Geometric and homological finiteness properties

Alex Suciu

Northeastern University

Mini-Workshop on
Interactions between low-dimensional topology and complex algebraic geometry

Mathematisches Forshungsinstitut, Oberwolfach

October 27, 2017

Finiteness properties for spaces and groups

- A recurring theme in topology is to determine the geometric and homological finiteness properties of spaces and groups.
- For instance, to decide whether a path-connected space X is homotopy equivalent to a CW-complex with finite k-skeleton.
- A group G has property F_k if it admits a classifying space $K(G,1)$ with finite k-skeleton.
    - F_1: G is finitely generated;
    - F_2: G is finitely presentable.
- G has property FP_k if the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ admits a projective $\mathbb{Z}G$-resolution which is finitely generated in all dimensions up to k.
- The following implications (none of which can be reversed) hold:

$$\begin{aligned} G \text{ is of type } \mathrm{F}_k &\Rightarrow G \text{ is of type } \mathrm{FP}_k \\ &\Rightarrow H_i(G,\mathbb{Z}) \text{ is finitely generated, for all } i \leqslant k \\ &\Rightarrow b_i(G) < \infty, \text{ for all } i \leqslant k. \end{aligned}$$

- Moreover, FP_k & $\mathrm{F}_2 \Rightarrow \mathrm{F}_k$.

BIERI–NEUMANN–STREBEL–RENZ INVARIANTS

- (Bieri–Neumann–Strebel 1987) For a f.g. group G, let
$$\Sigma^1(G) = \{\chi \in S(G) \mid \mathcal{C}_\chi(G) \text{ is connected}\},$$
where $S(G) = (\mathrm{Hom}(G,\mathbb{R})\backslash\{0\})/\mathbb{R}^+$ and $\mathcal{C}_\chi(G)$ is the induced subgraph of $\mathrm{Cay}(G)$ on vertex set $G_\chi = \{g \in G \mid \chi(g) \geqslant 0\}$.
- $\Sigma^1(G)$ is an open set, independent of generating set for G.
- (Bieri, Renz 1988)
$$\Sigma^k(G,\mathbb{Z}) = \{\chi \in S(G) \mid \text{the monoid } G_\chi \text{ is of type } \mathrm{FP}_k\}.$$
In particular, $\Sigma^1(G,\mathbb{Z}) = \Sigma^1(G)$.
- The Σ-invariants control the finiteness properties of normal subgroups $N \lhd G$ for which G/N is free abelian:
$$N \text{ is of type } \mathrm{FP}_k \iff S(G,N) \subseteq \Sigma^k(G,\mathbb{Z})$$
where $S(G,N) = \{\chi \in S(G) \mid \chi(N) = 0\}$. In particular:
$$\ker(\chi\colon G \twoheadrightarrow \mathbb{Z}) \text{ is f.g.} \iff \{\pm\chi\} \subseteq \Sigma^1(G).$$

- Fix a connected CW-complex X with finite k-skeleton, for some $k \geqslant 1$. Let $G = \pi_1(X, x_0)$.
- For each $\chi \in S(X) := S(G)$, set

$$\widehat{\mathbb{Z}G}_\chi = \Big\{\lambda \in \mathbb{Z}^G \mid \{g \in \operatorname{supp} \lambda \mid \chi(g) < c\} \text{ is finite, } \forall c \in \mathbb{R}\Big\}.$$

  This is a ring, contains $\mathbb{Z}G$ as a subring; hence, a $\mathbb{Z}G$-module.
- (Farber, Geoghegan, Schütz 2010)

$$\Sigma^q(X, \mathbb{Z}) := \{\chi \in S(X) \mid H_i(X, \widehat{\mathbb{Z}G}_{-\chi}) = 0,\ \forall\, i \leqslant q\}.$$

- (Bieri) G is of type $\mathrm{FP}_k \implies \Sigma^q(G, \mathbb{Z}) = \Sigma^q(K(G,1), \mathbb{Z})$, $\forall q \leqslant k$.

DWYER–FRIED SETS

- For a fixed $r \in \mathbb{N}$, the connected, regular covers $Y \to X$ with group of deck-transformations $\mathbb{Z}^r$ are parametrized by the Grassmannian of r-planes in $H^1(X, \mathbb{Q})$.
- Moving about this variety, and recording when $b_1(Y), \dots, b_i(Y)$ are finite defines subsets $\Omega_r^i(X) \subseteq \operatorname{Gr}_r(H^1(X, \mathbb{Q}))$, which we call the *Dwyer–Fried invariants* of X.
- These sets depend only on the homotopy type of X. Hence, if G is a f.g. group, we may define $\Omega_r^i(G) := \Omega_r^i(K(G, 1))$.

EXAMPLE

Let K be a knot in S^3. If $X = S^3 \backslash K$, then $\dim_{\mathbb{Q}} H_1(X^{\text{ab}}, \mathbb{Q}) < \infty$, and so $\Omega_1^1(X) = \{\text{pt}\}$. But $H_1(X^{\text{ab}}, \mathbb{Z})$ need not be a f.g. $\mathbb{Z}$-module.

THEOREM

Let G be a f.g. group, and $\nu\colon G \twoheadrightarrow \mathbb{Z}^r$ an epimorphism, with kernel Γ. Suppose $\Omega^k_r(G) = \emptyset$, and Γ is of type F_{k-1}. Then $b_k(\Gamma) = \infty$.

- Proof: Set $X = K(G, 1)$; then $X^\nu = K(\Gamma, 1)$. Since Γ is of type F_{k-1}, we have $b_i(X^\nu) < \infty$ for $i \leqslant k-1$. Since $\Omega^k_r(X) = \emptyset$, we must have $b_k(X^\nu) = \infty$.

It follows that $H_k(\Gamma, \mathbb{Z})$ is not f.g., and Γ is not of type FP_k.

COROLLARY

Let G be a f.g. group, and suppose $\Omega^3_1(G) = \emptyset$. Let $\nu\colon G \twoheadrightarrow \mathbb{Z}$ be an epimorphism. If the group $\Gamma = \ker(\nu)$ is f.p., then $b_3(\Gamma) = \infty$.

THE STALLINGS GROUP

- Let $Y = S^1 \vee S^1$ and $X = Y \times Y \times Y$. Clearly, X is a classifying space for $G = F_2 \times F_2 \times F_2$.
- Let $\nu\colon G \to \mathbb{Z}$ be the homomorphism taking each standard generator to 1. Set $\Gamma = \ker(\nu)$.
- Stallings (1963) showed that Γ is finitely presented:

$$\Gamma = \langle a, b, c, x, y \mid [x, a], [y, a], [x, b], [y, b], [a^{-1}x, c], [a^{-1}y, c], [b^{-1}a, c] \rangle$$

- Stallings then showed, via a Mayer-Vietoris argument, that $H_3(\Gamma, \mathbb{Z})$ is not finitely generated.
- Alternate explanation: $\Omega_1^3(X) = \emptyset$. Thus, by the previous Corollary, a stronger statement holds: $b_3(\Gamma)$ is not finite.

KOLLÁR'S QUESTION

QUESTION (J. KOLLÁR 1995)

Given a smooth, projective variety M, is the fundamental group $G = \pi_1(M)$ commensurable, up to finite kernels, with another group, π, admitting a $K(\pi, 1)$ which is a quasi-projective variety?

(Two groups, G_1 and G_2, are said to be *commensurable up to finite kernels* if there is a zig-zag of groups and homomorphisms connecting them, with all arrows of finite kernel and cofinite image.)

THEOREM (DIMCA–PAPADIMA–S. 2009)

For each $k \geqslant 3$, there is a smooth, irreducible, complex projective variety M of complex dimension $k-1$, such that $\pi_1(M)$ is of type F_{k-1}, but not of type FP_k.

Further examples given by Llosa Isenrich and Bridson (2016/17).

COMMUTATIVE DIFFERENTIAL GRADED ALGEBRAS

- Let $A = (A^{\bullet}, \mathrm{d})$ be a commutative, differential graded algebra over a field $\Bbbk$ of characteristic 0. That is:
  - $A = \bigoplus_{i \geqslant 0} A^i$, where A^i are $\Bbbk$-vector spaces.
  - The multiplication $\cdot : A^i \otimes A^j \to A^{i+j}$ is graded-commutative, i.e., $ab = (-1)^{|a||b|} ba$ for all homogeneous a and b.
  - The differential $\mathrm{d} \colon A^i \to A^{i+1}$ satisfies the graded Leibniz rule, i.e., $\mathrm{d}(ab) = \mathrm{d}(a)b + (-1)^{|a|} a\,\mathrm{d}(b)$.
- A CDGA A is of *finite-type* (or q*-finite*) if it is connected (i.e., $A^0 = \Bbbk \cdot 1$) and $\dim A^i < \infty$ for all $i \leqslant q$.
- $H^{\bullet}(A)$ inherits an algebra structure from A.
- A cdga morphism $\varphi \colon A \to B$ is both an algebra map and a cochain map. Hence, it induces a morphism $\varphi^* \colon H^{\bullet}(A) \to H^{\bullet}(B)$.

- A map $\varphi\colon A \to B$ is a *quasi-isomorphism* if φ^* is an isomorphism. Likewise, φ is a q-quasi-isomorphism (for some $q \geqslant 1$) if φ^* is an isomorphism in degrees $\leqslant q$ and is injective in degree $q+1$.
- Two cdgas, A and B, are *(q-)equivalent* ($\simeq_q$) if there is a zig-zag of (q-)quasi-isomorphisms connecting A to B.
- A cdga A is *formal* (or just *q-formal*) if it is (q-)equivalent to $(H^\bullet(A), d=0)$.
- A CDGA is *q-minimal* if it is of the form $(\bigwedge V, d)$, where the differential structure is the inductive limit of a sequence of Hirsch extensions of increasing degrees, and $V^i = 0$ for $i > q$.
- Every CDGA A with $H^0(A) = \Bbbk$ admits a *q-minimal model*, $\mathcal{M}_q(A)$ (i.e., a q-equivalence $\mathcal{M}_q(A) \to A$ with $\mathcal{M}_q(A) = (\bigwedge V, d)$ a q-minimal cdga), unique up to iso.

ALGEBRAIC MODELS FOR SPACES

- Given any (path-connected) space X, there is an associated Sullivan $\mathbb{Q}$-cdga, $A_{\mathrm{PL}}(X)$, such that $H^\bullet(A_{\mathrm{PL}}(X)) = H^\bullet(X, \mathbb{Q})$.
- An *algebraic* (*q*-)*model* (over $\Bbbk$) for X is a $\Bbbk$-cgda (A, d) which is (q-) equivalent to $A_{\mathrm{PL}}(X) \otimes_{\mathbb{Q}} \Bbbk$.
- If M is a smooth manifold, then $\Omega_{\mathrm{dR}}(M)$ is a model for M (over $\mathbb{R}$).
- Examples of spaces having finite-type models include:
  - Formal spaces (such as compact Kähler manifolds, hyperplane arrangement complements, toric spaces, etc).
  - Smooth quasi-projective varieties, compact solvmanifolds, Sasakian manifolds, etc.

CHARACTERISTIC VARIETIES

- Let $\widehat{G} = \mathrm{Hom}(G, \mathbb{C}^*) = H^1(X, \mathbb{C}^*)$ be the character group of $G = \pi_1(X)$.
- The *characteristic varieties* of X are the sets
$$\mathcal{V}^i(X) = \{\rho \in \widehat{G} \mid H_i(X, \mathbb{C}_\rho) \neq 0\}.$$
- If X has finite k-skeleton, then $\mathcal{V}^i(X)$ is a Zariski closed subset of the algebraic group $\widehat{G}$, for each $i \leqslant k$.
- The varieties $\mathcal{V}^i(X)$ are homotopy-type invariants of X.
- $\mathcal{V}^1(X)$ depends only on $G = \pi_1(X)$. Set $\mathcal{V}^i(G) := \mathcal{V}^i(K(G, 1))$. Then $\mathcal{V}^1(G) = \mathcal{V}^1(G/G'')$.

EXAMPLE (S.–YANG–ZHANG – 2015)

Let $f \in \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ be an Laurent polynomial with $f(1) = 0$. There is then a f.p. group G with $G_{\mathrm{ab}} = \mathbb{Z}^n$ such that $\mathcal{V}^1(G) = \mathbf{V}(f)$.

RESONANCE VARIETIES OF A CDGA

- Let $A = (A^{\bullet}, \mathrm{d})$ be a connected, finite-type CDGA over $\mathbb{C}$.
- For each $a \in Z^1(A) \cong H^1(A)$, we get a cochain complex,

$$(A^{\bullet}, \delta_a)\colon \; A^0 \xrightarrow{\delta_a^0} A^1 \xrightarrow{\delta_a^1} A^2 \xrightarrow{\delta_a^2} \cdots,$$

  with differentials $\delta_a^i(u) = a \cdot u + \mathrm{d}\, u$, for all $u \in A^i$.
- The *resonance varieties* of A are the affine varieties

$$\mathcal{R}^i(A) = \{a \in H^1(A) \mid H^i(A^{\bullet}, \delta_a) \neq 0\}.$$

- If X is a connected, finite-type CW-complex, we get the usual resonance varieties by setting $\mathcal{R}^i(X) := \mathcal{R}^i(H^{\bullet}(X, \mathbb{C}))$.

Infinitesimal finiteness obstructions

Question

Let X be a connected CW-complex with finite q-skeleton. Does X admit a q-finite q-model A?

Theorem

If X is as above, then, for all $i \leqslant q$:

- *(Dimca–Papadima 2014)* $\mathcal{V}^i(X)_{(1)} \cong \mathcal{R}^i(A)_{(0)}$.
  In particular, if X is q-formal, then $\mathcal{V}^i(X)_{(1)} \cong \mathcal{R}^i(X)_{(0)}$.
- *(Macinic, Papadima, Popescu, S. 2017)* $\mathrm{TC}_0(\mathcal{R}^i(A)) \subseteq \mathcal{R}^i(X)$.
- *(Budur–Wang 2017) All the irreducible components of* $\mathcal{V}^i(X)$ *passing through the origin of* $H^1(X, \mathbb{C}^*)$ *are* algebraic subtori.

Example

Let G be a f.p. group with $G_{\mathrm{ab}} = \mathbb{Z}^n$ and $\mathcal{V}^1(G) = \{t \in (\mathbb{C}^*)^n \mid \sum_{i=1}^n t_i = n\}$. Then G admits no 1-finite 1-model.

THEOREM (PAPADIMA–S. 2017)

Suppose X is $(q+1)$ finite, or X admits a q-finite q-model. Then $b_i(\mathcal{M}_q(X)) < \infty$, for all $i \leqslant q+1$.

COROLLARY

Let G be a f.g. group. Assume that either G is finitely presented, or G has a 1-finite 1-model. Then $b_2(\mathcal{M}_1(G)) < \infty$.

EXAMPLE

- Consider the free metabelian group $G = F_n / F_n''$ with $n \geqslant 2$.
- We have $\mathcal{V}^1(G) = \mathcal{V}^1(F_n) = (\mathbb{C}^*)^n$, and so G passes the Budur–Wang test.
- But $b_2(\mathcal{M}_1(G)) = \infty$, and so G admits no 1-finite 1-model (and is not finitely presented).

Bounding the Σ-invariants

- Let $\exp\colon H^1(X,\mathbb{C}) \to H^1(X,\mathbb{C}^*)$ be the coefficient homomorphism induced by $\mathbb{C} \to \mathbb{C}^*$, $z \mapsto e^z$.
- Given a Zariski closed subset $W \subset H^1(X,\mathbb{C}^*)$, set

$$\tau_1(W) = \{z \in H^1(X,\mathbb{C}) \mid \exp(\lambda z) \in W,\ \forall \lambda \in \mathbb{C}\}.$$

- $\tau_1(W)$ is a finite union of rationally defined linear subspaces.
- Set $\tau_1^{\Bbbk}(W) = \tau_1(W) \cap H^1(X,\Bbbk)$ for $\Bbbk \subset \mathbb{C}$; $\mathcal{W}^i(X) = \bigcup_{j\leqslant i} \mathcal{V}^j(X)$.

Theorem (Papadima–S. 2010)

$$\Sigma^i(X,\mathbb{Z}) \subseteq S(X) \backslash S(\tau_1^{\mathbb{R}}(\mathcal{W}^i(X))). \qquad (\dagger)$$

If X is formal, we may replace $\tau_1^{\mathbb{R}}(\mathcal{W}^i(X))$ with $\bigcup_{j\leqslant i} \mathcal{R}^j(X,\mathbb{R})$.

Example (Koban–McCammond–Meier 2015)

$$\Sigma^1(P_n) = \mathcal{R}^1(P_n,\mathbb{R})^{\complement}.$$

BOUNDING THE Ω-INVARIANTS

THEOREM (DWYER–FRIED 1987, PAPADIMA–S. 2010)

Let $\nu\colon \pi_1(X) \twoheadrightarrow \mathbb{Z}^r$ *be an epimorphism. Then* $\bigoplus_{i=0}^{k} H_i(X^\nu, \mathbb{C})$ *is finite-dimensional if and only if the algebraic torus* $\operatorname{im}\left(\hat{\nu}\colon \widehat{\mathbb{Z}^r} \hookrightarrow \widehat{\pi_1(X)}\right)$ *intersects* $\mathcal{W}^k(X)$ *in only finitely many points.*

COROLLARY (S. 2014)

$$\Omega^i_r(X) = \left\{P \in \operatorname{Gr}_r(H^1(X,\mathbb{Q})) \mid \dim\left(\exp(P\otimes\mathbb{C}) \cap \mathcal{W}^i(X)\right) = 0\right\}.$$

Given a homogeneous variety $V \subset \Bbbk^n$, the set $\sigma_r(V) = \left\{P \in \operatorname{Gr}_r(\Bbbk^n) \mid P \cap V \neq \{0\}\right\}$ is Zariski closed.

THEOREM (S. 2012/2014)

$$\Omega^i_r(X) \subseteq \operatorname{Gr}_r(H^1(X,\mathbb{Q})) \setminus \sigma_r(\tau_1^{\mathbb{Q}}(\mathcal{W}^i(X))).$$

If the upper bound for the Σ*-invariants is attained, then the upper bound for the* Ω*-invariants is also attained.*

RFRp GROUPS

- Let G be a f.g. group and let p be a prime.
- We say that G is *residually finite rationally p* if there exists a sequence of subgroups $G = G_0 > \cdots > G_i > G_{i+1} > \cdots$ such that
  1. $G_{i+1} \triangleleft G_i$.
  2. $\bigcap_{i\geqslant 0} G_i = \{1\}$.
  3. G_i / G_{i+1} is an elementary abelian p-group.
  4. $\ker(G_i \to H_1(G_i, \mathbb{Q})) < G_{i+1}$.
- The class of RFRp groups is closed under taking subgroups, finite direct products, and finite free products.
- Finitely generated free groups; closed, orientable surface groups; and right-angled Artin groups are RFRp, for all p.
- Finite groups and non-abelian nilpotent groups are *not* RFRp, for any p.

THEOREM (KOBERDA–S. 2016)

Let G be a f.g. group which is RFRp for some prime p. Then:

- *G is residually finite. In particular, if G is finitely presented, then G has a solvable word problem.*
- *G is torsion-free.*
- *G is residually torsion-free polycyclic.*

THEOREM

Let G be a f.p. group which is non-abelian and RFRp for infinitely many primes p. Then:

- *G is bi-orderable.*
- *The maximal k-step solvable quotients $G/G^{(k)}$ are not finitely presented, for any $k \geqslant 2$.*
- $\Sigma^1(G)^{\mathrm{c}} \neq \emptyset$.

LARGE GROUPS

A finitely generated group G is said to be *large* if there is a finite-index subgroup $H < G$ which surjects onto a free, non-cyclic group.

THEOREM (KOBERDA 2014)

An f.p. group G is large if and only if there exists a finite-index subgroup $K < G$ such that $\mathcal{V}^1(K)$ has infinitely many torsion points.

THEOREM (KS 2016)

Let G be a f.p. group which is non-abelian and RFRp for infinitely many primes p. Then G is large.

PROPOSITION (PS 2017, FOLLOWS FROM ARAPURA)

Let X be a quasi-projective manifold. Then $\pi_1(X)$ is large if and only if there is a finite cover $Y \to X$ and a regular, surjective map from Y to a smooth curve C with $\chi(C) < 0$, so that the generic fiber is connected.

BOUNDARY MANIFOLDS OF PLANE CURVES

- Let $\mathcal{C}$ be a (reduced) algebraic curve in $\mathbb{CP}^2$.
- The *boundary manifold* of $\mathcal{C}$ is defined as $M_{\mathcal{C}} = \partial T$, where T is a regular neighborhood of $\mathcal{C}$.
- $M = M_{\mathcal{C}}$ is a closed, oriented graph-manifold over a graph Γ.

EXAMPLE

Suppose $\mathcal{C}$ is smooth. Then $\mathcal{C} \cong \Sigma_g$, where $g = \binom{d-1}{2}$, and $d = \deg(\mathcal{C})$. Thus, $M_{\mathcal{C}}$ is a circle bundle over Σ_g with Euler number $e = d^2$.

In this example, $\pi_1(M)$ is not RFRp, for any prime p, provided $d \geqslant 2$.

EXAMPLE

Suppose $\mathcal{C} = C \cup L$ consists of a smooth conic and a transverse line. The graph Γ is a square, the vertex manifolds are thickened tori $S^1 \times S^1 \times I$, and $M_{\mathcal{C}}$ is the Heisenberg nilmanifold.

In this example, $\pi_1(M)$ is not RFRp, for any prime p.

QUESTION

For which plane algebraic curves $\mathcal{C}$ is the fundamental group of the boundary manifold $M_{\mathcal{C}}$ an RFRp group (for some p or all primes p)?

THEOREM (KS 2016)

Let $\mathcal{C}$ be an algebraic curve in $\mathbb{C}^2$, with boundary manifold M. Suppose that each irreducible component of $\mathcal{C}$ is smooth and transverse to the line at infinity, and all singularities of $\mathcal{C}$ are of type A. Then $\pi_1(M)$ is RFRp, for all primes p.

COROLLARY

If M is the boundary manifold of a line arrangement in $\mathbb{C}^2$, then $\pi_1(M)$ is RFRp, for all primes p.

CONJECTURE

Arrangement groups are RFRp, for all primes p.

ASSOCIATED GRADED LIE ALGEBRAS

- The *lower central series* of a group G is defined inductively by $\gamma_1 G = G$ and $\gamma_{k+1} G = [\gamma_k G, G]$.
- This forms a filtration of G by characteristic subgroups. The LCS quotients, $\gamma_k G/\gamma_{k+1} G$, are abelian groups.
- The group commutator induces a graded Lie algebra structure on

$$\mathrm{gr}(G, \Bbbk) = \bigoplus\nolimits_{k \geqslant 1} (\gamma_k G/\gamma_{k+1} G) \otimes_{\mathbb{Z}} \Bbbk.$$

- Assume G is finitely generated. Then $\mathrm{gr}(G)$ is also finitely generated (in degree 1) by $\mathrm{gr}_1(G) = H_1(G, \Bbbk)$.
- For instance, $\mathrm{gr}(F_n)$ is the free graded Lie algebra $\mathbb{L}_n := \mathrm{Lie}(\Bbbk^n)$.

HOLONOMY LIE ALGEBRAS

- Let A be a 1-finite cdga. Set $A_i = (A^i)^*$.
- Let $\mu^*\colon A_2 \to A_1 \wedge A_1$ be the dual to the multiplication map $\mu\colon A^1 \wedge A^1 \to A^2$.
- Let $d^*\colon A_2 \to A_1$ be the dual of the differential $d\colon A^1 \to A^2$.
- The *holonomy Lie algebra* of A is the quotient

$$\mathfrak{h}(A) = \mathrm{Lie}(A_1)/\langle \mathrm{im}(\mu^* + d^*)\rangle.$$

- For a f.g. group G, set $\mathfrak{h}(G) := \mathfrak{h}(H^\bullet(G, \Bbbk))$. There is then a canonical surjection $\mathfrak{h}(G) \twoheadrightarrow \mathrm{gr}(G)$, which is an isomorphism precisely when $\mathrm{gr}(G)$ is quadratic.

MALCEV LIE ALGEBRAS

- Let G be a f.g. group. The successive quotients of G by the terms of the LCS form a tower of finitely generated, nilpotent groups,

$$\cdots \longrightarrow G/\gamma_4 G \longrightarrow G/\gamma_3 G \longrightarrow G/\gamma_2 G = G_{ab} .$$

- (Malcev 1951) It is possible to replace each nilpotent quotient N_k by $N_k \otimes \Bbbk$, the (rationally defined) nilpotent Lie group associated to the discrete, torsion-free nilpotent group $N_k / \operatorname{tors}(N_k)$.
- The inverse limit, $\mathfrak{M}(G) = \varprojlim_k (G/\gamma_k G) \otimes \Bbbk$, is a prounipotent, filtered Lie group, called the *prounipotent completion* of G over $\Bbbk$.
- The pronilpotent Lie algebra

$$\mathfrak{m}(G) := \varprojlim_k \mathfrak{Lie}((G/\gamma_k G) \otimes \Bbbk),$$

endowed with the inverse limit filtration, is called the *Malcev Lie algebra* of G (over $\Bbbk$).

- By dualizing the canonical filtration of $\mathcal{M}_1(G)$, we obtain a tower of central extensions of finite-dimensional nilpotent Lie algebras,

$$\cdots \twoheadrightarrow \mathfrak{m}_{n+1} \twoheadrightarrow \mathfrak{m}_n \twoheadrightarrow \cdots \twoheadrightarrow \mathfrak{m}_1 = \{0\}\ ;$$

  $\mathfrak{m}(G)$ is isomorphic to the inverse limit of this tower.
- The group-algebra $\Bbbk G$ has a natural Hopf algebra structure, with comultiplication $\Delta(g) = g \otimes g$ and counit the augmentation map.
- (Quillen 1968) The I-adic completion of the group-algebra, $\widehat{\Bbbk G} = \varprojlim_k \Bbbk G / I^k$, is a filtered, complete Hopf algebra.
- An element $x \in \widehat{\Bbbk G}$ is called *primitive* if $\widehat{\Delta} x = x \hat{\otimes} 1 + 1 \hat{\otimes} x$. The set of all such elements, with bracket $[x, y] = xy - yx$, and endowed with the induced filtration, is a complete, filtered Lie algebra.
- We then have $\mathfrak{m}(G) \cong \operatorname{Prim}(\widehat{\Bbbk G})$ and $\operatorname{gr}(\mathfrak{m}(G)) \cong \operatorname{gr}(G)$.
- (Sullivan 1977) G is 1-formal $\Longleftrightarrow$ $\mathfrak{m}(G)$ is quadratic.

FINITENESS OBSTRUCTIONS FOR GROUPS

LEMMA

For $n \geqslant 2$, the graded vector space $\mathbb{L}''_n / [\mathbb{L}_n, \mathbb{L}''_n]$ is infinite-dimensional.

THEOREM (PS 2017)

Let G be a f.g. group which has a free, non-cyclic quotient. Then:

- *G/G'' is not finitely presentable.*
- *G/G'' does not admit a 1-finite 1-model.*

THEOREM (PS 2017)

A f.g. group G admits a 1-finite 1-model A if and only if $\mathfrak{m}(G)$ is the lcs completion of a finitely presented Lie algebra, namely,

$$\mathfrak{m}(G) \cong \widehat{\mathfrak{h}(A)}.$$