ for $H^1(A)$, and let $\{x_1, \dots, x_n\}$ be dual basis for $H_1(A) = H^1(A)^\vee$.
- Identify $\mathrm{Sym}(H_1(A))$ with $S = \mathbb{C}[x_1, \dots, x_n]$, the coordinate ring of the affine space $H^1(A)$.
- Define a cochain complex of free S-modules,

$$(A^\bullet \otimes S, \delta)\colon \quad \cdots \longrightarrow A^i \otimes S \xrightarrow{\delta^i} A^{i+1} \otimes S \xrightarrow{\delta^{i+1}} A^{i+2} \otimes S \longrightarrow \cdots,$$

where $\quad \delta^i(u \otimes s) = \sum_{j=1}^n e_j u \otimes s x_j + \mathrm{d}\, u \otimes s$.

- The specialization of $A \otimes S$ at $a \in H^1(A)$ coincides with (A, δ_a).
- The cohomology support loci $\widetilde{\mathcal{R}}^i(A) = \mathrm{supp}(H^i(A^\bullet \otimes S, \delta))$ are subvarieties of $H_1(A)$.

- Let $(A_\bullet \otimes S, \partial)$ be the dual chain complex.
- The homology support loci $\widetilde{\mathcal{R}}_i(A) = \operatorname{supp}(H_i(A_\bullet \otimes S, \partial))$ are subvarieties of $H_1(A)$.
- Using a result of [Papadima–S. 2014], we obtain:

THEOREM

For each $q \geqslant 0$, the duality isomorphism $H^1(A) \cong H_1(A)$ restricts to an isomorphism $\bigcup_{i\leqslant q} \mathcal{R}^i(A) \cong \bigcup_{i\leqslant q} \widetilde{\mathcal{R}}_i(A)$.

- We also have $\mathcal{R}^i(A) \cong \mathcal{R}_i(A)$.
- In general, though, $\widetilde{\mathcal{R}}^i(A) \ncong \widetilde{\mathcal{R}}_i(A)$.
- If $d = 0$, then all the resonance varieties of A are homogeneous.
- In general, though, they are not.

EXAMPLE

- Let A be the exterior algebra on generators a, b in degree 1, endowed with the differential given by $\mathrm{d}\, a = 0$ and $\mathrm{d}\, b = b \cdot a$.
- $H^1(A) = \mathbb{C}$, generated by a. Set $S = \mathbb{C}[x]$. Then:

$$A_\bullet \otimes S\colon \ S \xrightarrow{\partial_2 = \binom{0}{x-1}} S^2 \xrightarrow{\partial_1 = (x \ 0)} S \,.$$

- Hence, $H_1(A_\bullet \otimes S) = S/(x-1)$, and so $\widetilde{\mathcal{R}}_1(A) = \{1\}$. Using the above theorem, we conclude that $\mathcal{R}^1(A) = \{0, 1\}$.
- $\mathcal{R}^1(A)$ is a non-homogeneous subvariety of $\mathbb{C}$.
- $H^1(A_\bullet \otimes S) = S/(x)$, and so $\widetilde{\mathcal{R}}^1(A) = \{0\} \neq \widetilde{\mathcal{R}}_1(A)$.

RESONANCE VARIETIES OF A SPACE

- Let X be a connected, finite-type CW-complex.
- We may take $A = H^*(X, \mathbb{C})$ with $d = 0$, and get the usual resonance varieties, $\mathcal{R}^i(X) := \mathcal{R}^i(A)$.
- Or, we may take (A, d) to be a finite-type cdga, weakly equivalent to Sullivan's model $A_{\mathrm{PL}}(X)$, if such a cdga exists.
- If X is *formal*, then $(H^*(X, \mathbb{C}), d = 0)$ is such a finite-type model.
- Finite-type cdga models exist even for possibly non-formal spaces, such as nilmanifolds and solvmanifolds, Sasakian manifolds, smooth quasi-projective varieties, etc.

THEOREM (MACINIC, PAPADIMA, POPESCU, S. – 2013)

Suppose there is a finite-type CDGA (A, d) *such that* $A_{\mathrm{PL}}(X) \simeq A$. *Then, for each* $i \geqslant 0$, *the tangent cone at* 0 *to the resonance variety* $\mathcal{R}^i(A)$ *is contained in* $\mathcal{R}^i(X)$.

In general, we cannot replace $\mathrm{TC}_0(\mathcal{R}^i(A))$ by $\mathcal{R}^i(A)$.

EXAMPLE

- Let $X = S^1$, and take $A = \bigwedge(a, b)$ with $\mathrm{d}\, a = 0$, $\mathrm{d}\, b = b \cdot a$.
- Then $\mathcal{R}^1(A) = \{0, 1\}$ is not contained in $\mathcal{R}^1(X) = \{0\}$, though $\mathrm{TC}_0(\mathcal{R}^1(A)) = \{0\}$ is.

- A rationally defined CDGA (A, d) has *positive weights* if each A^i can be decomposed into weighted pieces A^i_α, with positive weights in degree 1, and in a manner compatible with the CDGA structure:
  1. $A^i = \bigoplus_{\alpha\in\mathbb{Z}} A^i_\alpha$.
  2. $A^1_\alpha = 0$, for all $\alpha \leqslant 0$.
  3. If $a \in A^i_\alpha$ and $b \in A^j_\beta$, then $ab \in A^{i+j}_{\alpha+\beta}$ and $\mathrm{d}\, a \in A^{i+1}_\alpha$.
- A space X is said to have positive weights if $A_{\mathrm{PL}}(X)$ does.
- If X is formal, then X has positive weights, but not conversely.

THEOREM (DIMCA–PAPADIMA 2014, MPPS)

Suppose there is a rationally defined, finite-type CDGA (A, d) with positive weights, and a q-equivalence between $A_{\mathrm{PL}}(X)$ and A preserving $\mathbb{Q}$-structures. Then, for each $i \leqslant q$,

1. *$\mathcal{R}^i(A)$ is a finite union of rationally defined linear subspaces of $H^1(A)$.*
2. *$\mathcal{R}^i(A) \subseteq \mathcal{R}^i(X)$.*

EXAMPLE

- Let X be the 3-dimensional Heisenberg nilmanifold.
- All cup products of degree 1 classes vanish; thus, $\mathcal{R}^1(X) = H^1(X, \mathbb{C}) = \mathbb{C}^2$.
- Model $A = \bigwedge(a, b, c)$ generated in degree 1, with $\mathrm{d}\, a = \mathrm{d}\, b = 0$ and $\mathrm{d}\, c = a \cdot b$.
- This is a finite-dimensional model, with positive weights: $\mathrm{wt}(a) = \mathrm{wt}(b) = 1$, $\mathrm{wt}(c) = 2$.
- Writing $S = \mathbb{C}[x, y]$, we get

$$A_\bullet \otimes S\colon \ \cdots \longrightarrow S^3 \xrightarrow{\begin{pmatrix} y & 0 & 0 \\ -x & 0 & 0 \\ 1 & -x & -y \end{pmatrix}} S^3 \xrightarrow{\begin{pmatrix} x & y & 0 \end{pmatrix}} S\ .$$

- Hence $H_1(A_\bullet \otimes S) = S/(x, y)$, and so $\mathcal{R}^1(A) = \{0\}$.

CHARACTERISTIC VARIETIES

- Let X be a finite-type, connected CW-complex.
  - $\pi = \pi_1(X, x_0)$: a finitely generated group.
  - $\mathrm{Char}(X) = \mathrm{Hom}(\pi, \mathbb{C}^*)$: an abelian, algebraic group.
  - $\mathrm{Char}(X)^0 \cong (\mathbb{C}^*)^n$, where $n = b_1(X)$.
- Characteristic varieties of X:

$$\mathcal{V}^i(X) = \{\rho \in \mathrm{Char}(X) \mid H^i(X, \mathbb{C}_\rho) \neq 0\}.$$

THEOREM (LIBGOBER 2002, DIMCA–PAPADIMA–S. 2009)

$$\tau_1(\mathcal{V}^i(X)) \subseteq \mathrm{TC}_1(\mathcal{V}^i(X)) \subseteq \mathcal{R}^i(X)$$

- Here, if $W \subset (\mathbb{C}^*)^n$ is an algebraic subset, then

$$\tau_1(W) := \{z \in \mathbb{C}^n \mid \exp(\lambda z) \in W, \text{ for all } \lambda \in \mathbb{C}\}.$$

- This is a finite union of rationally defined linear subspaces of $\mathbb{C}^n$.

THEOREM (DIMCA–PAPADIMA 2014)

Suppose $A_{\mathrm{PL}}(X)$ *is* q*-equivalent to a finite-type model* (A, d)*. Then* $\mathcal{V}^i(X)_{(1)} \cong \mathcal{R}^i(A)_{(0)}$*, for all* $i \leqslant q$*.*

COROLLARY

If X *is a* q*-formal space, then* $\mathcal{V}^i(X)_{(1)} \cong \mathcal{R}^i(X)_{(0)}$*, for all* $i \leqslant q$*.*

- A precursor to corollary can be found in work of Green–Lazarsfeld on the cohomology jump loci of compact Kähler manifolds.
- The case when $q = 1$ was first established in [DPS-2009].
- Further developments in work of Budur–Wang [2013].

THE TANGENT CONE THEOREM

THEOREM

Suppose $A_{\mathrm{PL}}(X)$ is q-equivalent to a finite-type CDGA *A. Then, $\forall i \leqslant q$,*

1. $\mathrm{TC}_1(\mathcal{V}^i(X)) = \mathrm{TC}_0(\mathcal{R}^i(A))$.
2. *If, moreover, A has positive weights, and the q-equivalence $A_{\mathrm{PL}}(X) \simeq A$ preserves $\mathbb{Q}$-structures, then $\mathrm{TC}_1(\mathcal{V}^i(X)) = \mathcal{R}^i(A)$.*

THEOREM (DPS-2009, DP-2014)

Suppose X is a q-formal space. Then, for all $i \leqslant q$,

$$\tau_1(\mathcal{V}^i(X)) = \mathrm{TC}_1(\mathcal{V}^i(X)) = \mathcal{R}^i(X).$$

COROLLARY

If X is q-formal, then, for all $i \leqslant q$,

1. *All irreducible components of $\mathcal{R}^i(X)$ are rationally defined subspaces of $H^1(X, \mathbb{C})$.*
2. *All irreducible components of $\mathcal{V}^i(X)$ which pass through the origin are algebraic subtori of $\mathrm{Char}(X)^0$, of the form $\exp(L)$, where L runs through the linear subspaces comprising $\mathcal{R}^i(X)$.*

The Tangent Cone theorem can be used to detect non-formality.

EXAMPLE

- Let $\pi = \langle x_1, x_2 \mid [x_1, [x_1, x_2]] \rangle$.
- Then $\mathcal{V}^1(\pi) = \{t_1 = 1\}$, and so $\tau_1(\mathcal{V}^1(\pi)) = \mathrm{TC}_1(\mathcal{V}^1(\pi)) = \{x_1 = 0\}$.
- On the other hand, $\mathcal{R}^1(\pi) = \mathbb{C}^2$, and so π is not 1-formal.

EXAMPLE (DPS 2009)

Let $\pi = \langle x_1, \dots, x_4 \mid [x_1, x_2], [x_1, x_4][x_2^{-2}, x_3], [x_1^{-1}, x_3][x_2, x_4] \rangle$. Then $\mathcal{R}^1(\pi) = \{z \in \mathbb{C}^4 \mid z_1^2 - 2z_2^2 = 0\}$: a quadric which splits into two linear subspaces over $\mathbb{R}$, but is irreducible over $\mathbb{Q}$. Thus, π is not 1-formal.

EXAMPLE (S.–YANG–ZHAO 2015)

Let π be a finitely presented group with $\pi_{\text{ab}} = \mathbb{Z}^3$ and

$$\mathcal{V}^1(\pi) = \big\{(t_1, t_2, t_3) \in (\mathbb{C}^*)^3 \mid (t_2 - 1) = (t_1 + 1)(t_3 - 1)\big\},$$

This is a complex, 2-dimensional torus passing through the origin, but this torus does not embed as an algebraic subgroup in $(\mathbb{C}^*)^3$. Indeed,

$$\tau_1(\mathcal{V}^1(\pi)) = \{x_2 = x_3 = 0\} \cup \{x_1 - x_3 = x_2 - 2x_3 = 0\}.$$

Hence, π is not 1-formal.

GYSIN MODELS

- Let X be a (connected) smooth quasi-projective variety.
- Let $\overline{X}$ be a "good" compactification, i.e., $X = \overline{X} \setminus D$, for some normal-crossings divisor $D = \{D_1, \dots, D_m\}$.
- Algebraic model: $A = A(\overline{X}, D)$ (Morgan's *Gysin model*): a rationally defined, bigraded CDGA, with $A^i = \bigoplus_{p+q=i} A^{p,q}$ and

$$A^{p,q} = \bigoplus_{|S|=q} H^p\Big(\bigcap_{k\in S} D_k, \mathbb{C}\Big)(-q)$$

- Multiplication $A^{p,q} \cdot A^{p',q'} \subseteq A^{p+p',q+q'}$ from cup-product in $\overline{X}$.
- Differential $\mathrm{d}\colon A^{p,q} \to A^{p+2,q-1}$ from intersections of divisors.
- Model has positive weights: $\mathrm{wt}(A^{p,q}) = p + 2q$.
- Improved version by Dupont [2013]: divisor D is allowed to have "arangement-like" singularities.

- Suppose $X = \Sigma$ is a connected, smooth algebraic curve.
- Then Σ admits a canonical compactification, $\overline{\Sigma}$, and thus, a canonical Gysin model, $A(\Sigma)$.

EXAMPLE

Let $\Sigma = E^*$ be a once-punctured elliptic curve. Then $\overline{\Sigma} = E$, and

$$A(\Sigma) = \bigwedge(a, b, e)/(ae, be)$$

where a, b are in bidegree $(1, 0)$ and e in bidegree $(0, 1)$, while $\mathrm{d}\, a = \mathrm{d}\, b = 0$ and $\mathrm{d}\, e = ab$.

THE TANGENT CONE THEOREM

THEOREM (BUDUR, WANG 2013)

Let X be a smooth quasi-projective variety. Then each characteristic variety $\mathcal{V}^i(X)$ is a finite union of torsion-translated subtori of $\mathrm{Char}(X)$.

THEOREM

Let $A(X)$ be a Gysin model for X. Then, for each $i \geqslant 0$,

$$\tau_1(\mathcal{V}^i(X)) = \mathrm{TC}_1(\mathcal{V}^i(X)) = \mathcal{R}^i(A(X)) \subseteq \mathcal{R}^i(X).$$

Moreover, if X is q-formal, the last inclusion is an equality, for all $i \leqslant q$.

EXAMPLE

Let X be the $\mathbb{C}^*$-bundle over $E = S^1 \times S^1$ with $e = 1$. Then $\mathcal{V}^1(X) = \{1\}$, and so $\tau_1(\mathcal{V}^1(X)) = \mathrm{TC}_1(\mathcal{V}^1(X)) = \{0\}$. On the other hand, $\mathcal{R}^1(X) = \mathbb{C}^2$, and so X is not 1-formal.

A holomorphic map $f\colon X \to \Sigma$ is *admissible* if f is surjective, has connected generic fiber, and the target Σ is a connected, smooth complex curve with $\chi(X) < 0$.

THEOREM (ARAPURA 1997)

The map $f \mapsto f^*(\operatorname{Char}(\Sigma))$ yields a bijection between the set $\mathcal{E}_X$ of equivalence classes of admissible maps $X \to \Sigma$ and the set of positive-dimensional, irreducible components of $\mathcal{V}^1(X)$ containing 1.

THEOREM (DP 2014, MPPS 2013)

$$\mathcal{R}^1(A(X)) = \bigcup_{f\in\mathcal{E}_X} f^*(H^1(A(\Sigma))).$$

THEOREM (DPS 2009)

Suppose X is 1-formal. Then $\mathcal{R}^1(X) = \bigcup_{f\in\mathcal{E}_X} f^*(H^1(\Sigma,\mathbb{C}))$. Moreover, all the linear subspaces in this decomposition have dimension $\geqslant 2$, and any two distinct ones intersect only at 0.

Hyperplane arrangements

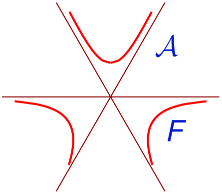
- An arrangement of hyperplanes is a finite set $\mathcal{A}$ of codimension-1 linear subspaces in $\mathbb{C}^n$.
- Its complement, $M(\mathcal{A}) = \mathbb{C}^\ell \setminus \bigcup_{H\in\mathcal{A}} H$, is a Stein manifold; thus, it is homotopic to a connected, finite cell complex of dimension n.
- The space $M = M(\mathcal{A})$ is formal, and so the OS-algebra $A = H^*(M, \mathbb{C})$ (with zero differential) is a model for M.

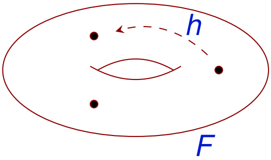
Theorem (Falk–Yuzvinsky 2007)

The set $\mathcal{E}_M$ is in bijection with multinets on sub-arrangements of $\mathcal{A}$. Each such k-multinet gives rise to a $(k-1)$-dimensional linear subspace of the resonance variety $\mathcal{R}^1(M) \subset H^1(M, \mathbb{C})$, and all components of $\mathcal{R}^1(M)$ arise in this fashion.

MILNOR FIBRATION

- For each $H \in \mathcal{A}$ let α_H be a linear form with $\ker(\alpha_H) = H$, and let $Q = \prod_{H \in \mathcal{A}} \alpha_H$ be a defining polynomial for $\mathcal{A}$.
- The restriction of the map $Q\colon \mathbb{C}^n \to \mathbb{C}$ to the complement is a smooth fibration, $Q\colon M \to \mathbb{C}^*$.
- The typical fiber of this fibration, $Q^{-1}(1)$, is called the *Milnor fiber* of the arrangement, and is denoted by $F = F(\mathcal{A})$.
- The monodromy diffeomorphism, $h\colon F \to F$, is given by $h(z) = \exp(2\pi \mathrm{i}/m)z$, where $m = |\mathcal{A}|$.





PROBLEM

Let $\mathcal{A}$ be a hyperplane arrangement, with Milnor fiber $F = F(\mathcal{A})$.

1. *Find a good compactification $\overline{F}$.*
2. *Does the monodromy $h\colon F \to F$ extend to a diffeomorphism $\bar{h}\colon \overline{F} \to \overline{F}$?*
3. *Write down an explicit presentation for the resulting Gysin model, $A(F)$.*
4. *Compute the resonance varieties $\mathcal{R}^i(A(F))$ and $\mathcal{R}^i(F)$, and decide whether they depend only on the intersection lattice of $\mathcal{A}$.*
5. *Decide whether these varieties coincide, and, if so, whether $F(\mathcal{A})$ is formal.*

Example (Zuber 2010)

- Let $\mathcal{A}$ be the arrangement in $\mathbb{C}^3$ defined by

$$Q = (z_1^3 - z_2^3)(z_1^3 - z_3^3)(z_2^3 - z_3^3).$$

- The variety $\mathcal{R}^1(M) \subset \mathbb{C}^9$ has 12 local components (from triple points), and 4 essential components (from 3-nets).
- One of these 3-nets corresponds to the rational map $\mathbb{CP}^2 \dashrightarrow \mathbb{CP}^1$, $(z_1, z_2, z_3) \mapsto (z_1^3 - z_2^3, z_2^3 - z_3^3)$.
- This map can be used to construct a 4-dimensional subtorus $T = \exp(L)$ inside $\operatorname{Char}(F(\mathcal{A})) = (\mathbb{C}^*)^{12}$.
- The linear subspace $L \subset H^1(F(\mathcal{A}), \mathbb{C})$ is not a component of $\mathcal{R}^1(F(\mathcal{A}))$.
- Thus, the tangent cone formula is violated, and so the Milnor fiber $F(\mathcal{A})$ is not 1-formal.

Elliptic arrangements

- An *elliptic arrangement* is a collection $\mathcal{A} = \{H_1, \dots, H_m\}$ of subvarieties in a product of elliptic curves E^n.
- Each $H_i \in \mathcal{A}$ is required to be of the form $H_i = f_i^{-1}(\zeta_i)$, for some $\zeta_i \in E$ and some homomorphism $f_i \colon E^{\times n} \to E$ given by

$$f_i(z_1, \dots, z_n) = \sum_{j=1}^{n} c_{ij} z_j \qquad (c_{ij} \in \mathbb{Z}).$$

- Let $\operatorname{corank} = n - \operatorname{rank}(c_{ij})$ and say $\mathcal{A}$ is *essential* if $\operatorname{corank}(\mathcal{A}) = 0$.

Theorem (Denham, S., Yuzvinsky 2014)

- *If $\mathcal{A}$ is essential, then the complement $M(\mathcal{A})$ is a Stein manifold.*
- *$M(\mathcal{A})$ is both a duality space and an abelian duality space of dimension $n + r$, where $r = \operatorname{corank}(\mathcal{A})$.*

- Let $L(\mathcal{A})$ denote the poset of all connected components of intersections elliptic hyperplanes from $\mathcal{A}$, ordered by inclusion.
- We say $\mathcal{A}$ is *unimodular* if all subspaces in $L(\mathcal{A})$ are connected.
- Let $A(\mathcal{A}) = \bigwedge(a_1, b_1, \dots, a_n, b_n, e_1, \dots, e_m)/I(\mathcal{A})$, where $I(\mathcal{A})$ is the ideal generated by the Orlik–Solomon relations among the $e_i's$, together with $f_i^*(a) \cdot e_i$ and $f_i^*(b) \cdot e_i$, for $1 \leqslant i \leqslant m$.
- Define $\mathrm{d}\colon A^{\bullet}(\mathcal{A}) \to A^{\bullet+1}(\mathcal{A})$ by setting $\mathrm{d}\, a_i = \mathrm{d}\, b_i = 0$ and $\mathrm{d}\, e_i = f_i^*(a) \cdot f_i^*(b)$.

THEOREM (BIBBY 2013)

Let $\mathcal{A}$ be an unimodular elliptic arrangement, and let $(A(\mathcal{A}), d)$ be the (rationally defined) CDGA from above. There is then a weak equivalence $A_{\mathrm{PL}}(M(\mathcal{A})) \simeq A(\mathcal{A})$ preserving $\mathbb{Q}$-structures.

Theorem

Let $\mathcal{A}$ be an unimodular elliptic arrangement. Then, for each $i \geqslant 0$,

$$\tau_1(\mathcal{V}^i(M(\mathcal{A}))) = \mathrm{TC}_1(\mathcal{V}^i(M(\mathcal{A}))) = \mathcal{R}^i(A(\mathcal{A})) \subseteq \mathcal{R}^i(M(\mathcal{A})),$$

with equality for $i \leqslant q$ if $M(\mathcal{A})$ if q-formal.

Problem

Let $\mathcal{A}$ be an unimodular elliptic arrangement, with complement $M(\mathcal{A})$ and intersection poset $L(\mathcal{A})$.

1. *Is the cohomology algebra $H^*(M(\mathcal{A}), \mathbb{C})$ determined by $L(\mathcal{A})$?*
2. *Are the resonance varieties $\mathcal{R}^i(A(\mathcal{A}))$ and $\mathcal{R}^i(M(\mathcal{A}))$ determined by $L(\mathcal{A})$?*
3. *Is there a combinatorial criterion to decide whether these varieties coincide, and, if so, whether $M(\mathcal{A})$ is formal?*

EXAMPLE

- Let $\mathcal{A}$ be the arrangement in $E^{\times 2}$ defined by the polynomial $f = z_1 z_2 (z_1 - z_2)$.
- Then $M(\mathcal{A}) = \mathrm{Conf}(E^*, 2)$, the configuration space of 2 labeled points on a punctured elliptic curve.
- Direct computation yields

$$\mathcal{R}^1(M(\mathcal{A})) = \{(x_1, x_2, y_1, y_2) \in \mathbb{C}^4 \mid x_1 y_2 - x_2 y_1 = 0\},$$

$$\mathcal{R}^1(A(\mathcal{A})) = \{x_1 = y_1 = 0\} \cup \{x_2 = y_2 = 0\} \cup \{x_1 + x_2 = y_1 + y_2 = 0\}.$$

- Thus, $M(\mathcal{A})$ is not 1-formal.

REFERENCES

- A.I. Suciu, *Around the tangent cone theorem*, arxiv:1502.02279.
- G. Denham, A.I. Suciu, S. Yuzvinsky, *Combinatorial covers and vanishing cohomology*, arxiv:1411.7981
- A. Dimca, S. Papadima, A.I. Suciu, *Algebraic models, Alexander-type invariants, and Green–Lazarsfeld sets*, Bull. Math. Soc. Sci. Math. Roumanie (to appear), arxiv:1407.8027.
- A. Măcinic, S. Papadima, R. Popescu, A.I. Suciu, *Flat connections and resonance varieties: from rank one to higher ranks*, arxiv:1312.1439.
- S. Papadima, A.I. Suciu, *Jump loci in the equivariant spectral sequence*, Math. Res. Lett. **21** (2014), no. 4, 863–883.
- S. Papadima, A.I. Suciu, *Non-abelian resonance: product and coproduct formulas*, in: Springer Proc. Math. Stat., vol. 96, 2014, pp. 269–280.
- A.I. Suciu, Y. Yang, G. Zhao, *Homological finiteness of abelian covers*, Ann. Sc. Norm. Super. Pisa Cl. Sci. **14** (2015), no. 1.