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Mini-Workshop: Interactions between Low-dimensional Topology and Complex Algebraic Geometry

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Abstract. Recent developments exhibit a strong connection between low-dimensional topology and complex algebraic geometry. A common theme is provided by the Alexander polynomial and its many avatars. The mini-Workshop brought together at Oberwolfach groups of researchers working in mostly separate areas, but sharing common interests in a vibrant, emerging field at the crossroads of Topology, Group Theory, and Geometry.

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Introduction by the Organisers

Overview. The organizers brought together at Oberwolfach two groups of researchers who have worked over the past few years, mostly independently, on questions strongly linked by the study of algebraic invariants that occur both in low-dimensional topology and in complex algebraic geometry. One group consisted of topologists who mostly study knots and links and closed 3-manifolds, while the other group consisted of geometers who mostly study algebraic plane curves, smooth projective varieties, or singularities. Several participants were recent either Ph.D. students or recent Ph.D.s, many of them on their first visit to MFO. In all, there were 17 mathematicians attending the mini-workshop (including the organizers), coming from the Canada, France, Germany, Hungary, India, Italy, Korea, Poland, Spain, and the United States.

The meeting allowed us to compare some closely related constructions and find some common ground within the scope of rather varied disciplinary perspectives. The relatively spontaneous format of the meeting made it possible to mix some informal and semi-expository talks and small group discussions with more formal announcements of recent developments, indicated in the abstracts that follow. The fact that we were a small group led to much more interaction during talks and to much more lively discussions than is usually experienced at bigger meetings. The very interactive nature of this mini-workshop was extremely beneficial for everybody since researchers with very different mathematical background came together.

Research themes. The (single variable) Alexander polynomials were first introduced to study the topology of knot complements in the 3-sphere. They are easier to handle than the corresponding fundamental groups, which are highly non-commutative in general. In a series of beautiful papers written in the 1980s and 1990s, Libgober had the idea to use the Alexander polynomials for the study of the topology of algebraic plane curve complements and, later on, that of complements of complex affine hypersurfaces with only isolated singularities. He showed that the Alexander polynomial of a plane curve complement is a fine enough invariant to detect Zariski pairs (i.e., pairs of plane curves which have homeomorphic regular neighborhoods, but non-homeomorphic complements), and, moreover, its zeros are among those of the local Alexander polynomials associated to the link pairs of singular points (hence they are all roots of unity).

In relation to an old question of Serre, such rigidity-type results for Alexander polynomials preclude many knot groups from being realized as fundamental groups of plane curve complements. Libgober's results were more recently extended to hypersurfaces with arbitrary singularities by Maxim, Dimca, Liu, and others. This fact generated a flurry of activities in many exciting directions.

In the past few years, Alexander polynomials of plane curve complements have found deep applications in algebraic geometry, singularity theory, and number theory. For example, Cogolludo and Libgober established an intriguing connection between the rank of the Mordell-Weil group of certain isotrivial elliptic threefolds with base $\mathbb{P}^2$ and the vanishing order of the Alexander polynomial of the reduction of the discriminant of the elliptic fibration.

The Alexander polynomial also describes the algebraic monodromy Milnor fibration associated to an arrangement of lines in $\mathbb{P}^2$. Recent work of Papadima and Suciu shows that such an Alexander polynomial is determined by the combinatorics of the arrangement if the lines intersect only in double and triple points, but the general case remains open.

The classical Alexander polynomial of knots and links have been generalized by Ozsváth-Szabó to a whole package of 3-manifold invariants, called the Heegaard–Floer invariants. These objects are structurally much harder to deal with than Alexander polynomials, but they also contain significantly more information. The study of interactions between Heegaard–Floer invariants and algebraic geometry has been an increasingly active area of research. Hereby the interaction goes both ways.

We first outline how Heegaard–Floer homology can inform our understanding of plane curve singularities. Campillo, Delgado, and Gusein-Zade made a connection between the Alexander polynomial of the link of a plane curve singularity and the semigroup of the singular point. On the other hand, the Alexander polynomial of an algebraic knot determines the Heegaard–Floer chain complex of this knot. These two ingredients allowed Borodzik and Livingston to draw a connection between Heegaard–Floer invariants of algebraic links with the relevant semigroups. As a result, based on the fundamental inequality for the d-invariants of Ozsváth and Szabó, they were able to prove the conjecture of Fernández de Bobadilla, Luengo, Melle-Hernández, and Némethi on the semigroup distribution property of a rational cuspidal curve in $\mathbb{CP}^2$.

Conversely, algebraic geometry can further our understanding of Heegaard–Floer theory. For instance, in 2005 Némethi introduced lattice homology, which is a combinatorial object associated to a resolution graph of a surface singularity. Different resolutions of the same singular point yield isomorphic homology groups, hence the lattice homology is an invariant of a surface singularity. It was proved by Némethi that the lattice homology is isomorphic to the Heegaard–Floer homology of the link of the singularity, as long as the resolution graph is almost rational.

Structure of the mini-workshop. The schedule of the meeting comprised 16 lectures of one hour each, including 14 research talks, and a 3-lectures series by A. Némethi. Speakers presented recent progress on open problems in some of the above-mentioned research themes.

For instance, recall that a Kähler group is a group that can be realized as the fundamental group of a connected, compact Kähler manifold. It is still an open problem (attributed to Serre) to completely classify Kähler groups. In his talk, M. Mj explained how to use the theory of cuts developed by Delzant–Gromov to decide completely which 3-manifold groups are Kähler, which one-relator groups are Kähler, and so on. P. Py discussed his work with T. Delzant on a (virtual) classification of cubulable Kähler groups, i.e., Kähler groups which admit actions on CAT(0) cubical complexes.

Several talks were dedicated to the study of homological duality properties of complex algebraic manifolds and of their fundamental groups, and various applications. For instance, G. Denham presented his recent work with A. Suciu, in which they show that complements of linear, toric, and elliptic arrangements are both duality and abelian duality spaces. Such spaces have good vanishing properties for their cohomology with rank-one local system coefficients. Other examples of abelian duality spaces were discussed in the talk of Y. Liu (who presented joint work with L. Maxim and B. Wang), by making use of the formalism of perverse sheaves. In a different vein, A. Suciu presented his work with S. Papadima on homological and geometric finiteness properties of groups, and related work with T. Koberda on the RFRp property of groups.

Fundamental groups of quasi-projective manifolds were also the main topic of talks by J.I. Cogolludo-Agustín and E. Artal Bartolo. More precisely, Cogolludo-Agustín presented his joint work with A. Libgober on the asymptotic behavior of certain invariants (Alexander-type invariants and the number of surjections onto free groups) of the fundamental groups of complements of divisors on smooth projective surfaces, whereas Artal Bartolo discussed his joint work with J.I. Cogolludo-Agustín and J. Martín-Morales on arithmetic Zariski tuples.

Several deep connections between low-dimensional topology and complex geometry were established and emphasized. For instance, M. Borodzik explained his joint project with J. Hom and A. Schinzel on how to use invariants coming from involutive Heegard Floer theory to restrict possible configurations of singular points of a planar rational cuspidal curve. Furthermore, A. Némethi presented in his lecture series a beautiful interplay between algebraic geometry (via the theory of local complex normal surface singularities), low-dimensional topology (Heegaard Floer homology and foliations), and group theory (left-orderability property of the fundamental group).

A number of talks targeted topics in low-dimensional topology. For example, J.C. Cha gave a brief overview of the use of Cheeger–Gromov's L^2 ρ-invariants in dimensions three and four. E. Toffoli presented his results on ρ-invariants for manifolds with boundary, with applications to concordance. Concordance of knots and links was also the subject of S. Harvey's talk, who reported on her joint work with C. Leidy on pure braids, Whitney towers, and 0-solvability.

Other aspects of the interplay between topology and complex geometry were also present in the talks by T. Koberda and L. Flapan. Specifically, Koberda discussed the relationship between the algebraic structure of a group G and the possible degrees of regularity of faithful action of G on a compact, one-dimensional manifold, with applications to complex geometry. Flapan presented her recent results on the study of monodromy of Kodaira fibrations, and obtaining new obstructions on groups which can be realized as monodromy groups of such fibrations.

Concluding remarks. The mini-Workshop was the ideal place to present, discuss and further develop the ideas and results that are currently emerging in the different research groups that were brought together at Oberwolfach. Spending a concentrated and highly intense week in a relatively small group allowed for in-depth and continuing discussions during lectures and , in particular with new acquaintances. These opportunities (difficult to find at larger meetings) were enhanced by the diversity of backgrounds of the participants. This speaks to the fact that the usual, more rigid conference climate was superseded by an open and creative workshop atmosphere.

There was general agreement that the mini-workshop created an effective and stimulating research atmosphere. The work initiated at Oberwolfach is continuing now in several research groups. The free flow of ideas and the intense interactions at the meeting gave rise to new projects, which should start bearing fruit in the not too distant future.

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Workshop: Mini-Workshop: Interactions between Low-dimensional Topology and Complex Algebraic Geometry

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Abstracts

Cheeger–Gromov L^2 ρ-invariants and low dimensional topology

Jae Choon Cha

In this extended abstract, we give a brief introduction, for non-experts, on some recent results in dimension three and four which use the Cheeger–Gromov L^2 ρ-invariant as a key ingredient. In what follows, manifolds are compact and oriented.

Cheeger-Gromov ρ-invariants. In [11], for a closed $(4k-1)$-manifold M and a group homomorphism $\phi\colon \pi_1(M) \to G$, Cheeger and Gromov defined a real number $\rho^{(2)}(M, \phi)$. Unless specified otherwise, M will always be 3-dimensional in what follows. Briefly, $\rho^{(2)}(M, \phi)$ is the difference of the η-invariant of the signature operator on M and its L^2-analogue on the G-cover of M associated to ϕ. By index theoretic arguments, it can be shown that the above definition is equivalent to the L^2-signature defect of a 4-manifold bounded by M over a group containing G. Indeed, using the L^2-induction property, Weinberger observed that the $\rho^{(2)}(M, \phi)$ can always *defined* in this way. For more details on the definition of $\rho^{(2)}(M, \phi)$, the readers are referred to, for instance, [11, 13, 10, 14, 7, 1, 3].

In dimension 4: Disk embedding and knot concordance. In the landmark work of Cochran, Orr, and Teichner [13], it was first shown that certain Cheeger–Gromov invariants give obstructions to topological knot concordance. Since then, there have been remarkable amount of new results and applications which use the Cheeger–Gromov invariant in this direction. The state-of-the-art technology which generalizes Cochran–Orr–Teichner gives obstructions from Cheeger–Gromov invariants over *amenable* groups. We recall that a (discrete) group is *amenable* if it admits a finitely additive invariant mean. For a prime p, a group is called *locally p-indicable* if every finitely generated nontrivial subgroup surjects onto $\mathbb{Z}/p\mathbb{Z}$. For a knot K in S^3, let M_K be the zero-framed surgery manifold. The following result appeared in [1] and is proven using the results and ideas in [7].

Theorem 1 (Amenable signature theorem [1])**.** *Suppose K is a slice knot and W is the exterior of a slicing disk for K in D^4. Note that $\partial W = M_K$. Suppose G is an amenable group which is locally indicable for some p, and $\phi\colon \pi_1(M_K) \to G$ is a homomorphism factoring through $\pi_1(W)$. Then $\rho^{(2)}(M_K, \phi) = 0$.*

We remark that there are other versions of Theorem 1, for homology cobordism of closed 3-manifolds [7], and more generally Whitney tower cobordism of bordered 3-manifolds [2].

Among the applications, there are results on the structure of knot concordance group, homology cobordism, link concordance, double concordance, Whitney towers and gropes. For instance see [8, 9, 4, 17, 5, 18, 19]. A recent interesting application is a result on the *smooth* knot concordance: in [6], a Whitney tower version of the amenable signature theorem is combined with the Heegaard Floer d-invariants to show that the bipolar filtration of Cochran, Harvey and Horn [12] has rich structures which are invisible to the eyes of modern smooth invariants including the s, d, τ, ϵ, ν^+ and Υ-invariants.

In dimension 3: Universal bound and complexity. In the original work of Cheeger and Gromov [11], they proved the following result, which played an important role in their proof of index theoretic results:

Theorem 2 (Cheeger-Gromov [11])**.** *For each smooth closed $(4k-1)$-manifold M, there is a constant C_M such that $|\rho^{(2)}(M,\phi)| \le C_M$ for every $\phi\colon \pi_1(M) \to G$.*

We remark that the codomain G of ϕ is allowed to arbitrarily vary.

It was first observed by Cochran and Teichner [13] that the existence of C_M, which is often called the universal bound, is exetremely useful in 4-dimensional applications. From this a natural question arises: *can we understand C_M from a topological viewpoint?* Since the approach of Cheeger and Gromov used deep analytic arguments, it was difficult to obtain topological information.

Recently, in [3], a topological argument which proves the existence of C_M was first given. For 3-manifolds, furthermore, we have *quantitative* results.

Theorem 3 (Cha [3])**.** *Suppose M is a closed 3-manifold admitting a triangulation, namely a simplicial complex structure, with n tetrahedra. Then,*

$$|\rho^{(2)}(M,\phi)| \le 363\,090 \cdot n \quad \text{for any } \phi\colon \pi_1(M) \to G.$$

In [3], we also present explicit quantitative universal bounds from Heegaard splittings and from surgery presentations. Our universal bounds are shown to be asymptotically optimal.

The above results have many applications which promote existence results on concordance, homology cobordism, Whitney towers and gropes to explicit constructions. For this purpose one uses Theorem 3 as an upper bound for the absolute value of the Cheeger–Gromov invariants.

While its proof is fully 4-dimensional, the statement of Theorem 3 is purely 3-dimensional. This gives unexpected 3-dimensional applications, which are obtained by using the inequality as a lower bound of the number of tetrahedra. In the literature, the *complexity* $c(M)$ of a 3-manifold M is defined to be the minimum number of tetrahedra in a pseudo-simplicial triangulation, that is, a collection of standard tetrahedra together with affine identifications of faces whose quotient space is M. Finding an efficient lower bound of the complexity has been regarded as a hard problem. For instance, even for many lens spaces, the determination of the complexity is left open. The following conjecture, which is stated for the simplest case, is due to Jaco-Rubinstein [16] (see also Matveev [20]).

Conjecture. For $n > 3$, $c(L(n,1)) = n-3$.

It is known that the conjecture holds for even n [15], and that $c(L(n,1)) \le n-3$ holds [16]. The opposite inequality for odd n is open.

By taking the second barycentric subdivision, one easily sees that the complexity is bilipschitz equivalent to the minimal number of tetrahedra in a triangulation. From this, we obtain an immediate consequence of Theorem 3:

Theorem 4 (Cha [3])**.** *For every* $\phi\colon \pi_1(M) \to G$, $c(M) \geq \dfrac{1}{209\,139\,840} \cdot |\rho^{(2)}(M,\phi)|$.

Using this, and by computing $\rho^{(2)}(L(n,1),\mathrm{id})$, the following is proven in [3]:

$$\frac{1}{627\,419\,520} \cdot (n-3) \leq c(L(n,1)) \leq n-3.$$

This confirms the conjecture asymptotically.

For the proof of Theorem 3, we develop some new techniques and tools, including a quantitative geometric approach to the Atiyah–Hirzebruch bordism spectral sequence, and an algebraic notion of controlled chain homotopy [3]. They appear intriguing on their own.

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Links of rational singularities, L-spaces and LO fundamental groups

András Némethi

In this talk we wish to connect three areas of mathematics, algebraic geometry (especially, the theory of local complex normal surface singularities), low dimensional topology (Heegaard Floer homology and foliations), and group theory (left-orderable property). There are well-defined interplays between them: links of such singularities are oriented 3-manifolds, whose fundamental groups (with minor exceptions) characterize the corresponding 3-manifolds and the topology of the singularity. We show that certain basic objects (fundamental in classification procedures in these three rather independent theories) can be identified in a surprising way. In singularity theory we target the rational singularities; by definition they are those germs with vanishing geometric genus. This vanishing (although it is analytic in nature) was characterized combinatorially by the plumbing graph of the link by Artin and Laufer (graphs satisfying the property are called 'rational graphs') [1, 2, 18]. In 3–dimensional topology we consider the family of L-spaces, introduced by Ozsváth and Szabó, they are characterized by the vanishing of the reduced Heegaard Floer homology, and are key fundamental objects in recent developments in topology [25, 26]. Being a rational singularity link, or an L-space, will be compared with the left-orderability of the corresponding fundamental groups.

In fact, the link M of a complex normal surface singularity (X, o) is a special plumbed 3-manifold: oriented S^1-fibrations over orientable base spaces are plumbed along a connected, negative definite graph. In this note we will be interested only in rational homology sphere 3-manifolds, hence the corresponding plumbing graphs are trees of S^2's. The connection between singularity theory and topology imposed by the link had deep influences in both directions and created several bridges. One of them is the introduction of the *lattice cohomology* $\{\mathbb{H}^q(M)\}_{q\geq 0}$ of such 3-manifolds by the author [23] (see also [22]). Although $\mathbb{H}^*(M)$ is defined combinatorially from the graph, it can be compared with several analytic invariants, e.g with the geometric genus as well. In particular, in [22, 23] the author proved:

Theorem 1. *(X, o) is a rational singularity if and only if the reduced lattice cohomology of its link M satisfies $\mathbb{H}^0_{red}(M) = 0$; or, equivalently, $\mathbb{H}^*_{red}(M) = 0$.*

On the other hand, in [23] the author formulated the following conjecture.

Conjecture 2. *The Heegaard Floer homology and the lattice cohomology of M are isomorphic (up to a shift in degrees):*

$$HF^+_{red,even/odd}(-M,\sigma) = \oplus_{q\ even/odd}\ \mathbb{H}^q_{red}(M,\sigma)[-d(M,\sigma)],$$

where $\sigma \in \mathrm{Spin}^c(M)$, and $d(M,\sigma)$ is the d-invariant of $HF^+(M,\sigma)$.

In particular, the above conjecture predicts that $HF^+_{red}(M) = 0$ (that is, M is an L-space) if and only if $\mathbb{H}^*_{red}(M) = 0$, which is equivalent with the rationality of the graph by Theorem 1.

The goal of the present note is to prove the above prediction:

Theorem 3. *A singularity link is an L-space if and only if the singularity is rational.*

In fact, one direction of the statement is already known. The author introduced the notion of 'bad vertices' of a graph [22, 24]. In this way, a graph without bad vertices is rational; a graph with one bad vertex is a graph that becomes rational after a '(negative) surgery at that vertex'. In particular, the number of bad vertices measures how far the graph is from being rational. Related to Conjecture 2 in [23] the author proved:

Theorem 4. *If the number of bad vertices of the plumbing graph is ≤ 1 then Conjecture 2 is true.*

This was generalized in [28] by Ozsváth, Stipsicz and Szabó for two bad vertices.

Since the above theorem applies for rational links, Theorems 1 and 4 imply that the link of a rational singularity is an L-space.

The opposite direction was obstructed by the lack of characterizations of the L-spaces (at least in some language that can be reformulated inside of singularity theory). This obstruction was broken recently by several results in this direction, whose final form is the main result of Hanselman, J. Rasmussen, S. D. Rasmussen and Watson [15]:

Theorem 5. *If M is a closed, connected orientable graph manifold then the following are equivalent:*

(i) *M is not an L-space;*
(ii) *M has left-orderable (LO) fundamental group;*
(iii) *M admits a C^0 coorientable taut foliation.*

Recall that a group G is left-orderable if there exists a strict total ordering $<$ of G such that $g < h$ implies $fg < fh$ for all $f, g, h \in G$. (By convention, the trivial group is not LO.)

The equivalence (ii)⇔(iii) was established by Boyer and Clay [7]. The implication (iii)⇒(i) was proved independently by Kazez and Roberts in [17, Corollary 1.6], by Bowden in [5], and by Boyer and Clay in [8] (see also [27] for the smooth case). The equivalence (i)⇔(ii) was conjectured by Boyer, Gordon and Watson [6]. The above Theorem 5 was the final answer to this conjecture. For the history and partial contributions see the introduction and references from [15], and the references therein.

Theorem 5 allows us to reformulate the remaining implication of Theorem 3 as follows: if M is the link of a non-rational singularity then $\pi_1(M)$ is LO, hence not an L-space.

Theorems 3 and 5 combined provide:

Corollary 6. *If M is the link of a normal surface singularity (that is, if M is the plumbed manifold associated with a connected, negative definite graph), then the following are equivalent:*

(i) *M is the link of a non-rational singularity (i.e., the graph is not a 'rational graph');*
(ii) *M is not an L-space;*
(iii) *M has left-orderable (LO) fundamental group;*
(iv) *M admits a C^0 coorientable taut foliation.*

Remark 7. An integral homology sphere M is a rational link if and only if $M = S^3$ or $M = \Sigma(2,3,5)$, the link of the Brieskorn E_8-singularity $\{x^2 + y^3 + z^5 = 0\}$, see e.g. [21]. In particular, if M is an integral homology sphere singularity link, not of type S^3 or $\Sigma(2,3,5)$, then by Corollary 6 (ii)-(iii)-(iv) above are automatically satisfied. (See [11] for left-orderability of $\pi_1(M)$ when M is an integral homology 3-sphere which is an irreducible and toroidal graph manifold. Also, in [3] Boileau and Boyer proved that an integral homology sphere graph manifold not homeomorphic to either S^3 or $\Sigma(2,3,5)$ admits a horizontal foliation; this basically implies the statement of Theorem 5 for this subclass.)

Remark 8. In order to show the flavour of the proof, we list the main ingredients.

(A) The characterization of rational graphs via Laufer's algorithm (Laufer's computation sequence), and also the graph-combinatorics of bad vertices.

(B) A theorem of Boyer, Rolfsen and Wiest [4], which states that for a compact, irreducible, 3-manifold M, the fundamental group $\pi_1(M)$ is LO if and only if there exists a non-trivial homomorphism $\pi_1(M) \to L$, where L is an LO group. In particular, since $\mathbb{Z}^r$ is LO for any $r \in \mathbb{Z}_{>0}$, if $H_1(M, \mathbb{Q}) \neq 0$ then using the abelianization map we obtain that $\pi_1(M)$ is LO.

(C) A theorem of Clay, Lidman and Watson [11] regarding the behavior of LO property with respect to free products with amalgamation (more precisely, with respect to the decomposition of M along a torus and closing the pieces along 'LO-slopes').

(D) The equivalences (i)$\Leftrightarrow$(ii) from Theorem 5 above (combined with Theorem 4 at the 'induction start'.).

Applications. Several results valid from singularity theory can be reinterpreted via the above correspondence in terms of L-spaces. E.g., since rational graphs are stable with respect to taking subgraphs, or decreasing the decorations of the vertices, we obtain:

Corollary 9. *Negative definite plumbing graphs of plumbed L-spaces are stable with respect to taking subgraphs, or decreasing the decorations of the vertices.*

Using stability with respect to finite coverings we obtain the following.

Corollary 10. *Assume that we have a finite covering $M_1 \to M_2$ of graph 3-manifolds associated with connected negative definite plumbing graphs. The covering is either unbranched, or it is branched with branch locus $B_2 \subset M_2$. In the second case we assume that M_2 admits a negative definite plumbing representation, such that all the connected components of B_2 are represented by S^1-fibers of Seifert fibrations on the pieces of the JSJ decomposition.*

Then M_2 is an L-space whenever M_1 is an L-space.

This statement follows also from Corollary 3 of [15], where the authors use a similar stability condition valid for LO groups. The proof of the above corollary emphasizes the perfect parallelism of analytic geometry (analytic coverings and rational singularities) with LO behaviour of fundamental groups along topological coverings.

One can find easily (even non-branched) coverings when M_2 is an L-space but M_1 is not.

Example 11. (Coverings) *Let $K \subset S^3$ be an embedded algebraic link (the link of an isolated plane curve singularity). The cyclic $\mathbb{Z}_n$ covering of S^3 branched along K is an L-space if and only if*

- *$n = 2$ and K is the link of an A-D-E (simple) plane curve singularity, or*
- *$n > 2$ and K is the torus link $T_{2,m}$ with $\frac{1}{m} + \frac{1}{n} > \frac{1}{2}$.*

Example 12. (The Seifert fibered case) *The link of a weighted homogeneous normal surface singularity is a Seifert 3-manifold. In [29] Pinkham computed the geometric genus for such singularity in terms of the Seifert invariants in the case when the link is a rational homology sphere. The vanishing of the corresponding expression provides a numerical rationality criterion in terms of Seifert invariant. Hence, the main result provides a new criterion for the topological properties* (ii)-(iii)-(iv) *from Corollary 6. Here is this new numerical criterion.*

Assume that the star-shaped graph has $\nu \geq 3$ legs, the central vertex v_0 is decorated by e_0, and the i-th leg by $-b_{i1}, \dots, -b_{is_i}$, where

$$[b_{i1}, \dots, b_{is_i}] = b_{i1} - \cfrac{1}{b_{i2} - \cdots} = \alpha_i/\omega_i$$

is the (Hirzebruch) continued fraction with $b_{ij} \geq 2$. The positive integers $\{(\alpha_i, \omega_i)\}_{i=1}^{\nu}$ are the Seifert invariants with $0 < \omega_i < \alpha_i$, $\gcd(\alpha_i, \omega_i) = 1$. (Here v_0 is connected to the vertices decorated by $-b_{i1}$.) We assume that the graph is negative definite, that is, $e := e_0 + \sum_i \omega_i/\alpha_i < 0$. Then, by [29]*, M is non-rational if and only if*

$$\sum_i \lfloor -l\omega_i/\alpha_i \rfloor \leq le_0 - 2 \quad \textit{for at least one } l \in \mathbb{Z}_{\geq 0}.$$

This looks very different than the previous criteria used in topology, e.g., for the existence of foliations, results of Eisenbud, Hirsch, Jankins, Neumann, Naimi [12, 16, 20, 19]*. Let us recall it for $\nu = 3$.*

Following [16, 19] *we say that* $(x, y, z) \in (\mathbb{Q} \cap (0, 1))^3$ *is realizable if there exist coprime integers* $m > a > 0$ *such that up to a permutation of* x, y, z *one has* $x < a/m$, $y < (m - a)/m$, $z < 1/m$.

Then $M(\Gamma)$ *admits a coorientable transversal foliation if and only if one of the following holds:*

$$e_0 = -1 \quad \text{and} \quad \{\beta_i/\alpha_i\}_{i=1,2,3} \text{ is realizable;}$$
$$e_0 = -2 \quad \text{and} \quad \{(\alpha_i - \beta_i)/\alpha_i\}_{i=1,2,3} \text{ is realizable.}$$

A direct arithmetical proof of the equivalence of these two criteria will be proved in another note. Arithmetical properties behind this equivalence can also be found in [9, 13, 10], *via 'ziggurats'and 'rotation numbers'.*

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Local systems on arrangements of hypersurfaces

Graham Denham
(joint work with Alexander I. Suciu)

1. **Summary.** This talk is based on the preprint [DS17]. We consider smooth, complex quasi-projective varieties U which admit a compactification with a boundary which is an arrangement of smooth algebraic hypersurfaces. If the hypersurfaces intersect locally like hyperplanes, and the relative interiors of the hypersurfaces are Stein manifolds, we prove that the cohomology of certain local systems on U vanishes. As an application, we show that complements of linear, toric, and elliptic arrangements are both duality and abelian duality spaces.

2. **Abelian duality and local systems.** It has long been recognized that complements of complex hyperplane arrangements satisfy certain vanishing properties for homology with coefficients in local systems. We revisited this subject in our joint work with Sergey Yuzvinsky, [DSY16, DSY17], in a more general context.

Let X be a connected, finite-type CW complex, with fundamental group G. Following Bieri and Eckmann [BE73], we say that X is a *duality space* of dimension n if $H^q(X, \mathbb{Z}G) = 0$ for $q \neq n$ and $H^n(X, \mathbb{Z}G)$ is non-zero and torsion-free. We also say that X is an *abelian duality space* of dimension n if the analogous condition, with the coefficient G-module $\mathbb{Z}G$ replaced by $\mathbb{Z}G^{\mathrm{ab}}$ is satisfied. Noteworthily, these properties impose significant conditions on the cohomology of local systems on X.

Let $\Bbbk$ be an algebraically closed field. The group $\widehat{G} = \mathrm{Hom}_{\mathrm{Gps}}(G, \Bbbk^*)$ of $\Bbbk$-valued multiplicative characters of G is an algebraic group, with identity the trivial representation $\mathbb{1}$. The *characteristic varieties* $\mathcal{V}^q(X, \Bbbk)$ are the subsets of $\widehat{G}$ consisting of those characters ρ for which $H^q(X, \Bbbk_\rho) \neq 0$. We highlight an interesting consequence of the abelian duality space property, which we established in [DSY17]: If

X is an abelian duality space of dimension n, then the characteristic varieties of X *propagate*, that is,

$$\{\mathbb{1}\} = \mathcal{V}^0(X, \Bbbk) \subseteq \mathcal{V}^1(X, \Bbbk) \subseteq \cdots \subseteq \mathcal{V}^n(X, \Bbbk),$$

or, equivalently, if $H^p(X, \Bbbk_\rho) \neq 0$ for some $\rho \in \widehat{G}$, then $H^q(X, \Bbbk_\rho) \neq 0$ for all $p \leq q \leq n$.

3. **Arrangements of smooth hypersurfaces.** Davis, Januszkiewicz, Leary, and Okun showed in [DJLO11] that complements of (linear) hyperplane arrangements are duality spaces. More generally, we proved in [DSY17] that complements of both linear and elliptic arrangements are duality and abelian duality spaces.

Our goal here is to further generalize these results to a much wider class of *arrangements of hypersurfaces*, by which we mean a collection of smooth, irreducible, codimension 1 subvarieties which are embedded in a smooth, connected, complex projective algebraic variety, and which intersect locally like hyperplanes. We isolate a subclass of such arrangements whose complements enjoy the aforementioned duality properties, and therefore have vanishing twisted cohomology in the appropriate range.

Theorem 1. *Let U be a connected, smooth, complex quasi-projective variety of dimension n. Suppose U has a smooth compactification Y for which*

(1) *Components of the boundary $D = Y \setminus U$ form an arrangement of hypersurfaces $\mathcal{A}$;*
(2) *For each submanifold X in the intersection poset $L(\mathcal{A})$, the complement of the restriction of $\mathcal{A}$ to X is a Stein manifold.*

Then U is both a duality space and an abelian duality space of dimension n.

An important consequence of this theorem is that the characteristic varieties of such “recursively Stein” hypersurface complements propagate. As another application, we prove the following ‘generic vanishing of cohomology’ result.

Theorem 2. *Let U be as above, and let $G = \pi_1(U)$.*

(1) *If A is a finite-dimensional representation of G over a field $\Bbbk$, and if $A^{\gamma_g} = 0$ for all g in a building set $\mathcal{G}_X$, where $X \in L(\mathcal{A})$, then $H^i(U, A) = 0$ for all $i \neq n$.*
(2) *$H^i(U, \ell_2 G) = 0$ for all $i \neq n$, where $\ell_2 G$ denotes the left $\mathbb{R}[G]$-module of square-summable functions on G.*

Consequently, the cohomology groups of U with coefficients in a ‘generic’ local system vanish in the range below n. Likewise, the ℓ_2-Betti numbers of U are all zero except in dimension n. One application is to Euler characteristic: using ℓ_2-Betti numbers, we see that $(-1)^n \chi(U) \geq 0$, where $\chi(U)$ is the (usual) topological Euler characteristic of U. Related vanishing results can be found in two recent preprints by Liu, Maxim, and Wang [LMW17a, LMW17b].

4. **Linear, elliptic, and toric arrangements.** The theory of hyperplane arrangements originates in the study of configuration spaces and braid groups. Here we consider a broader class of hypersurface arrangements of current interest.

Theorem 3. *Suppose that $\mathcal{A}$ is one of the following:*

(1) *An affine-linear arrangement in $\mathbb{C}^n$, or a hyperplane arrangement in $\mathbb{CP}^n$;*
(2) *A non-empty elliptic arrangement in E^n;*
(3) *A toric arrangement in $(\mathbb{C}^*)^n$.*

Then the complement $M(\mathcal{A})$ is both a duality space and an abelian duality space of dimension $n-r$, $n+r$, and n, respectively, where r is the corank of the arrangement.

As mentioned previously, the first two statements already appeared in our paper [DSY17]; at the time, however, we were unable to address the third one. Since then, De Concini and Gaiffi [DCG16] have constructed a compactification for toric arrangements which is compatible with our approach. The claim that the complement of a toric arrangement is a duality space was first reported by Davis and Settepanella in [DS13, Theorem 5.2]. However, a serious gap appeared in the proof: see Davis [Dav15]. Part of our motivation here, then, is to provide an independent alternative, as well as a uniform proof of the three claims above.

As a consequence of Theorem 3, the characteristic varieties propagate for all linear, elliptic and toric arrangements. The formality of linear and toric arrangement complements implies that their resonance varieties propagate, as well. In the linear case, a more refined propagation of resonance property was established by Budur in [Bu11].

If $\mathcal{A}$ is an affine complex arrangement, work of Kohno [Koh86], Esnault, Schechtman, Varchenko [ESV92], and Schechtman, Terao, Varchenko [STV95] gives sufficient conditions for a local system $\mathbb{L}$ on $M(\mathcal{A})$ to insure the vanishing of the cohomology groups $H^i(M(\mathcal{A}), \mathbb{L})$ for all $i < \operatorname{rank}(\mathcal{A})$. Similar conditions for the vanishing of cohomology of with coefficients in rank 1 local systems were given by Levin and Varchenko [LV12] for elliptic arrangements, and by Esterov and Takeuchi [ET17] for certain toric hypersurface arrangements. In turn, we obtain a unified set of generic vanishing conditions for cohomology of local systems on complements of arrangements of smooth, complex algebraic hypersurfaces.

The ℓ_2-cohomology of a linear arrangement also vanishes outside of the middle (real) dimension: this is a result of Davis, Januszkiewicz and Leary [DJL07]. The same claim for toric arrangements appears in [DS13]; however, the argument given there has the same gap mentioned above. As part of our approach here, we also obtain vanishing results for ℓ_2-cohomology of hypersurface arrangements.

5. **Orbit configuration spaces.** As a second application, we obtain an almost complete characterization of the duality and abelian duality properties of ordered orbit configuration spaces on Riemann surfaces. We will not define orbit configuration spaces here; however, we remind the reader that the classical configuration spaces are recovered by taking Γ to be the trivial group.

Theorem 4. *Suppose Γ is a finite group that acts freely on a Riemann surface $\Sigma_{g,k}$ of genus g with k punctures. Let $F_\Gamma(\Sigma_{g,k}, n)$ be the orbit configuration space of n ordered, disjoint Γ-orbits.*

(1) *If $k > 0$, then $F_\Gamma(\Sigma_{g,k}, n)$ is both a duality space and an abelian duality space of dimension n.*
(2) *If $k = 0$, then $F_\Gamma(\Sigma_g, n)$ is a duality space of dimension $n+1$, provided $g \geq 1$, and is an abelian duality space of dimension $n+1$ if $g = 1$.*
(3) *If $g = k = 0$, then $F(\Sigma_g, n)$ is neither a duality space nor an abelian duality space. It is not an abelian duality space if $g \geq 2$ and $n \leq 2$.*

Hence the characteristic varieties propagate for the orbit configuration spaces $F_\Gamma(\Sigma_{g,k}, n)$, where either $k \geq 1$, or $k = 0$ and $g = 1$, for any finite group Γ acting freely on $\Sigma_{g,k}$.

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Propagation property and codimension bound for the jump loci of smooth quasi-projective variety

Yongqiang Liu
(joint work with Laurentiu Maxim, Botong Wang)

One central topic in singularity theory is to understand the topology of smooth complex algebraic varieties. The smooth complex algebraic varieties exhibit extra special properties attached to their topological invariants. One essential useful invariant for us is the cohomology jump loci.

Let X be a smooth complex quasi-projective varieties of complex dimension n. Consider the moduli space of rank 1 local systems on X: $\mathrm{Hom}(\pi_1(X), \mathbb{C}^*)$. The degree i cohomology jump loci of X are defined by

$$\mathcal{V}^i(X) := \{\rho \in \mathrm{Hom}(\pi_1(X), \mathbb{C}^*) \mid \dim H^i(X, \mathbb{C}_\rho) \neq 0\}.$$

Here $\mathbb{C}_\rho$ is the corresponding rank 1 $\mathbb{C}$ local system on X. These jump loci are closed sub-varieties of $\mathrm{Hom}(\pi_1(X), \mathbb{C}^*)$ and homotopy invariants of X.

Let $f\colon X \to Y$ be an algebraic proper map between two complex algebraic varieties (e.g., the Albanese map associated to X, when X is smooth and projective). Set $r(f) = \dim X \times_Y X - \dim X$. This number $r(f)$ is known as the defect of semi-smallness of f [3, page 575]. If $r(f) = 0$ (e.g., a closed embedding), f is called semi-small.

Theorem 1. *Let X be a smooth projective variety of complex dimension n. Assume that X admits an algebraic proper semi-small map $f\colon X \to A$ from X to an abelian variety A. Then we have the following results:*

- *Propagation property:*
$$\mathcal{V}^0(X) \subseteq \cdots \subseteq \mathcal{V}^{n-1}(X) \subseteq \mathcal{V}^n(X) \supseteq \mathcal{V}^{n+1}(X) \supseteq \cdots \supseteq \mathcal{V}^{2n}(X).$$
- *Codimension lower bound:*
$$\operatorname{codim} \mathcal{V}^{n\pm i}(X) \geq 2i \text{ for any } 0 \leq i \leq n.$$

This theorem is an undergoing work by the authors. The proof uses the decomposition theorem of [1] and the Bhatt–Schnell–Scholze's result [2] for perverse sheaves on complex abelian varieties.

Theorem 2. [7, Theorem 1.7, Theorem 1.9] *Let X be a smooth quasi-projective variety of complex dimension n. Assume that the mixed Hodge structure on $H^1(X, \mathbb{Q})$ is pure of type $(1,1)$, or equivalently, there exists a smooth compactification $\overline{X}$ of*

X with $b_1(\overline{X}) = 0$. If X admits a proper semi-small map $f : X \to T$ (e.g., a finite map or a closed embedding), where $T = (\mathbb{C}^*)^N$ is a complex affine torus, then we have the following properties:

- *Propagation property:*
$$\mathcal{V}^0(X) \subseteq \cdots \subseteq \mathcal{V}^{n-1}(X) \subseteq \mathcal{V}^n(X).$$
- *Codimension lower bound:*
$$\operatorname{codim} \mathcal{V}^{n-i}(X) \geq i \text{ for any } 0 \leq i \leq n.$$

The proof of Theorem 2 relies on decomposition theorem [1] and Gabber and Loeser's result [6] for perverse sheaves on complex affine torus. Theorem 2 is inspired by Denham–Suciu–Yuzvinsky's work [4, 5] about abelian duality spaces.

Theorem 3. [7, Theorem 1.9] *Let X be a smooth complex quasi-projective variety of dimension n. Assume that the mixed Hodge structure on $H^1(X, \mathbb{Q})$ is pure of type $(1, 1)$, or equivalently, there exists a smooth compactification $\overline{X}$ of X with $b_1(\overline{X}) = 0$. If X admits a proper semi-small map $f\colon X \to T = (\mathbb{C}^*)^N$ (e.g., a finite map or a closed embedding), then X is an abelian duality space of dimension n.*

Example 4. *Let $T = (\mathbb{C}^*)^n$ be the n-dimensional complex affine torus. Take a finite collection of irreducible hypersurfaces in T, say $V_1, \ldots, V_r$. Set $X = T - \bigcup_{i=1}^r V_i$. Then X admits a closed embedding to $(\mathbb{C}^*)^{n+r}$. It is easy to check that the mixed Hodge structure on $H^1(X, \mathbb{Q})$ is pure of type $(1, 1)$. So X is an abelian duality space of dimension n, and the properties listed in Theorem 2 hold for the cohomology jump loci of X.*

Theorem 5. [7, Theorem 6.1] *Let X be a compact Kähler manifold. Then X is an abelian duality space if and only if X is a compact complex torus. In particular, abelian varieties are the only complex projective manifolds that are abelian duality spaces.*

Inspired by the above theorem, we ask the following question:

Question 6. *Does there exist a closed orientable manifold that is an abelian duality space, but not a real torus?*

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Monodromy of Kodaira Fibrations

Laure Flapan

A Kodaira fibration is a non-isotrivial fibration $f\colon S \to B$ from a smooth algebraic surface S to a smooth algebraic curve B such that for every $b \in B$, the fiber $F_b := f^{-1}(b)$ is also a smooth algebraic curve. The non-isotriviality of the fibration, meaning the property that not all fibers F_b are isomorphic as algebraic varieties, ensures that the fundamental group $\pi_1(B)$ does not act trivially on the fibers.

Such fibrations were originally constructed by Kodaira [5] as a way to show that, unlike the topological Euler characteristic, the signature σ of a manifold, meaning the signature of the intersection form on the middle homology of the manifold, is not multiplicative for fiber bundles. More precisely, for any fiber bundle $\phi\colon X \to Y$ with fiber F_y, the topological Euler characteristic χ_{top} satisfies $\chi_{\text{top}}(X) = \chi_{\text{top}}(Y)\chi_{\text{top}}(F_y)$. Prior to Kodaira's construction, Chern–Hirzebruch–Serre [2] had shown that if $\pi_1(Y)$ acts trivially on the fiber F_y, then the signature also satisfies $\sigma(X) = \sigma(Y)\sigma(F)$. Kodaira's construction proved that this hypothesis about the fundamental group was necessary, since for any Kodaira fibration $f\colon S \to B$, the surface S has signature $\sigma(S) > 0$, while the algebraic curves F_b and B have antisymmetric intersection form and thus satisfy $\sigma(F_b) = \sigma(B) = 0$.

Any Kodaira fibration $f\colon S \to B$ induces a short exact sequence of fundamental groups

$$1 \to \pi_1(F_b) \to \pi_1(S) \to \pi_1(B) \to 1. \tag{1}$$

In fact, Kotschick shows in [6, Proposition 1] that any compact complex surface S whose fundamental group fits into an exact sequence of the form (1) satisfies $\chi_{\text{top}}(S) = \chi_{\text{top}}(B)\chi_{\text{top}}(F_b)$ if and only if the sequence (1) is induced by a Kodaira fibration $S \to B$. Is is thus natural to ask:

Question 1. *For which extensions*

$$1 \to \pi_1(F_b) \to G \to \pi_1(B) \to 1$$

as in (1)*, is the group G the fundamental group of a Kodaira surface?*

Note that Kodaira in [5] together with Kas in [4] show that a Kodaira fibration must have base curve B of genus at least 2 and have fibers F_b of genus at least 3. The short exact sequence (1) induces a homomorphism

$$\rho\colon \pi_1(B) \to \mathrm{Mod}(F_b) \subset O(\pi_1(F_b))$$

into the mapping class group of F_b. Because the genus g of F_b is at least 3, the center $Z(\pi_1(F_b))$ is trivial [3]. Hence, since extensions with outer action ρ

are parametrized by $H^2(\pi_1(B), Z(\pi_1(F_b)))$, it follows that the homomorphism ρ completely determines the extension (1) [1, Corollary 6.8].

Letting $V = H_1(F_b, \mathbb{Z}) \otimes \mathbb{Q}$, the homomorphism ρ induces a homomorphism

$$\overline{\rho} : \pi_1(B) \to GL(V).$$

Observe that the Kodaira fibration $f \colon S \to B$ determines a map $B \to \mathcal{M}_g$ to the moduli space of curves of genus g, sending a point $b \in B$ to the curve F_b. Since, by assumption, the fibration f is non-isotrivial, the map $B \to \mathcal{M}_g$ has 1-dimensional image. Then by the Torelli theorem, the induced map $B \to \mathcal{A}_g$ to the moduli space of principally polarized abelian varieties that sends b to $H_1(F_b, \mathbb{Z})$ has 1-dimensional image. In particular, the variation of $\mathbb{Q}$-Hodge structures $R_1 f_* \mathbb{Q}$ is not locally constant. If the monodromy representation $\overline{\rho}$ had finite image, then, by Schmid's rigidity theorem [7, Theorem 7.24], the variation of $\mathbb{Q}$-Hodge structures $R_1 f_* \mathbb{Q}$ would be locally constant. Hence the representation $\overline{\rho}$, and consequently the homomorphism ρ as well, must have infinite image.

To better understand this image, note that any variation of $\mathbb{Q}$-Hodge structures $\mathbb{V}$ over a connected algebraic variety Y will yield a monodromy respresentation $\Phi \colon \pi_1(Y, y) \to GL(V_y)$ for $y \in Y$.

Definition 2. The *connected monodromy group* T of the variation $\mathbb{V}$ is the connected component of the identity of the smallest $\mathbb{Q}$-algebraic subgroup of $GL(V_y)$ containing the image of $\pi_1(Y, y)$.

Thus one way to approximate Question 1 about which groups can arise as the fundamental group of a Kodaira surface is to ask:

Question 3. *Which groups can arise as the connected monodromy group of a Kodaira fibration?*

The main result of this talk provides an answer to this question in the case of a Kodaira fibration whose fibers have genus 3. We give precise characterizations of the possible connected monodromy groups as $\mathbb{Q}$-algebraic groups, which can beroughly summarized by the following:

Theorem 4. *The connected monodromy group of a genus* 3 *Kodaira fibration must be isomorphic over* $\mathbb{C}$ *to one of the following groups:*

(1) $Sp(6)$
(2) $SL(2) \times SL(2) \times SL(2)$
(3) $Sp(4)$
(4) $SU(3)$
(5) $SL(2) \times SL(2)$

Moreover, groups (1), (2), *and* (3) *arise from Kodaira fibrations obtained as general complete intersection curves in a subvariety of* $\mathcal{A}_3$ *whose points all have endomorphisms by a specified ring. Groups* (4) *and* (5) *cannot arise in this way and thus are not known to actually occur.*

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Applications of cuts in low dimensions

Mahan Mj

The theory of cuts developed by Delzant-Gromov has a number of applications for low dimensional Kaehler groups and may be used to decide completely which 3-manifold groups are Kähler, which one-relator groups are Kähler and so on. We shall discuss some of these applications.

A standard theme in the theory of Kähler groups has been:

Question 1. *Take your favorite class of groups. (Our favorite classes occur naturally in geometric group theory or low-dimensional topology.) Which of them are Kähler/projective/quasiprojective?*

Examples of such "favorite" classes include 3-manifold groups and one-relator groups. Dimca and Suciu [DS] proved:

Theorem 2. *Let G be the fundamental group of a closed 3-manifold. Then G is Kähler if and only if G is finite.*

In earlier work with Biswas and Seshadri [BMS] we generalized Theorem 2 to the following general set-up:

$$1 \longrightarrow N \xrightarrow{i} G \xrightarrow{q} Q \longrightarrow 1 \tag{1}$$

is an exact sequence of finitely generated groups, G is a Kähler group and Q is a 3-manifold group.

In subsequent work with Biswas [BM], we proved that infinite one-relator Kähler groups are precisely fundamental groups of (complex) one dimensional orbifolds with at most one cone-point:

Theorem 3. *Let G be an infinite one-relator group. Then G is Kähler if and only if it is isomorphic to*

$$\langle a_1\,, b_1\,, \cdots, a_g\,, b_g \mid (\prod_{i=1}^{g}[a_i\,, b_i])^n\rangle\,,$$

where g and n are some positive integers.

The principal tool from Kähler groups is the theory of stable cuts of Delzant-Gromov [DG].

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Cubulable Kähler groups

Pierre Py

(joint work with Thomas Delzant)

The purpose of this talk was to explain the main results of the preprint entitled *Cubulable Kähler groups* (arXiv:1609.08474), which is a joint work with Thomas Delzant.

Recall that a Kähler group is a group that can be realized as the fundamental group of a compact Kähler manifold. We study actions of these groups on CAT(0) *cubical complexes*. These polyhedral complexes have recently played a fundamental role in geometric group theory and 3-dimensional topology. We refer for instance to [1, 2] for an introduction to these complexes and to these recent developments. The reason to think that actions of Kähler groups on CAT(0) cubical complexes can be classified comes from earlier results on relative ends or filtered ends of Kähler groups, see [3].

In the following, we say that a group is cubulable if it acts properly discontinuously and cocompactly on a CAT(0) cubical complex. Our first main result is the following, which virtually classifies cubulable Kähler groups.

Theorem 1. *If a Kähler group Γ is cubulable it has a finite index subgroup Γ_1 which is isomorphic to a direct product of surface groups, possibly with an Abelian factor.*

If now X is a projective manifold with cubulable fundamental group, one can virtually describe X up to biholomorphisms, asusming that it is aspherical. Namely, we prove:

Theorem 2. *Let X be an aspherical smooth projective manifold. Assume that the fundamental group of X is cubulable. Then X has a finite cover X_1 which is biholomorphic to a direct product of compact Riemann surfaces, possibly with an Abelian variety as a factor:*

$$X_1 \simeq \Sigma_1 \times \cdots \times \Sigma_r \times A.$$

Our work contains a slightly weaker statement in the case where X is only assumed to be compact Kähler, instead of projective.

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Involutive Heegaard Floer homology and rational cuspidal curves

Maciej Borodzik

(joint work with Jen Hom and Andrzej Schinzel)

We use invariants of Hendricks and Manolescu coming from involutive Heegaard Floer theory to restrict possible configurations of singular points of a planar rational cuspidal curve. This is a joint project with Jen Hom and Andrzej Schinzel.

Let $C \subset \mathbb{C}P^2$ be a rational cuspidal curve of degree d. We denote by $K_1, \dots, K_n$ the links of its singular points. Set $K = K_1 \# \cdots \# K_n$. The boundary Y of the tubular neighborhood N of C is easily seen to be the surgery on K with coefficient d^2. On the other hand, the difference $W = \mathbb{C}P^2 \setminus N$ is a rational homology sphere whose boundary is Y. We use the following result of Ozsváth and Szabó.

Theorem 1 ([7])**.** *For any spin-c structure $\mathfrak{s}$ on Y that extends to a spin-c structure on W we have $d(Y, \mathfrak{s}) = 0$.*

It is possible to compute the d-invariants of Y from the semigroups of singular points of C. For a singular point z_i of C let S_i be the semigroup of the singular point. For an integer m write $R_i(m) = \#(S_i \cap [0, m))$. This is the semigroup counting function, used e.g. in [4].

The function $R(m)$ is defined as the infimal convolution $R = R_1 \diamond R_2 \diamond \cdots \diamond R_n$, where

$$I \diamond J(m) := \min_{k+l=m} I(k) + J(l).$$

In [3] the d-invariants of Y were expressed in terms of the function R. In connection with Theorem 1 the following result is obtained.

Theorem 2 (see [3])**.** *Let C be a rational cuspidal curve and R as above. Then for any $j = 0, \dots, d-1$ we have $R(jd+1) = \frac{(j+1)(j+2)}{2}$.*

The result is very closely related to the conjecture stated in [4].

The main problem with Theorem 2 is that it gives strongest restrictions for unicuspidal curves, and it becomes substantially weaker if $n > 1$. We illustrate a general phenomenon with a simple example, given also in [2].

Example 3. *For $i \geq 1$ set K_i to be the torus knot $T(2, 2i+1)$ and let S_i be the semigroups of the plane curve singular point with local equation $x^2 - y^{2i+1} = 0$ (the link of this singular point is precisely K_i). The following relation is easy to prove.*

$$R_1 \diamond R_5 = R_2 \diamond R_4 = R_3 \diamond R_3. \tag{1}$$

However, there exists a rational cuspidal curve of degree 5 with singular points whose links are $T(2,5)$ and $T(2,9)$ (this corresponds to $R_2 \diamond R_4$), but there are no rational cuspidal curves whose singular points have links $T(2,3)$ and $T(2,11)$ or $T(2,7)$ and $T(2,7)$ (see [6] *and references therein). Theorem 2 does not distinguish from any of these three cases because of the equality* (1)*.*

In [1] there is a general approach for creating connected sums of algebraic knots having the same R-function.

In [5] Hendricks and Manolescu defined involutive Floer homology. Based on a very general construction they constructed two invariants $\overline{d}$ an $\underline{d}$ for any closed oriented 3-manifold Y that is a rational homology sphere, equipped with a spin structure. The following result is obtained in [5].

Theorem 4. *Suppose $(Y, \mathfrak{s})$ is a rational homology 3-sphere with a choice of a spin structure. If Y bounds a rational homology ball W and $\mathfrak{s}$ extends to a spin structure over W, then $\overline{d}(Y, \mathfrak{s}) = \underline{d}(Y, \mathfrak{s}) = 0$.*

In the case $Y = S^3_{d^2}(K)$ and $W = \mathbb{C}P^2 \setminus N$, there is a canonical spin structure on Y. It is shown in [2] that this spin structure extends over W if and only if the degree d is an odd number.

A much harder task is to unfold the condition $\overline{d}(Y, \mathfrak{s}) = \underline{d}(Y, \mathfrak{s}) = 0$. In [2] only the case $n = 2$ is dealt with and an answer is given in terms of the V_0 invariants of Rasmussen [8]. The following result relies on the connected sum formula for the involutive invariants by Zemke [9].

Theorem 5 (see [2])**.** *Suppose K_1 and K_2 are algebraic knots of genera g_1 and g_2 respectively. Then $V_0(K_1 \# K_2) = V_0(K_1) + V_0(K_2)$ if and only if for $q > 2(g_1 + g_2)$ we have $\overline{d}(Y, \mathfrak{s}) = \underline{d}(Y, \mathfrak{s})$, where $Y = S^3_q(K)$ and $\mathfrak{s}$ is the canonical spin structure on Y.*

We remark that a direct analogue of Theorem 5 for the sum of more than two algebraic knots does not hold.

The V_0-invariant of a connected sum K of algebraic knots is equal to $R(g)$, where g is the genus of K. Given the definition of the convolution, the condition $V_0(K_1 \# K_2) = V_0(K_1) + V_0(K_2)$ can be rephrased as

$$R_1(g_1) + R_2(g_2) = \min_{i+j=g_1+g_2} R_1(i) + R_2(j),$$

where R_i, $i = 1, 2$, is the semigroup counting function for the knot K_i and g_i is the genus.

Putting together Theorem 4 and Theorem 5 we obtain the following result.

Theorem 6 (see [2])**.** *Let C be a rational cuspidal curve of odd degree with two singular points K_1 and K_2. Then $V_0(K_1\#K_2) = V_0(K_1) + V_0(K_2)$.*

As a corollary we are able to distinguish between the three cases discussed in Example 3 using the V_0 invariants.

Corollary 7. *There exists no rational cuspidal curve of degree 5 having two singular points with links $T(2,3)$ and $T(2,11)$ or $T(2,7)$ and $T(2,7)$.*

We conclude by noticing that Theorem 6 does not have yet a counterpart in Seiberg–Witten theory (in the spirit of [4]).

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Triangular curves and cyclotomic Zariski tuples

Enrique Artal Bartolo

(joint work with José Ignacio Cogolludo-Agustín, Jorge Martín-Morales)

Let $\mathcal{C} \subset \mathbb{P}^2$ be a projective plane curve over the complex numbers. The *combinatorics* of $\mathcal{C}$ is the topological type of $(T(\mathcal{C}), \mathcal{C})$, where $T(\mathcal{C})$ is a regular neighborhood of $\mathcal{C}$ in $\mathbb{P}^2$; an alternative definition (and closer to the word *combinatorics*) is the following. Let $\sigma\colon S \to \mathbb{P}^2$ be the (minimal) composition of a sequence of blow-ups such that $\sigma^{-1}(\mathcal{C})$ is a simple normal crossing divisor (SNC-*morphism* for $\mathcal{C}$). We associate to $\mathcal{C}$ the dual graph Γ of $\sigma^{-1}(\mathcal{C})$ where each vertex v is weighted with its self-intersection e_v and its genus $[g_v]$; moreover, the vertices corresponding with the irreducible components of the strict transform of $\mathcal{C}$ are marked. The *combinatorics* of $\mathcal{C}$ is the isomorphism type of the weighted marked graph Γ.

Definition 1. Let $\mathcal{C}_1, \ldots, \mathcal{C}_r \subset \mathbb{P}^2$, $r \geq 2$. These curves form a *Zariski tuple* if they share the same combinatorics and the pairs $(\mathbb{P}^2, \mathcal{C}_i)$ and $(\mathbb{P}^2, \mathcal{C}_j)$ are not homemorphic, for $1 \leq i < j \leq r$.

The first example of Zariski pair corresponds to the combinatorics of an irreducible sextic curve with six ordinary cusps. There are two such curves $\mathcal{C}_1, \mathcal{C}_2$ such that $\pi_1(\mathbb{P}^2 \setminus \mathcal{C}_1) \cong \mathbb{Z}/2 * \mathbb{Z}/3$ (the cusps lie on a conic) and $\pi_1(\mathbb{P}^2 \setminus \mathcal{C}_2) \cong \mathbb{Z}/2 \times \mathbb{Z}/3$ (they do not); they were constructed by Zariski in [3, 4]. In modern words, he also showed that they did not have the same Alexander polynomial. Since then many Zariski pairs and tuples have been constructed and the main invariants used to distinguish their topology are Alexander polynomials, characteristic varieties, existence of finite covers, etc. While they describe topological properties, most of these invariants are actually of algebraic nature. Let us consider the following new definition.

Definition 2. Let $\mathcal{C}_1, \dots, \mathcal{C}_r \subset \mathbb{P}^2$, $r \geq 2$. These curves form an *arithmetic Zariski tuple* if they form a Zariski tuple and if there exists a curve $\mathcal{C} \subset \mathbb{P}^2(\mathbb{K})$, $\mathbb{K}$ a number field, such that $\mathcal{C}_j = \phi_j(\mathcal{C})$, where $\phi_j : \mathbb{K} \hookrightarrow \mathbb{C}$ is a field embedding.

As a consequence the above invariants are useless to distinguish these tuples, as it is the case for the algebraic fundamental group (the profinite completion of the topological fundamental group). Other invariants (like fundamental group, braid monodromy, truncated Alexander Invariant, linking number, arithmetic of lattices) have served to find arithmetic Zariski pairs by many authors. We complete the last definition.

Definition 3. A *cyclotomic Zariski tuple* $\mathcal{C}_1, \dots, \mathcal{C}_r \subset \mathbb{P}^2$ is an arithmetic Zariski tuple where $\mathbb{K} = \mathbb{Q}(\zeta_d)$, the cyclotomic field of primitive roots of order $d \neq 1, 2, 3, 4, 6$, where $r = \frac{\phi(d)}{2}$, corresponding to the primitive roots up to complex conjugation.

Our goal is to construct cyclotomic Zariski tuples for any d; the existence of such tuples were not known for general d.

Definition 4. Let $\mathscr{C}$ be a combinatorics. The *realization space* of $\mathscr{C}$ is the space $\Sigma_{\mathscr{C}}$ formed by the curves $\mathcal{C} \subset \mathbb{P}^2$ whose combinatorics is $\mathscr{C}$ (it is a constructible subset of the projective space of curves of degree $\deg \mathscr{C}$).

It can be proved that two curves in the same connected component of $\Sigma_{\mathscr{C}}$ are isotopic (a generalization of Randell's lattice isotopy for line arrangements). Hence, in order to look for Zariski pair candidates we need combinatorics with non-connected realization space. This is where *triangular curves* apperar. For $d \geq 2$, let us consider the combinatorics $\mathscr{C}_d$ of irreducible curves C of degree $2d$ such that $\#\operatorname{Sing}(C) = 3$ and for each $P \in \operatorname{Sing}(C)$, the topological type of (C, P) is the one of the germ of $\{u^d = v^{d+1}\}$ at the origin of $\mathbb{C}^2$ (the irreducibility condition comes from the fact that all the singularities are locally irreducible). A Newton polygon computation yields the following result.

Proposition 5. *For $d \geq 3$ the space $\Sigma_{\mathscr{C}_d}$ has $\left\lceil \frac{d+1}{2} \right\rceil$ connected components labelled by the d-roots of unity (up to complex conjugation); for $d = 2$, there is only one such component associated to the root -1.*

The curves of $\Sigma_{\mathscr{C}_d}$ associated with a d-root ζ are defined by the following polynomials (up to a change of coordinates):

$$F_{d,\zeta} := z^d(y+\zeta x)^d + y^d(x+z)^d + x^d(y+z)^d + \sum_{\substack{i+j+k=2d\\ i,j,k<d}} a_{i,j,k} x^i y^j z^k$$

for generic values of the free coefficients. We define two related combinatorics. We consider the combinatorics $\tilde{\mathscr{C}_d}$ formed by curves with four irreducible components $\mathcal{C}\cup\mathcal{X}\cup\mathcal{Y}\cup\mathcal{Z}$, where $\mathcal{C}\in\mathscr{C}_d$ and $\mathcal{X},\mathcal{Y},\mathcal{Z}$ are the lines joining the three singular points (note that these points cannot be aligned and these lines are not in the tangent cone of the singular points by Bézout's Theorem). Finally, let $\mathscr{D}_d$ be the combinatorics formed by a smooth curve $\mathcal{D}$ of degree d, and three lines *in general position* intersecting each one $\mathcal{D}$ at one point.

Remark 6. The curves in $\tilde{\mathscr{C}_d}$ and $\mathscr{D}_d$ since they are exchanged by a standard Cremona transformation on the three lines of the curve. In particular they share the same complement. Note also that there are natural correspondences between the connected components of $\mathscr{C}_d$, $\tilde{\mathscr{C}_d}$ and $\mathscr{D}_d$; they will be denoted adding ω as subindex.

Theorem 7. *Let $\mathcal{A}_{d,\omega}\in\mathscr{D}_{d,\omega}$, $\mathcal{B}_{d,\omega}\in\tilde{\mathscr{C}}_{d,\omega}$ and $\mathcal{C}_{d,\omega}\in\mathscr{C}_{d,\omega}$ for $d\geq 2$, $\omega^d=1$ and $(d,\omega)\neq(2,1)$.*

- *If $(d,\omega)=(2,-1)$, then $\pi_1(\mathbb{P}^2\setminus\mathcal{C}_{2,-1})$ is non-abelian of order 12 (in particular $\pi_1(\mathbb{P}^2\setminus\mathcal{A}_{2,-1})\cong\pi_1(\mathbb{P}^2\setminus\mathcal{B}_{2,-1})$ are non-abelian).*
- *If $(d,\omega)=(3,1)$, then $\pi_1(\mathbb{P}^2\setminus\mathcal{C}_{3,1})\cong\mathbb{Z}/2*\mathbb{Z}/3$ (in particular $\pi_1(\mathbb{P}^2\setminus\mathcal{A}_{3,1})\cong\pi_1(\mathbb{P}^2\setminus\mathcal{B}_{3,1})$ are non-abelian).*
- *For the rest of the cases $\pi_1(\mathbb{P}^2\setminus\mathcal{A}_{d,\omega})\cong\pi_1(\mathbb{P}^2\setminus\mathcal{B}_{d,\omega})$ is abelian (in particular $\pi_1(\mathbb{P}^2\setminus\mathcal{C}_{d,\omega})\cong\mathbb{Z}/2\times\mathbb{Z}/3$).*

While some of these results were already known [3, 1], these fundamental groups can be computed using a Kummer cover and computing the fundamental group of the complement of the line arrangement $xyz(x+y+z)(x+y)(x+z)(y+z)=0$.

In order to distinguish the topology of these curves we need another invariant based on the one introduced in [2]. Let $\mathcal{C}=\mathcal{C}_1\cup\dots\cup\mathcal{C}_r$ be an irreducible curve with its decomposition in irreducible components of degrees $d_1,\dots,d_r$; in particular $H_1(\mathbb{P}^2\setminus\mathcal{C};\mathbb{Z})$ is generated by meridians $\mu_{\mathcal{C}_1},\dots,\mu_{\mathcal{C}_r}$ sch that $\mu_{\mathcal{C}_1}^{d_1}\cdot\ldots\cdot\mu_{\mathcal{C}_r}^{d_r}=1$. Let $\sigma\colon S\to\mathbb{P}^2$ be the SNC-morphism for $\mathcal{C}$; for each exceptional component E, let us denote by $\mu_E\in H_1(\mathbb{P}^2\setminus\mathcal{C};\mathbb{Z})$ the meridian of E. Fix a non-trivial torsion character $\xi\colon H_1(\mathbb{P}^2\setminus\mathcal{C};\mathbb{Z})\to\mathbb{C}^*$. Let $\mathcal{E}=\{D_1,\dots,D_s\}$ be rational irreducible components of $\sigma^{-1}(\mathcal{C})$ such that $\xi(\mu_D)=1$ if either $D\in\mathcal{D}$ or D intersects one of the divisors in $\mathcal{D}$. Assume also that the dual graph of $\mathcal{D}$ has a non-trivial cycle γ. We can suitably *push* γ to $\mathbb{P}^2\setminus\mathcal{C}$ and the value $\mathcal{I}(\mathcal{C},\xi,\gamma)$ is independent of γ and, moreover, it is a topological invariant of the *oriented-ordered* topology of $(\mathbb{P}^2,\mathcal{C})$.

Let us apply this to a curve $\mathcal{A}_{d,\omega}\in\mathscr{D}_{d,\omega}$, $\omega^d=1$, $\Im\omega\geq 0$. Let us consider a character $\xi: H_1(\mathbb{P}^2\setminus\mathcal{D}_\omega;\mathbb{Z})\to\mathbb{C}^*$ defined by $\xi(\mu_{\mathcal{D}})=\exp\left(\frac{2i\pi}{d}\right)$ and $\xi(\mu_{\mathcal{X}})=\xi(\mu_{\mathcal{Y}})=\xi(\mu_{\mathcal{Z}})=1$. The strict transforms of $\mathcal{X},\mathcal{Y},\mathcal{Z}$ satisfy the above hypotheses; let us choose the cycle γ defined by the cyclic order $\mathcal{X},\mathcal{Y},\mathcal{Z}$.

Proposition 8. *With the above notations, $\mathcal{I}(\mathcal{A}_{d,\omega}, \xi, \gamma) = \omega$. In particular, $\{\mathcal{A}_{d,\omega} \mid \omega^d = 1, \Im\omega > 0\}$ form a Zariski tuple. Moreover, for any divisor e of d, $e \neq 1,2,3,4,6$, the curves $\{\mathcal{A}_{d,\omega} \mid \Phi_e(\omega) = 1, \Im\omega \geq 0\}$ form a cyclotomic Zariski tuple (Φ_e is the e-cyclotomic polynomial).*

The idea of the proof is quite simple. One considers a smooth model of the ramified covering associated to ξ, which turns out to be a smooth projective surface in $\mathbb{P}^3$ of degree d, and to track the preimages of the lines $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$, and interpret their behavior in terms of monodromy of the covering. The same ideas provide Zariski pairs for $\tilde{\mathscr{C}_d}$; in both cases, the presence of the connecting lines is essential, so these ideas cannot directly be used to distinguish the topological properties of the curves in $\mathscr{C}_d$.

Conjecture 9. *$\mathcal{C}_{d,\omega} \in \mathscr{C}_{d,\omega}$, $\omega^d = 1$, $\Im\omega \geq 0$. Then, $\{\mathcal{A}_{d,\omega} \mid \omega^d = 1, \Im\omega \geq 0\}$ form a Zariski tuple and for any divisor e of d, $e \neq 1,2,3,4,6$, $\{\mathcal{A}_{d,\omega} \mid \Phi_e(\omega) = 1, \Im\omega \geq 0\}$ form a cyclotomic Zariski tuple.*

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Rho invariants for manifolds with boundary and low dimensional topology

Enrico Toffoli

Given a smooth oriented odd-dimensional manifold M^{2k-1} and a unitary representation $\alpha\colon \pi_1(M) \to U(n)$, the Atiyah–Patodi–Singer (APS) rho invariant is a real number $\rho(M,\alpha)$ extending the concept of *signature defect*. Namely, whenever we can find a manifold W^{2k} with $\partial W = M$ and such that α extends to a representation of $\pi_1(W)$, we have [APS76]

$$\rho(M,\alpha) = \operatorname{sign}_\alpha(W) - n\operatorname{sign}(W), \tag{1}$$

where $\operatorname{sign}_\alpha(W)$ is the signature of the twisted intersection pairing on $H_k(W;\mathbb{C}^n_\alpha)$. The rho invariant $\rho(M,\alpha)$ is defined in terms of the spectra of the twisted and untwisted signature operators on M associated to to some Riemannian metric, but it turns out to be a diffeomorphism invariant.

In knot theory, APS rho invariants of the closed manifold obtained through a zero-framed surgery of a knot K or link L, denoted respectively by M_K or M_L, were used to give obstructions to concordance [Lev94, Fri04, Fri05]. For knots, in the easiest case of a one dimensional representations $\pi_1(M_K) \to U(1)$, the rho invariant $\rho(M_K, \alpha)$ coincides with the Levine–Tristram signature calculated at the image through α of a meridian.

Kirk and Lesch introduced a generalization to manifolds with boundary of the APS rho invariants [KL03, KL04]. Given a smooth oriented odd-dimensional manifold X^{2k-1} with boundary Σ^{2k-2}, a unitary representation $\alpha\colon \pi_1(X) \to U(n)$, a Riemannian metric g on Σ and Legrangian subspaces $V^\alpha \subseteq H^*(\Sigma; \mathbb{C}^n_\alpha)$, $V^\tau \subseteq H^*(\Sigma; \mathbb{C}^n)$, their invariant is a real number $\rho(X, \alpha, g, V^\alpha, V^\tau)$ depending on all of its variables. Rho invariants for manifolds with boundary relate to APS rho invariants according to gluing formulas: Kirk and Lesch computed the correction term between the APS rho invariant of a closed manifold M which is split into a union $X \cup_\Sigma Y$ along a hypersurface and the sum of the rho invariants of X and Y.

We apply the Kirk–Lesch rho invariants to low dimensional topology in the context of 3-*manifolds with framed toroidal boundary.* These are couples $(X\mathcal{F})$, where X is an oriented 3-manifold X whose boundary Σ is a disjoint union of tori, and $\mathcal{F}$ is an ordered basis for the first homology group of each of these tori. In this setting, given a representation $\alpha\colon \pi_1(X) \to U(n)$, we define

$$\rho(X, \alpha, \mathcal{F}) := \rho(X, \alpha, g_{\mathcal{F}}, V^\alpha_{\mathcal{F}}, V^\tau_{\mathcal{F}}),$$

where the Riemannian metric and the Lagrangians subspaces are obtained from the framing $\mathcal{F}$ in a natural way. The Lagrangians are chosen in order to make the gluing formulas as easy as possible.

Given a link $L \subseteq S^3$, its exterior X_L is a 3-manifold with toroidal boundary that we can equip with the framing $\mathcal{F}_L$ given by preferred longitudes and meridians. Given a representation $\alpha\colon \pi_1(X_L) \to U(n)$, we can define the link invariant

$$\rho(L, \alpha) := \rho(X_L, \mathcal{F}_L, \alpha).$$

We have then the following result.

Proposition 1. *If $\alpha\colon \pi_1(X_L) \to U(n)$ factors through $\pi_1(M_L)$, then $\rho(L, \alpha) = \rho(M_L, \alpha)$.*

This means that $\rho(L, \alpha)$ coincides with $\rho(M_L, \alpha)$ whenever the second invariant is defined, inheriting all of its concordance properties. The advantage of considering $\rho(L, \alpha)$ instead of $\rho(M_L, \alpha)$ lies in the fact that representations of $\pi_1(X_L)$ which do not factor through $\pi_1(M_L)$ can also be considered. This might turn particularly useful for studying concordance of links with non-vanishing linking numbers.

For 1-dimensional representations, the rho invariant $\rho(L, \alpha)$ can be explicitly compared to another well-known invariant. We recall that Cimasoni–Florens signatures are a multivariable version of the Levine–Tristram signatures [CF08]. They are integers $\sigma_L(\omega)$ defined for an r-component link $L \subseteq S^3$ and an r-tuple $\omega \in (S^1 \setminus \{1\})^r$ and they are invariant under link concordance and 1-solvable concordance for most values of ω [CNT17]. The following result implies that $\rho(L, \alpha)$ (for 1-dimensional α's) has the same concordance properties as the Cimasoni–Florens signatures.

Theorem 2. *Let $\alpha\colon \pi_1(X_L) \to U(1) = S^1$ the representation sending the i-th meridian of L to ω_i for all i. Then $\rho(L,\alpha) = \sigma_L(\omega)$ up to a correction term that only depends on ω and the linking numbers of L.*

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Pure braids, Whitney towers, and 0-solvability

Shelly Harvey

(joint work with JungHwan Park, Arunima Ray)

Let $\mathcal{C}^m$ be the string link concordance group of m-component links and P_m be the group of isotopy classes of pure braids with m-components. It is well-known that P_m is a subgroup of C^m and that two pure braids are isotopic if and only if they have the same Milnor's invariants. In this report, we investigate the relationship between the subgroup of pure braids and the n-solvable filtration, $\{\mathcal{F}_n\}_n$, of $\mathcal{C}^m$.

Recall that the n-solvable filtration of C^m is a highly non-trivial, and somewhat algebraically defined filtration of C^m. For instance, even for knots, it is known that each of the successive quotients of $\mathcal{C}^m$, $\mathcal{F}_n/\mathcal{F}_{n.5}$, contains of copy of $\mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty$ [CHL09, CHL11]. Recall that the knot concordance group is an abelian group (albeit, non finitely generated and far from being understood). However, for links (when $m \geq 2$), the string link concordance group is non-abelian. For a specific example, the Borromean rings are known to the be the closure of a commutator of the generators of the pure braid group; this is known not to be slice since it has Milnor's triple invariants non-zero. A natural question to ask is the following.

Question 1. *Are the successive quotients of the n-solvable filtration of $\mathcal{C}^m$, namely*

$$\mathcal{F}_n^m/\mathcal{F}_{n.5}^m$$

non-abelian for $m \geq 2$ and $n \geq 0$?

Note that Martin showed that $\mathcal{F}_{-0.5}/\mathcal{F}_0$ is abelian and is classified certain by Milnors invariants of length at most 4 [Ma16]. See also [MO17].

We could also how much of the non-commutativity of the link concordance group is captured by pure braids. It was shown by Kirk–Livingston–Wang the P_m is not normal for $m \geq 3$ and that $C^m/Ncl(P_m) \neq 0$ [KLW]. They did using the Gassner representation. Recently, M. Kuzbary show that for $m = 2$ that C^2/P_2 is non-abelian using the Sato–Levine invariant.

Question 2. *Is $C^m/Ncl(P_m)$ non-abelian for $m \geq 3$?*

To begin to approach these problems, we investigate $P_m \cap \mathcal{F}_n^m$, starting with small n. By Martin's theorem, it follows that $(P_m)^{(2)} \subset \mathcal{F}_0^m$ for all m, where $(P_m)^{(2)}$ is the 2^{nd} term of the derived series of P_m. For any group G, recall that $G^{(n)}$, the derived series of G, is defined inductively by $G^{(0)} = G$ and $G^{(n+1)} = [G^{(n)}, G^{(n)}]$.

Question 3. *Is $(P_m)^{(n+2)} \subset \mathcal{F}_n^m$ for $n \geq 1$?*

We show that $P_m^{(2)}$ lives inside a smaller, more geometrically defined subgroup. For each m and n, we define the group of m-compomnent (string) links bounded by symmetric Whitney towers of height n, denoted $\mathcal{W}_n^m$. We show that elements of $P^{(2)}$ bounds height 2 Whitney towers.

Theorem 4. *For each m, $P_m^{(2)} \subset \mathcal{W}_2^m$.*

There is a much smaller and more mysterious subgroup of $\mathcal{F}_0^m$, the subgroup of (string) links bounding *symmetric* gropes of height 2, denoted $\mathcal{G}_2^m$. It would be interesting to know if every element in $P_m^{(2)}$ bounds a symmetric grope of height 2!

Question 5. *Is $(P_m)^{(2)} \subset \mathcal{G}_2^m$?*

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Regularity and group actions and applications to complex geometry

Thomas Koberda
(joint work with Sang-hyun Kim)

In this talk, we discuss the relationship between the algebraic structure of a group G and the possible degrees of regularity of faithful action of G on a compact one–manifold, with applications to complex geometry. We generally follow the results of [1] and [4]. There is some overlap in the talk and in the report with the speaker's previous talk at MFO in December 2016, workshop 1649.

It is a standard fact that a countable group arises as a subgroup of the orientation preserving homeomorphisms of the interval or of the circle if and only if it is left orderable or cyclically orderable, respectively. This algebraically characterizes countable subgroups of these continuous groups. If the level of regularity is increased, it is much harder to decide if a group admits a faithful of that level of regularity.

We illustrate this point as follows: it is a classical result of Nielsen that if S is a surface of genus at least two with one marked point, then the mapping class group $\mathrm{Mod}(S)$ acts faithfully by orientation-preserving homeomorphisms on the circle. This action has intrinsic non-differentiability, and Farb–Franks [2] (and independently Ghys) showed that sufficiently complicated mapping class groups admit no faithful C^2 actions on the circle or on the interval (in fact, they show that all such actions are trivial). Parwani [6] showed that sufficiently complicated mapping class groups admit no faithful C^1 actions on the circle.

The main result of [1] shows that no finite index subgroup of a mapping class group admits a faithful C^2 action on a compact one–manifold, except in certain sporadic degenerate cases. This completely answers a question of Labourie. It is important to note that the category of compact one–manifolds is more natural than just the circle or the interval, since it is easy to produce a finite extension of a free group which is not cyclically orderable, even though such a group should naturally act on a compact one–manifold with a high level of regularity.

The tool which allows us to study finite index subgroups of mapping class groups is their right-angled Artin subgroups, as studied by the author in [5]. Once a right-angled Artin group occurs in a given group, it persists inside of all finite index subgroups of that group. In [1], it is proved that the right-angled Artin group on the graph P_4, the path on four vertices, admits no faithful C^2 action on a compact one-manifold. This result can be used to characterize the mapping class groups which admit finite index subgroups with faithful C^2 actions on a compact one-manifold. Moreover, this result proves that braid groups on four or more strands and many other natural examples of groups cannot admit faithful C^2 actions on a compact one–manifold, even after passing to a finite index subgroup.

The main result of [4], which generalizes the main result of [1], is that if G is a group which is not virtually metabelian then the group $(G \times \mathbb{Z}) * \mathbb{Z}$ admits no faithful C^2 action on a compact one–manifold. Since this theorem applies when

G is a nonabelian free group, it is not difficult to see that the result of [4] in fact subsumes the main result of [1].

Among the corollaries of the result of [4] which are not implied by the main result of [1] is the classification of right-angled Artin groups which admit faithful C^2 actions on a compact one–manifold. They are exactly the finite direct products of finite free products of free abelian groups. Moreover, there are implications beyond mapping class groups and right-angled Artin groups. Namely, it is a straightforward consequence of the main result of [4] that if F denotes Thompson's group F, then the group $F * \mathbb{Z}$ admits no faithful C^2 action on a compact one–manifold, even though F itself is topologically conjugate into the group of C^∞ diffeomorphisms of the interval, by a result of Ghys–Sergiescu [3].

If F_2 denotes the free group on two generators, then the group $(F_2 \times \mathbb{Z}) * \mathbb{Z}$ is a "poison" subgroup for any group which one wishes to act by C^2 diffeomorphisms on a compact manifold. This group occurs often in complex geometry: for instance, pure braid groups on four or more strands contain $(F_2 \times \mathbb{Z}) * \mathbb{Z}$, and they are fundamental groups of complements of hyperplane arrangements in complex projective space. Many arrangement groups do act faithfully by even C^∞ diffeomorphisms on every one–manifold, for example the arrangement groups which are isomorphic to direct products of free groups, or more generally which can be realized as subgroups of direct products of free groups. However, any arrangement group which contains a copy $(F_2 \times \mathbb{Z}) * \mathbb{Z}$ is precluded from admitting any such action.

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Asymptotic properties on fundamental groups of quasiprojective surfaces

José Ignacio Cogolludo-Agustín

(joint work with Anatoly Libgober)

The present talk is concerned with the asymptotic behavior of certain invariants of the fundamental group of complements of divisors on smooth projective surfaces.

Such invariants are mainly two: Alexander type invariants and the number of surjections onto free groups. By asymptotic behavior we mean the behavior of such invariants as the degree of the divisor increases.

The main focus of our interest will be curves in the complex projective plane, but similar questions can be asked about general smooth projective surfaces.

Alexander type invariants. Let G be a finitely presented group, which in our setting will be the fundamental group $\pi_1(S \setminus D)$ of the complement of a reduced divisor D in a smooth projective surface S. Its variety of characters, $\mathbb{T}_G = \operatorname{Hom}(G, \mathbb{C}^*)$ has a natural structure of an abelian variety as a finite disjoint union of translated complex tori $\mathbb{T}_G = \bigsqcup_{t\in \operatorname{Tors}(G)} \xi_t \cdot (\mathbb{C}^*)^{b_1(G)}$ of dimension $b_1(G) = \operatorname{rank} G/G'$, where $G/G' \simeq \mathbb{Z}^{b_1(G)} \times \operatorname{Tors}(G)$. Each character $\rho \in \mathbb{T}_G$ defines a maximal ideal in the ring $R_G = \mathbb{C}[G/G']$, that is, $\operatorname{MaxSpec}(R_G) \simeq \mathbb{T}_G$. On the other hand, the group G/G' acts on G'/G'' by conjugation. This turns $A_G = \mathbb{C}[G'/G'']$ into an R_G-module called the *Alexander invariant* of G. The support of this module defines then a subset $V(G)$ of $\mathbb{T}_G$ called the *characteristic variety* of G.

In case $G = \pi_1(\mathbb{S}^3 \setminus K)$ where $K \subset \mathbb{S}^3$ is a knot in $\mathbb{S}^3$, then $G/G' = \mathbb{Z}$ and $R_G = \mathbb{C}[t^{\pm 1}]$ is hence a Principal Ideal Domain. In this case, $A_G = R_G^r \times R_G/(\lambda_1) \times \cdots \times R_G/(\lambda_n)$ and $V(G)$ is given by the set of zeroes of the Alexander polynomial $\Delta_K(t) = \prod_{i=1}^n \lambda_i$.

Note that any surjection $G_1 \twoheadrightarrow G_2$ induces an injection $V(G_2) \hookrightarrow V(G_1)$ which should be understood via the inclusion of characters $\mathbb{T}_{G_2} \subset \mathbb{T}_{G_1}$. In the particular case of $G = \pi_1(S\setminus D)$, note that if $D = D_1 \cup D_2$, then the inclusion $S\setminus D \hookrightarrow S\setminus D_2$ induces a surjection $G \twoheadrightarrow G_2 = \pi_1(S \setminus D_2)$. An irreducible component V of $V(G)$ (which will also be denoted as $V(D)$) is called *essential* if it is not the image of any $V' \subset V(D_2) \hookrightarrow V(D)$.

For simplicity we will assume from now on that S is simply connected and consider $D = D_1 \cup \cdots \cup D_r$ descomposition of D in irreducible components. The existence of components $V \subset V(D)$, $G = \pi_1(S\setminus D)$ of positive codimension one are connected with the existence of a pencil $\phi\colon S \dashrightarrow \mathbb{P}^1$ such that and $D_i = \phi^{-1}(p_i)$ for some $p_1, \ldots, p_r \in \mathbb{P}^1$. Such a divisor will be called in *pencil position.*

As a consequence of a remarkable result by Yuzvinsky and Pereira-Yuzvinsky [13, 19], if $S = \mathbb{P}^2$ and D is a union of lines, then $V(D)$ has an essential component of dimension $k > 3$ if and only if D is in pencil position, that is, a union of $k+1$ concurrent lines.

Question. Is this result specific for union of lines or is it more general?

Surjections onto free groups. The complexity of a group G can be messured by the ammount of homomorphisms onto free groups. In the particular case of $G = \pi_1(S\setminus D)$, a fundamental theorem by Arapura [2], which has been generalized and strengthened in several directions [4, 6] implies that any positive dimensional component $V \subset V(D)$ of dimension k is a (possibly translated) pull-back of a component $V' \subset V(\mathbb{P}^1 \setminus \{p_0, \ldots, p_k\})$ for a pencil map $S \dashrightarrow \mathbb{P}^1$. The condition about the possible translation of the component V' can be avoided by using the orbifold structure on $\mathbb{P}^1 \setminus \{p_0, \ldots, p_k\}$ induced by the existence of multiple fibers in the pencil; see [3, 4].

As a consequence, the number of different homomorphisms of G onto a non-abelian free group $\mathbb{F}_k$, can be recovered as the number of different pencil maps $\phi\colon S \dashrightarrow \mathbb{P}^1$ where $\phi^{-1}(\{p_0, \ldots, p_k\}) \subset D$. Moreover, ϕ is called *essential* if $\phi^{-1}(\{p_0, \ldots, p_k\}) = D$.

k-reducible divisors. From the previous discussion, it is convenient to distinguish a curve in pencil position and a curve with an essential pencil map. In the former, each fiber $\phi^*(p_i)$ is an irreducible curve of the same degree, whereas in the latter this is not necessarily the case. For instance the twelve lines joining the nine inflexion points of a smooth cubic have an essential pencil map, which is the corresponding pencil of cubics. These twelve lines are the union of four fibers of such a pencil. However, this curve of degree twelve is not in a pencil position since the lines are not convergent.

It is thus convenient to study possible bounds on the number of reducible curves in linear systems on smooth surfaces. In order to describe them one needs the concept of k-reducibility. A divisor $C = \sum_{i=1} n_i C_i$ in $\mathbb{P}^2$ is called a *k-reducible* divisor if its irreducible components C_i have degree at most k.

Note that 1-reducible curves are classically known as *completely reducible* curves or *totally reducible* [8, Question 11.6]. As was mentioned above, the maximal number of completely reducible members of a primitive base-component-free pencil of curves of degree $d > 1$ is 4. Recall that a pencil is called *primitive* if its generic member is irreducible and *base-component free* if two generic members do not share an irreducible component. A systematic study of essential pencil maps of line arrangements can be found in [11, 10].

Let $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ denote a primitive base-component-free pencil of curves in $\mathbb{P}^2$ and let $\rho_{d,k}(\varphi) \in \mathbb{Z}_{>0}$ denote the number of k-reducible divisors in φ. Since $\rho_{d,k}(\varphi) \leq d^2 - 1$ (see [15, Satz C]), the maximum $\rho_{d,k}(\mathbb{P}^2) = \max\{\rho_{d,k}(\varphi)\}$ is well defined, but in principle it depends quadratically on d. However, by the previous discussion $\rho_{d,1}(\mathbb{P}^2) = 4$ gives a universal bound on the number of completely reducible divisors of a pencil of curves of degree d on $\mathbb{P}^2$ for $d \geq 2$.

The purpose of this talk is to analyze the extent of this remarkable property in two directions. On the one hand, one can analyze the existence of a universal bound for $\rho_{d,k}(\mathbb{P}^2)$ under some conditions for d. On the other hand, one can study possible analogues of $\rho_{d,1}$ for other surfaces.

Main Theorem A: As for the first question, we will show that $\rho_{d,k}(\mathbb{P}^2)$ has a universal bound whenever $d \geq 2k$ –which generalizes $d \geq 2$ for $k = 1$.

Main Theorem B: As for the second question we will prove that the pencils for which $\rho_{(3,3),1}(\mathbb{P}^1 \times \mathbb{P}^1) = 4$ are not unique (as conjectured for $\mathbb{P}^2$) and that $\rho_{d,1}(S_d)$ can be arbitrarily large for general surfaces in $\mathbb{P}^3$.

The condition $d \geq 2k$ is essential. Otherwise at most a linear bound in the degree of the linear system can be given. Ruppert described in [15] the existence of a pencil of curves of any degree d with exactly $3(d-1)$ reducible fibers. Consider the net $\mathcal{N}$ in $\mathbb{P}^2$ given by the following curves $\mathcal{C}_\lambda$ of degree d defined by the equation:

$$F_\lambda(x_0,x_1,x_2)=\lambda_0x_0(x_1^{d-1}-x_2^{d-1})+\lambda_1x_1(x_2^{d-1}-x_0^{d-1})+\lambda_2x_2(x_0^{d-1}-x_1^{d-1})$$

for any $\lambda=[\lambda_0:\lambda_1:\lambda_2]\in\mathbb{P}^2$. The curve

$$S(\lambda)=(\lambda_0^{d-1}-\lambda_1^{d-1})(\lambda_1^{d-1}-\lambda_2^{d-1})(\lambda_2^{d-1}-\lambda_0^{d-1})$$

in $\mathcal{N}$ defines the intersection with the discriminant. Moreover, any curve F_λ satisfying $S(\lambda)=0$ is reducible and contains a line. If $L(\lambda)$ is in general position with respect to $S(\lambda)$, then $L(\lambda)$ defines a pencil with exactly $3(d-1)$ reducible fibers.

As a consequence of the main Theorem A, a linear bound on the number of reducible members of a primitive base-component-free pencil of degree d is obtained as $3(d-1)$ (see [8, Question 11.6]). This improves the original bound given by Poincaré [14] as $(2d-1)^2+2d+2$ and the more recent one by Ruppert [15] given as d^2-1. This result has been extended either for arbitrary characteristic [12] or higher dimensions [16, 18, 17] or for some particular type of pencils coming from polynomial maps $f-\lambda z^d$ such as in [9, 12, 18, 1, 5, 7].

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Geometric and homological finiteness properties

Alexander I. Suciu

1. **Finiteness properties of spaces and groups.** A recurring theme in topology is to determine the geometric and homological finiteness properties of spaces and groups. For instance, one would like to decide whether a path-connected space X is homotopy equivalent to a CW-complex with finite k-skeleton. In this spirit, a group G is said to have property F_k if it admits a classifying space $K(G,1)$ with finite k-skeleton; likewise, G is said to have property FP_k if the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ admits a projective $\mathbb{Z}G$-resolution which is finitely generated in all dimensions up to k. If G is of type F_k then it is of type FP_k; the converse does not hold in general, but properties FP_k and F_2 together imply property F_k.

In [1], Bieri, Neumann, and Strebel associated to every finitely generated group G a subset $\Sigma^1(G)$ of the unit sphere $S(G)$ in the real vector space $\mathrm{Hom}(G,\mathbb{R})$. This "geometric" invariant of the group G is cut out of the sphere by open cones, and is independent of a finite generating set for G. Shortly after, Bieri and Renz introduced a nested family of higher-order invariants, $\{\Sigma^i(G,\mathbb{Z})\}_{i\geq 1}$, which record the finiteness properties of normal subgroups of G with abelian quotients. In [8], Farber, Geoghegan and Schütz further extended these definitions: to each connected, finite-type CW-complex X, they assign a sequence of invariants, $\{\Sigma^i(X,\mathbb{Z})\}_{i\geq 1}$, living in the unit sphere $S(X) \subset H^1(X,\mathbb{R})$. The sphere $S(X)$ can be thought of as parametrizing all free abelian covers of X, while the Σ-invariants (which are again open subsets), keep track of the geometric finiteness properties of those covers.

Another tack was taken by Dwyer and Fried in [7]. Instead of looking at all free abelian covers of X at once, they fix the rank, say r, of the deck-transformation group, and view the resulting covers as being parametrized by the rational Grassmannian $\mathrm{Gr}_r(H^1(X,\mathbb{Q}))$. Inside this Grassmannian, they consider the subsets $\Omega^i_r(X)$, consisting of all covers for which the Betti numbers up to degree i are finite, and show how to determine these sets in terms of the support varieties of the relevant Alexander invariants of X. Unlike the Σ-invariants, though, the Ω-invariants need not be open subsets, see [7, 21].

The Dwyer–Fried sets depend only on the homotopy type of X. Hence, if G is a finitely generated group, we may define $\Omega^i_r(G) := \Omega^i_r(K(G,1))$. Let now $\nu\colon G \twoheadrightarrow \mathbb{Z}^r$ be an epimorphism. As shown in [21], the following holds: If $\Omega^k_r(G)=\emptyset$ and $\Gamma := \ker(\nu)$ is of type F_{k-1}, then $b_k(\Gamma)=\infty$. To see how this works in a concrete example, let $Y = S^1 \vee S^1$; then $X = Y^{\times 3}$ is a classifying space for $G = F_2^3$. Let $\nu\colon G \to \mathbb{Z}$ be the homomorphism taking each standard generator to 1. Stallings showed in [18] that the group $\Gamma = \ker(\nu)$ is finitely presented, and that $H_3(\Gamma, \mathbb{Z})$ is not finitely generated. Using our machinery, we compute that $\Omega^3_1(X) = \emptyset$; and so, by the above, a stronger statement holds: $b_3(\Gamma)$ is not finite.

Theorem 1 ([6])**.** *For each $k \geq 3$, there is a smooth, complex projective variety M of complex dimension $k-1$ such that $\pi_1(M)$ is of type* F_{k-1}*, but not* FP_k*.*

This theorem answers in the negative a question of Kollár [12]. Some of the arguments that go into the proof are streamlined in [21]. Further examples of projective groups with exotic finiteness properties can be found in recent work of Llosa Isenrich and Bridson [13, 14, 2].

2. **Bounds on the Σ- and Ω-invariants.** Let $\widehat{G} = \mathrm{Hom}(G, \mathbb{C}^*) = H^1(X, \mathbb{C}^*)$ be the algebraic group of complex characters of $G = \pi_1(X)$. The *characteristic varieties* of X are the sets

$$\mathcal{V}^i(X) = \{\rho \in \widehat{G} \mid H_i(X, \mathbb{C}_\rho) \neq 0\}.$$

If the CW-complex X has finite k-skeleton, then $\mathcal{V}^i(X)$ is a Zariski closed subset of the algebraic group $\widehat{G}$, for each $i \leq k$. The varieties $\mathcal{V}^i(X)$ are homotopy-type invariants of X; moreover, $\mathcal{V}^1(X)$ depends only on $G = \pi_1(X)$. If we set $\mathcal{V}^i(G) := \mathcal{V}^i(K(G,1))$, then $\mathcal{V}^1(G) = \mathcal{V}^1(G/G'')$.

Let $\exp\colon H^1(X, \mathbb{C}) \to H^1(X, \mathbb{C}^*)$ be the coefficient homomorphism induced by the map $\mathbb{C} \to \mathbb{C}^*$, $z \mapsto e^z$. Given a Zariski closed subset $W \subset H^1(X, \mathbb{C}^*)$, let $\tau_1(W)$ be the 'exponential tangent cone' to W, i.e., the set of $z \in H^1(X, \mathbb{C})$ for which $\exp(\lambda z) \in W$, for all $\lambda \in \mathbb{C}$. As shown in [5], this set is a finite union of rationally defined linear subspaces. Furthermore, put $\tau_1^{\Bbbk}(W) = \tau_1(W) \cap H^1(X, \Bbbk)$ for $\Bbbk = \mathbb{Q}$ or $\mathbb{R}$, and write $\mathcal{W}^i(X) = \bigcup_{j \leq i} \mathcal{V}^j(X)$.

Theorem 2 ([16])**.** $\Sigma^i(X, \mathbb{Z}) \subseteq S(X) \setminus S(\tau_1^{\mathbb{R}}(\mathcal{W}^i(X))$.

For $i = 1$, equality holds for all right-angled Artin groups [16], as well as pure braid groups [9]. In general, though, the above inclusion is strict, even in the case of complements of hyperplane arrangements [20].

Given a homogeneous variety $V \subset \Bbbk^n$, the locus of r-planes in $\Bbbk^n$ intersecting V non-trivially, $\sigma_r(V)$, is a Zariski closed subset of the Grassmannian $\mathrm{Gr}_r(\Bbbk^n)$.

Theorem 3 ([19, 21])**.** $\Omega^i_r(X) \subseteq \mathrm{Gr}_r(H^1(X, \mathbb{Q})) \setminus \sigma_r\big(\tau_1^{\mathbb{Q}}(\mathcal{W}^i(X))\big)$.

Furthermore, if the upper bound for the Σ^i-invariants is attained, then the upper bound for the Ω^i_r-invariants is also attained, for all r, see [20].

3. **Infinitesimal finiteness obstructions.** Let A be commutative differential graded $\mathbb{C}$-algebra (for short, a CDGA). We say that A is *q-finite* if it is connected (i.e., $A^0 = \mathbb{C} \cdot 1$) and $\sum_{i \leq q} \dim A^i < \infty$. Two CDGAs A and B have the same q-type (written $A \simeq_q B$) if there is a zig-zag of CDGA maps connecting A and B, with each such map inducing isomorphisms in homology up to degree q and a monomorphism in degree $q+1$. Every CDGA A with $H^0(A) = \mathbb{C}$ admits a *q-minimal model*, $\mathcal{M}_q(A)$, unique up to isomorphism; see [23].

A q-*model* for a space X is a CDGA A with the same q-type as Sullivan's CDGA of piecewise polynomial, complex-valued forms on X [23]. Examples of spaces having finite-type models include formal spaces (such as compact Kähler manifolds, hyperplane arrangement complements, etc), smooth quasi-projective varieties, compact solvmanifolds, and Sasakian manifolds.

For each $a \in Z^1(A) \cong H^1(A)$, we construct a cochain complex, $(A^\bullet, \delta_a)$, with differentials $\delta_a^i \colon A^i \to A^{i+1}$, $u \mapsto a \cdot u + \mathrm{d}\, u$. The *resonance varieties* of A are the sets

$$\mathcal{R}^i(A) = \{a \in H^1(A) \mid H^i(A^\bullet, \delta_a) \neq 0\}.$$

If A is q-finite, these sets are Zariski closed, for all $i \leq q$. Given a connected, finite-type CW-complex X, we obtain the usual resonance varieties by setting $\mathcal{R}^i(X) := \mathcal{R}^i(H^\bullet(X, \mathbb{C}))$.

Theorem 4. *Let X be a connected CW-complex with finite q-skeleton which admits a q-finite q-model A. Then, for all $i \leq q$:*

- [4] $\mathcal{V}^i(X)_{(1)} \cong \mathcal{R}^i(A)_{(0)}$. *Hence, if X is q-formal, then $\mathcal{V}^i(X)_{(1)} \cong \mathcal{R}^i(X)_{(0)}$.*
- [15] $\mathrm{TC}_0(\mathcal{R}^i(A)) \subseteq \mathcal{R}^i(X)$.
- [3] *All the irreducible components of $\mathcal{V}^i(X)$ passing through the identity are* algebraic subtori *of $\pi_1(X)\widehat{\ }$.*

The just-mentioned result of Budur and Wang [3] yields a powerful obstruction for the existence of (partially) finite models for spaces and groups. For instance, if G is a finitely presented group with $G_{\mathrm{ab}} = \mathbb{Z}^n$ and $\mathcal{V}^1(G) = \{t \in (\mathbb{C}^*)^n \mid \sum_{i=1}^n t_i = n\}$, then G admits no 1-finite 1-model. In a recent preprint with S. Papadima, we provide a completely different obstruction.

Theorem 5 ([17])**.** *Suppose X is $(q+1)$ finite, or X admits a q-finite q-model. Then $b_i(\mathcal{M}_q(X)) < \infty$, for all $i \leq q+1$.*

Corollary 6 ([17])**.** *Let G be a finitely generated group. Assume that either G is finitely presented, or G has a* 1*-finite* 1*-model. Then $b_2(\mathcal{M}_1(G)) < \infty$.*

For instance, let $G = F_n/F_n''$ be the free metabelian group of rank $n \geq 2$. Then $\mathcal{V}^1(G) = \mathcal{V}^1(F_n) = (\mathbb{C}^*)^n$, and so G passes the Budur–Wang test. Yet $b_2(\mathcal{M}_1(G)) = \infty$, and so, by Corollary 6, this group admits no 1-finite 1-model, and no finite presentation. More generally, we have the following result.

Theorem 7 ([17])**.** *Let G be a finitely generated group which has a free, non-cyclic quotient. Then G/G'' is not finitely presentable, and does not admit a* 1*-finite* 1*-model.*

We also reinterpret the condition that a group G admits a 1-finite 1-model in terms of the Malcev Lie algebra $\mathfrak{m}(G)$, which is the set of primitive elements in the completion of the group algebra $\mathbb{Q}G$ with respect to the filtration by powers of the augmentation ideal; see for instance [22] and reference therein.

Theorem 8 ([17])**.** *A finitely generated group G admits a* 1*-finite* 1*-model if and only if $\mathfrak{m}(G)$ is the lower central series completion of a finitely presented Lie algebra.*

4. **RFRp groups, finiteness, and largeness.** In recent work with T. Koberda, we modify Agol's celebrated definition of RFRS groups, as follows. Let G be a finitely generated group and let p be a prime. We say that G is *residually finite rationally* p if there exists a descending sequence of subgroups $\{G_i\}_{i\geq 0}$ such that $G_0 = G$; $G_{i+1} \triangleleft G_i$; $\bigcap_{i\geq 0} G_i = \{1\}$; G_i/G_{i+1} is an elementary abelian p-group; and $\ker(G_i \to H_1(G_i, \mathbb{Q})) < G_{i+1}$. The class of RFR$p$ groups is closed under taking subgroups, finite direct products, and finite free products. Such groups are residually finite, torsion-free, and residually torsion-free polycyclic.

Theorem 9 ([11])**.** *Let G be a finitely presented, non-abelian group which is RFRp for infinitely many primes p. Then G is bi-orderable; the maximal k-step solvable quotients $G/G^{(k)}$ are not finitely presented, for any $k \geq 2$; and $\Sigma^1(G) \neq S(G)$.*

Surface groups and right-angled Artin groups are RFRp, for all p, but finite groups and non-abelian nilpotent groups are *not* RFRp, for any p. We show in [11] that a large class of groups occurring at the interface between complex algebraic geometry and low-dimensional topology enjoy the RFRp property. More precisely, let $\mathcal{C}$ be an algebraic curve in $\mathbb{C}^2$, with boundary manifold M. Suppose that each irreducible component of $\mathcal{C}$ is smooth and transverse to the line at infinity, and all singularities of $\mathcal{C}$ are of type A. Then $\pi_1(M)$ is RFRp, for all p.

A finitely generated group G is said to be *large* if there is a finite-index subgroup $H < G$ which surjects onto a free, non-cyclic group. As shown in [10], a finitely presented group G is large if and only if there exists a finite-index subgroup $K < G$ such that $\mathcal{V}^1(K)$ has infinitely many torsion points.

Theorem 10 ([11])**.** *Let G be a finitely presented group which is non-abelian and RFRp for infinitely many primes p. Then G is large.*

The following result from [17] (based on foundational work of Arapura) gives a geometric interpretation of largeness within the class of quasi-projective groups.

Proposition 11 ([17])**.** *Let X be a quasi-projective manifold. Then $\pi_1(X)$ is large if and only if there is a finite cover $Y \to X$ and a regular, surjective map from Y to a smooth curve C with $\chi(C) < 0$, so that the generic fiber is connected.*

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