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Mini-Workshop: Cohomology Rings and Fundamental Groups of Hyperplane Arrangements, Wonderful Compactifications, and Real Toric Varieties

Organised by
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Abstract. The purpose of this workshop was to bring together researchers with a common interest in the objects mentioned in the title from, respectively, the points of view of toric and tropical geometry, arrangement theory, and geometric group theory.

Mathematics Subject Classification (2010): Primary: 14M25, 14T05, 20F55, 52C35, 55N91, 57M07. Secondary: 05E25, 06A07, 13F55, 14C15, 14H10, 20F36, 20J05, 32S22, 52B20, 55P62, 57M05.

Introduction by the Organisers

Overview. This Mini-Workshop was organized by G. Denham (London, Canada) and A. Suciu (Boston, USA). The participants were drawn together by overlapping interests in combinatorial constructions in algebraic geometry, topology, and group theory. Several participants were recent Ph.D.s, some of them on their first visit to MFO. In all, there were 17 people attending the mini-workshop (including the organizers), coming from the United States, Canada, Germany, Great Britain, Italy, and Russia.

The meeting allowed us to compare some closely related constructions and find some common ground within the scope of rather varied disciplinary perspectives. The relatively spontaneous format of the meeting made it possible to mix some informal and semi-expository talks and small group discussions with more formal announcements of recent developments, indicated in the abstracts that follow.

Research themes. Some of the mathematical objects of interest included moment-angle complexes and their generalizations, as well as real and complex toric varieties; complex hyperplane and subspace arrangements; fundamental groups such as right-angled Artin groups and arrangement groups; tropical and wonderful compactifications.

One of the themes explored at the meeting was a wider accessibility of ideas from tropical geometry, as applied to the constructions above. The abstract by María Angélica Cueto makes this more precise and provides a quick introduction to the subject. In particular, the construction of De Concini and Procesi's wonderful compactification of the complement of a union of hyperplanes via toric geometry (an example of Tevelev's tropical compactifications) was reviewed: the abstract of Eva-Maria Feichtner mentions a solution to a problem posed by Corrado De Concini which was obtained at the workshop. Through informal discussions, the relationship between the Chow rings of the wonderful models with those of the compactifying toric varieties was brought into sharper focus. Furthermore, Diane Maclagan described how methods of tropical geometry could apply to describe the effective cone of the wonderful compactification.

The theory of toric varieties and torus-equivariant topology were implicit ingredients in much of what took place at the meeting. They were, in fact, the focus of the talks on equivariant (co)homology given by Matthias Franz and Hal Schenck. The wonderful models are also closely related to the toric varieties defined by classical root systems, such as the Hessenberg varieties of symmetric, isospectral tridiagonal matrices. These spaces and their cohomology rings admit Coxeter group actions; one approach to understanding their homology uses the representation theory of the reflection groups, together with a subtle comparison with the relevant wonderful compactifications. One of the abstracts by Alex Suciu describes another, totally different approach, based on the topological interpretation of smooth toric varieties pioneered by Davis and Januszkiewicz, and on some recent developments in toric topology.

One of the most fruitful ideas to arise from the theory of hyperplanes arrangements is that of turning the cohomology ring of a space into a family of cochain complexes, parametrized by the cohomology group in degree one, and extracting certain "resonance" varieties from these data, as the loci where the cohomology of those cochain complexes jumps. The abstract of Dan Cohen mentions a solution to a 12-year old conjecture, expressing the ranks of the Chen groups of an arrangement in terms of the dimensions of the components of the resonance varieties. In a more combinatorial vein, the abstract by Mike Falk describes various connections between resonance varieties of arrangements, multinets, Bergman fans, and tropical varieties. Finally, another abstract by Alex Suciu describes a stratification of the Grassmannian of m-planes in the second exterior power of a vector space, that keeps track of the corresponding resonance schemes.

Some of the themes from this mini-workshop turned out to overlap with those of the mini-workshop *Topology of Real Singularities and Motivic Aspects*, which led to some interesting discussions between members of the respective groups. In particular, Ian Leary had reported on the ℓ^2-cohomology of hyperplane complements, and Laurentiu Maxim was able to join this group for an afternoon and present some complementary results for affine hypersurface complements.

Concluding remarks. Spending a concentrated and highly intense week in a relatively small group allowed for in-depth and continuing conversations, in particular with new acquaintances. These opportunities (difficult to find at larger meetings) were enhanced by the diversity of backgrounds of the participants. This speaks to the fact that the usual, more rigid conference climate was superseded by an open and creative workshop atmosphere.

There was general agreement that the mini-workshop created an effective and stimulating research atmosphere. During the week of the workshop, and soon thereafter, some progress was made in solving old and new problems. The work initiated at Oberwolfach is continuing now in several research groups. The intense interactions at the meeting gave rise to new projects, which should start bearing fruit in the not too distant future.

Mini-Workshop: Cohomology Rings and Fundamental Groups of Hyperplane Arrangements, Wonderful Compactifications, and Real Toric Varieties

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Abstracts

Chen ranks and resonance

Daniel C. Cohen
(joint work with Henry K. Schenck)

Chen Ranks. Let G be a finitely presented group, with commutator subgroup $G' = [G, G]$, and second commutator subgroup $G'' = [G', G']$. The Chen groups of G are the lower central series quotients $\mathrm{gr}_k(G/G'')$ of G/G''. These groups were introduced by K.T. Chen in [1], so as to provide accessible approximations of the lower central series quotients of a link group. For example, if $G = F_n$ is the free group of rank n (the fundamental group of the n-component unlink), the Chen groups are free abelian, and their ranks, $\theta_k(G) = \operatorname{rank} \mathrm{gr}_k(G/G'')$, are given by $\theta_k(F_n) = (k-1)\binom{k+n-2}{k}$ for $k \geq 2$. In particular, $\theta_k(F_2) = k - 1$ for $k \geq 2$.

Let P_n be the Artin pure braid group on n strands, the fundamental group of the configuration space of n ordered points in $\mathbb{C}$. The Chen groups of P_n are free abelian, and their ranks are given by $\theta_k(P_n) = \binom{n+1}{4}(k-1)$ for $k \geq 3$, see [3].

Resonance. Let $A = H^*(G;\mathbb{C})$. For $a \in A^1$, since $a \cup a = 0$, multiplication by a provides A with the structure of a (cochain) complex: $A^0 \xrightarrow{a} A^1 \xrightarrow{a} A^2 \xrightarrow{a} \dots$ The (first) resonance variety of G is $\mathcal{R}^1(G) = \{a \in A^1 \mid H^1(A, a) \neq 0\}$, a homogeneous algebraic subvariety in $A^1 = H^1(G;\mathbb{C})$.

The group G is said to be 1-formal if the Malcev Lie algebra of G is quadratic (see [9] for details). For any finitely generated 1-formal group, Dimca-Papadima-Suciu [5] show that all irreducible components of the resonance variety $\mathcal{R}^1(G)$ are linear subspaces of A^1. In particular, this holds for an arrangement group, the fundamental group $G(\mathcal{A}) = \pi_1(M(\mathcal{A}))$ of the complement of a complex hyperplane arrangement, as previously shown by a number of authors, including Libgober-Yuzvinsky [8], who additionally show that the irreducible components of $\mathcal{R}^1(G(\mathcal{A}))$ are "projectively disjoint" – they meet only at the origin in A^1.

For example, if $G = P_n$ is the pure braid group, the fundamental group of the complement of the braid arrangement, $\mathcal{R}^1(P_n)$ is a union of $\binom{n+1}{4}$ 2-dimensional linear subspaces of $A^1 = H^1(P_n;\mathbb{C})$, see for instance [4]. Note that, for $k \geq 3$, $\theta_k(P_n) = \binom{n+1}{4}\theta_k(F_2)$. This, and many other examples, led to the following.

Conjecture 1 (Suciu [12]). *If $G = G(\mathcal{A})$ is an arrangement group, then, for $k \gg 0$, $\theta_k(G) = \sum_{r\geq 2} h_r \theta_k(F_r)$, where h_r is the number of irreducible components of dimension r in $R^1(G)$.*

In this talk, we announce a positive resolution of this conjecture. Some additional terminology is required to state the general result.

Recall that $A = H^*(G;\mathbb{C})$, and let $\mu\colon A^1 \wedge A^1 \to A^2$ be the cup product map, $\mu(a \wedge b) = a \cup b$. A non-zero subspace $U \subseteq A^1$ is said to be p-isotropic with respect to the cup product map if the restriction of μ to $U \wedge U$ has rank p.

A group G is said to be a commutator-relators group if it admits a presentation $G = F/R$, where F is a finitely generated free group and R is the normal closure of a finite subset of $[F, F]$. For such a group, the resonance variety $\mathcal{R}^1(G)$ may be realized as the variety defined by the annihilator of the linearized Alexander invariant B of G, a module over the polynomial ring $S = \mathrm{Sym}(H_1(G;\mathbb{C})) = \mathbb{C}[x_1, \dots, x_n]$, $\mathcal{R}^1(G) = V(\mathrm{ann}(B))$. We can thus view $\mathcal{R}^1(G)$ as a scheme.

Theorem 1. *Let G be a finitely presented, 1-formal, commutator-relators group. Assume that the components of $\mathcal{R}^1(G)$ are (i) 0-isotropic, (ii) projectively disjoint, and (iii) reduced (viewing $\mathcal{R}^1(G)$ as a scheme). Then, for $k \gg 0$,*

$$\theta_k(G) = \sum_{r\geq 2} h_r(k-1)\binom{r+k-2}{k} = \sum_{r\geq 2} h_r\theta_k(F_r),$$

where h_r is the number of irreducible components of dimension r in $R^1(G)$.

Examples illustrating the necessity of the hypotheses in the theorem include:

(1) The Heisenberg group $G = \langle a, b \mid [a,[a,b]], [b,[a,b]]\rangle$ is not 1-formal. Here, $\mathcal{R}^1(G) = H^1(G;\mathbb{C})$ is 0-isotropic since the cup product is trivial. But $\theta_k(G) \neq \theta_k(F_2)$. Since G is nilpotent, $\theta_k(G) = 0$. See [5].

(2) The fundamental group G of a closed, orientable surface of genus $g \geq 2$ is 1-formal. But $\mathcal{R}^1(G) = H^1(G;\mathbb{C})$ is not 0-isotropic, and $\theta_k(G) \neq \theta_k(F_{2g})$. See [9].

(3) Let Γ be the graph with vertex set $\mathcal{V} = \{1,2,3,4,5\}$ and edge set $\mathcal{E} = \{12, 13, 24, 34, 45\}$, and let $G = G_\Gamma$ be the corresponding right angled Artin group. The resonance variety is the union of two 3-dimensional subspaces in $H^1(G;\mathbb{C})$ which are not projectively disjoint, and $\theta_k(G) \neq 2\theta_k(F_3)$. See [10].

(4) Let $G = \langle a, b, c, d \mid [b,c], [a,d], [c,d], [a,c][d,b]\rangle$. As a variety, $\mathcal{R}^1(G)$ is a 2-dimensional subspace of $H^1(G;\mathbb{C})$. But $\mathcal{R}^1(G)$ is not reduced, and $\theta_k(G) \neq \theta_k(F_2)$. We do not know if this group is 1-formal.

Arrangement groups satisfy the hypotheses of the theorem, as does the "group of loops," see below. Examples illustrating the utility of the theorem include:

(1) Let PB_n denote the type B pure Artin group, the fundamental group of the complement of the type B Coxeter arrangement in $\mathbb{C}^n$. Analysis of $\mathcal{R}^1(PB_n)$ yields $\theta_k(PB_n) = \left[16\binom{n}{3} + 9\binom{n}{2}\right](k-1) + \binom{n}{2}(k^2-1)$ for $k \gg 0$.

(2) Let $P\Sigma_n$ be the McCool group of basis conjugating automorphisms of the free group of rank n, also known as the group of loops. Analysis of $\mathcal{R}^1(P\Sigma_n)$ (see [2]) yields $\theta_k(P\Sigma_n) = \binom{n}{2}(k-1) + \binom{n}{3}(k^2-1)$ for $k \gg 0$.

Discussion. We discuss some elements of the proof of the theorem.

As noted previously, since G is 1-formal, work of Dimca-Papadima-Suciu [5] implies that $\mathcal{R}^1(G)$ is a union of linear subspaces in $A^1 = H^1(G;\mathbb{C})$. Since G is additionally a commutator-relators group, work of Papadima-Suciu [9] implies that the Chen ranks of G are given by the Hilbert series of the linearized Alexander invariant B of G, $\sum_{k\geq 2}\theta_k(G)t^k = \mathrm{Hilb}(B,t)$ (with appropriate degree conventions).

Assume that G is minimally generated by n elements, so that $A^1 = H^1(G;\mathbb{C}) \cong \mathbb{C}^n$, generated by $e_1, \dots, e_n$. Let $E = \bigwedge A^1$ be the exterior algebra on A^1, and let I be the ideal in E generated by $\ker(\mu\colon A^1 \wedge A^1 \to A^2)$, the kernel of the cup product map in degree 2. The linearized Alexander invariant B of G admits a presentation $S \otimes I^3 \xrightarrow{\partial} S \otimes I^2 \to B \to 0$, where the map ∂ is dual to the map $S \otimes I^2 \to S \otimes I^3$ given by multiplication by $x = \sum_{i=1}^n x_i \otimes e_i \in S \otimes E^1$.

If L is an irreducible component of $\mathcal{R}^1(G)$, let I_L be the ideal in E generated by $\bigwedge^2 L$, a subideal of I. Associated to I_L, we have a "local" linearized Alexander invariant B_L and a surjection $B \to B_L$. This yields an exact sequence of S-modules

$$0 \longrightarrow K \longrightarrow B \longrightarrow \oplus B_L \longrightarrow C \longrightarrow 0,$$

the direct sum over all irreducible components L of $\mathcal{R}^1(G)$. One can check that $\mathrm{Hilb}(B_L, t) = \sum_{k \geq 2} \theta_k(F_r)$ if $\dim(L) = r$. To prove the theorem, it suffices to show that the kernel K and cokernel C above have finite length.

If $G = G(\mathcal{A})$ is an arrangement group, Schenck-Suciu [11] show that C has finite length. This argument extends. Showing that K has finite length is more involved. One part of this is the following. Given $L \subset \mathcal{R}^1(G)$ as above, let J_L be the ideal in E generated by $\{q \in I \mid \ell \wedge q \in I_L \forall \ell \in L\}$.

Lemma 1. *$I_L = J_L$ if and only if L is reduced.*

Arrangement groups and the basis-conjugating automorphism group are known to satisfy all the hypotheses of the theorem, except possibly the condition that all components of the resonance variety are reduced. For this, by the lemma, it suffices to show that $I_L = J_L$ for each (irreducible) $L \subset \mathcal{R}^1(G)$. This can be done directly in the case where $G = P\Sigma_n$. For $G = G(\mathcal{A})$ an arrangement group, this can be done using the structure of resonance varieties of arrangement groups uncovered by work of Falk [6], Libgober-Yuzvinsky [8], and Falk-Yuzvinsky [7].

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An introduction to Tropical Geometry and Tropical Compactifications

María Angélica Cueto

The field of tropical geometry began as a framework to link amoebas, logarithmic limit sets [1], and (real) algebraic geometry. It synthesized and boosted the pioneering work of Bieri–Groves [2] and Viro's "patchworking" techniques to construct real algebraic varieties by "cutting and pasting" [3, 7]. In its ten years of existence, it has brought on truly explosive development, establishing deep connections with enumerative algebraic geometry, symplectic and analytic geometry, number theory, dynamical systems, mathematical biology, statistical physics, random matrix theory, and mathematical physics.

Tropical geometry can be considered as algebraic geometry over the semifield $(\mathbb{R}, \min, +)$. It is a polyhedral version of classical algebraic geometry: algebraic varieties are replaced by weighted, balanced polyhedral complexes, in order to answer open questions or to derive simpler proofs of classical results. These objects preserve just enough data about the original varieties to remain meaningful, while discarding much of their complexity. There are many approaches to this subject: valuation theory, logarithmic limits sets in the sense of Bergman, and Gröbner theory. Here, we choose the first perspective.

Throughout this talk, we consider an algebraically closed field K with a non-trivial valuation $\operatorname{val}\colon K^* = K \smallsetminus \{0\} \to \mathbb{R}$. Here, by valuation we mean a function that satisfies the following properties:

(1) $\operatorname{val}(f\,g) = \operatorname{val}(f) + \operatorname{val}(g)$, for any pair $f, g \in K^*$,
(2) $\operatorname{val}(f + g) \geq \min\{\operatorname{val}(f), \operatorname{val}(g)\}$.

It is not hard to show that if $\operatorname{val}(f) \neq \operatorname{val}(g)$, then the second condition above is an equality, i.e. $\operatorname{val}(f+g) = \min\{\operatorname{val}(f), \operatorname{val}(g)\}$. By declaring $\operatorname{val}(0) = \infty$, we can extend the valuation to all K.

Our favorite example of a valued field (K, val) as above is given by the Puiseux series $K = \mathbb{C}\{\{t\}\}$, whose elements are Laurent polynomials in $t^{1/n}$, where we let $n \in \mathbb{N}$. The valuation of a series is given by its lowest exponent.

Definition 1. Given an algebraically closed valued field (K, val) as above, and a subvariety $Y \subset (K^*)^n$, we define:

$$\mathcal{T}Y = \operatorname{closure}\{(\operatorname{val}(y_1), \ldots, \operatorname{val}(y_n)) : \underline{y} = (y_1, \ldots, y_n) \in Y\} \subset \mathbb{R}^n,$$

where the closure is taken with respect to the Euclidean Topology in $\mathbb{R}^n$.

Example 1. We consider the line $Y = \{x + y + 1 = 0\} \subset (\mathbb{C}\{\{t\}\}^*)^2$. Notice that this variety is defined over $\mathbb{C}$, i.e. its defining equation has complex coefficients. We can rewrite it as

$$Y = \{(a, -1 - a) : a \in \mathbb{C}\{\{t\}\} \smallsetminus \{0, -1\}\}.$$

By definition, we get

$$(1) \qquad (\mathrm{val}(a), \mathrm{val}(-1 - a)) = \begin{cases} (\mathrm{val}(a), 0) & \text{if } \mathrm{val}(a) > 0, \\ (\mathrm{val}(a), \mathrm{val}(a)) & \text{if } \mathrm{val}(a) < 0, \\ (0, \mathrm{val}(a + 1)) & \text{if } \mathrm{val}(a + 1) > 0, \\ (0, 0) & \text{otherwise.} \end{cases}$$

After taking closure, we obtain the picture on the left-hand side of Figure 1. It has the structure of a one dimension pure polyhedral fan, glued out of a point and three half rays. Each one of the pieces on the right correspond to the four different cases in (1). ◇

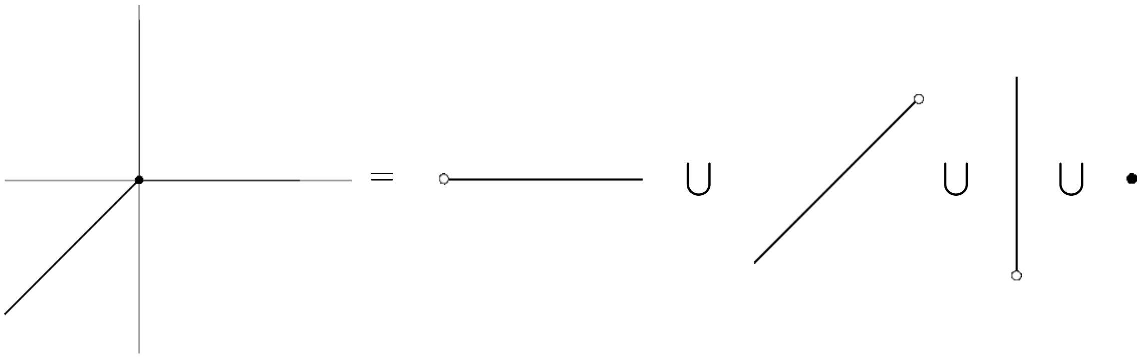

FIGURE 1. Tropicalization of the line $Y = \{x + y + 1 = 0\}$ in the 2-dimensional algebraic torus over $\mathbb{C}\{\{t\}\}$.

If we pick an irreducible subvariety $Y \subset (\mathbb{C}\{\{t\}\}^*)^n$ defined over $\mathbb{C}$, then $\mathcal{T}Y$ is a pure rational weighted balanced polyhedral fan. These weights encode the intersection theory on the toric variety X_Σ associated to the fan $\Sigma = \mathcal{T}Y$ [5]. This fan reflects many property of the variety Y, including its dimension and degree (in the projective case). In 2007 Tevelev showed that, surprisingly, the support of this fan gives a method for constructing nice compactifications of Y inside toric varieties [6]. In particular, it allows us to answer the following question:

Question 1. Given a subvariety Y of the algebraic torus $T = (\mathbb{C}^*)^n$ and a toric variety X_Σ with dense torus T, consider the closure $\overline{Y}$ of Y inside X_Σ. Which T-orbits of X_Σ does $\overline{Y}$ intersect?

Tevelev's result says that $\overline{Y}$ intersects the T-orbit of a cone $\sigma \in \Sigma$ if an only if $\mathcal{T}(\overline{Y} \cap T)$ intersects the relative interior of σ [6, Lemma 2.2]. Therefore, if we choose a non-complete fan Σ supported on $\mathcal{T}(\overline{Y} \cap T) \subset \mathbb{R}^n$, then the a priori partial compactification $\overline{Y} \subset X_\Sigma$ is indeed compact. Moreover, the T-orbit of an s-dimensional cone in Σ intersects $\overline{Y}$ in codimension s [6, Propositions 2.3 ad 2.5].

We conclude by interpreting the weights in $\mathcal{T}Y$ in terms of $\overline{Y}$. If the fan Σ is smooth and refines the Gröbner fan structure on $\mathcal{T}Y$, then the weights on the maximal cones of $\mathcal{T}(\overline{Y} \cap T)$ reflect the intersection theory on X_Σ. More precisely, the weight on a maximal cone σ is the intersection number of the class of $\overline{Y}$ and the class of the closure of the T-orbit associated to σ [4, Lemma 9.2]. We illustrate these facts with the example of a line in 2-space.

Example 2. Consider the line $Y = \{x+y+1=0\}$ as in Example 1 and its naïve compactification $\overline{Y} = \{x+y+z=0\}$ inside the toric variety $\mathbb{CP}^2$. We analyze the intersection of $\overline{Y}$ with each of the seven torus orbits in $\mathbb{CP}^2$.

(1) $\overline{Y}$ intersects the T-orbit of the cone $\{(0,0)\}$ in Y and $\mathcal{T}Y$ contains this cone. The intersection has codimension 0 in $\overline{Y}$.
(2) The T-orbit associated to the cone $\mathbb{R}\langle e_1\rangle$ intersects $\overline{Y}$ at the point $\{(1 : 0 : -1)\}$ with multiplicity one, and $\mathcal{T}Y$ contains this 1-dimensional cone. This cone has weight one. This set has codimension 1 in $\overline{Y}$. A similar situation occurs with the cones $\mathbb{R}\langle e_2\rangle$ and $\mathbb{R}\langle -e_1 - 2_2\rangle$.
(3) Finally, notice that $\mathcal{T}Y$ does not contain any of the two dimensional cones $\mathbb{R}\langle e_1, e_2\rangle$, $\mathbb{R}\langle e_1, -e_1 - e_2\rangle$ nor $\mathbb{R}\langle e_2, -e_1 - e_2\rangle$, and none of the three torus fixed points $(1:0:0), (0:1:0), (0:0:1)$ belong to $\overline{Y}$. Thus, the intersection is empty and its codimension in $\overline{Y}$ is 2. ◇

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Resonance varieties and tropical geometry

Michael J. Falk

In this informal talk we sketch a connection between resonance varieties and tropical geometry, and in the process introduce some of the main notions to be studied in the mini-workshop, namely cohomology jumping loci, multinets and the associated pencils of hypersurfaces, and Bergman fans. The material comes from joint work with Dan Cohen, Graham Denham, and Alexander Varchenko [CDFV12], and work in progress with Eva-Maria Feichtner.

Let $\mathcal{A} = \{H_0, \ldots, H_n\}$ be an arrangement of hyperplanes in $\mathbb{P}^\ell$, with H_i defined by the linear homogeneous form $\alpha_i \colon \mathbb{C}^{\ell+1} \to \mathbb{C}$ for $0 \le i \le n$. Let $\omega_j = \mathrm{d}\log(\alpha_j) = \frac{\mathrm{d}\alpha_j}{\alpha_j}$. The cohomology of the complement $M = \mathbb{P}^\ell - \bigcup_{i=0}^n H_i$ is isomorphic to the algebra $A^\cdot$ of differential forms generated by $\{\sum_{j=0}^n \lambda_j \omega_j \mid \sum_{j=0}^n \lambda_j = 0\}$. Given $a \in A^1$, one has a cochain complex

$$(A^\cdot, a) : 0 \to A^0 \xrightarrow{a\wedge -} \cdots \xrightarrow{a\wedge -} A^\ell \to 0.$$

The p^{th} resonance variety $\mathcal{R}^p(\mathcal{A})$ is $\{a \in A^1 \mid H^p(A^\cdot, a \wedge -) \neq 0\}$. These are subtle invariants of the graded ring $A^\cdot$. For example, degree-one resonance varieties distinguish the pure braid group from a product of free groups, where the betti numbers and lower central series quotients coincide.

We are interested in completely decomposable cocycles in the complex $(A^\cdot, a)$, or, more precisely, collections of linearly independent one-forms $a_i = \sum_{j=0}^n \lambda_{ij}\omega_j$, $0 \le i \le p < \ell$, satisfying the relation

$$a_0 \wedge \cdots \wedge a_p = 0. \tag{1}$$

For each i, $a_0 \wedge \cdots \wedge \widehat{a}_i \wedge \cdots \wedge a_p$ is a decomposable cocycle in $(A^\cdot, a_i)$. The $(p+1)$-dimensional subspace $D \subseteq A^1$ spanned by $\{a_0, \ldots, a_p\}$ is *singular*, in the sense that the map $\bigwedge^{p+1}(D) \to A^{p+1}$ is trivial. If $\mathcal{A}$ is p-generic, i.e., $\mathcal{A}$ has only normal-crossing singularities through codimension p, then this implies $a_0 \wedge \cdots \wedge \widehat{a}_i \wedge \cdots \wedge a_p$ is not a coboundary in $(A^\cdot, a_i)$, hence represents a nonzero element of $H^p(A^\cdot, a_i)$. Thus in this case the subspace D is contained in $\mathcal{R}^p(\mathcal{A})$.

Clearly every element of $\mathcal{R}^1(\mathcal{A})$ comes from relation (1) with $p = 1$, and $\mathcal{R}^1(\mathcal{A})$ is the union of the corresponding singular subspaces. Moreover, according to [FY07], every such relation $a_0 \wedge a_1 = 0$ with support equal to $\mathcal{A}$ arises from a multinet structure on $\mathcal{A}$. A (d,k)-*multinet* is a partition $\mathcal{A} = \mathcal{A}_1 \coprod \cdots \coprod \mathcal{A}_k$, $k \ge 3$, and a multiplicity function $m \colon \mathcal{A} \to \mathbb{Z}_{>0}$, satisfying

(1) $\sum_{H \in \mathcal{A}_i} m(H) = d$ for $1 \le i \le k$;
(2) for each $X \in \mathcal{X}$, $\sum_{H \in \mathcal{A}_i, H \supseteq X} m(H)$ is independent of i; and
(3) for each i, $\bigcup_{H \in \mathcal{A}_i} H - \bigcup_{X \in \mathcal{X}} X$ is connected.

Here $\mathcal{X}$ is the set of codimension-two intersections $X = H \cap H'$ where H and H' lie in different blocks of the partition. Existence of a multinet on $\mathcal{A}$ is equivalent to the existence of a pencil of projective hypersurfaces having no fixed components, connected generic fiber, and $k \ge 3$ singular fibers which are completely decomposable, i.e., are unions of hyperplanes with multiplicities, and whose components comprise $\mathcal{A}$. The blocks of the partition are the sets of components of the completely reducible fibers, and $m(H)$ is the multiplicity of H in the corresponding fiber. If $m(H) = 1$ for all H and the sum in (ii) is equal to one for all $X \in \mathcal{X}$, then $\mathcal{A} = \mathcal{A}_1 \coprod \cdots \coprod \mathcal{A}_k$ is called a (k,d)-*net*. Geometric constraints imply that there are no (k,d)-multinets for $k \geq 5$ - see [Yuz09] and the references therein. The only known example with $k = 4$ comes from the Hesse pencil of nonsingular cubics, whose singular fibers make up the Hessian arrangement [OT92].

One of the few known examples with multiplicities different from one is provided by the Coxeter arrangement of type B_3, with defining polynomial

$$xyz(x^2 - y^2)(y^2 - z^2)(z^2 - x^2).$$

One considers the three polynomials $\Phi_0 = x^2(y^2 - z^2)$, $\Phi_1 = y^2(z^2 - x^2)$, and $\Phi_2 = z^2(x^2 - y^2)$. These quartics lie in a pencil because $\Phi_0 + \Phi_1 + \Phi_2 = 0$. The rational map

$$\Phi = [\Phi_0 : \Phi_1 : \Phi_2] \colon \mathbb{P}^2 \to \mathbb{P}^2$$

has image the line Σ given by $w_0 + w_1 + w_2 = 0$. Φ is regular on M, and $\Phi(M)$ is the complement Σ_0 in Σ of the three coordinate lines. The logarithmic forms $\tau_0 = \mathrm{d}\log(y_0/y_2)$ and $\tau_1 = \mathrm{d}\log(y_1/y_2)$ are regular on Σ_0 and $\tau_0 \wedge \tau_1 = 0$ because $\dim(\Sigma_0) = 1$. Then the pullbacks $a_0 = \Phi^*(\tau_0) = \mathrm{d}\log(\Phi_0/\Phi_2)$ and $a_1 = \Phi^*(\tau_1) = \mathrm{d}\log(\Phi_1/\Phi_2)$ satisfy $a_0 \wedge a_1 = 0$, and they span a component of $\mathcal{R}^1(\mathcal{A})$. The corresponding multinet has $k = 3$ classes of $d = 4$ lines each, counting multiplicities, corresponding to the factors of Φ_i, $i = 0, 1, 2$, with the coordinate lines each given multiplicity two. In this case these are the only singular fibers of Φ, hence $\Phi \colon M \to \Sigma_0$ is a fiber bundle.

In case the weight vectors $(\lambda_{i0}, \ldots, \lambda_{in})$ lie in $\mathbb{Z}^{n+1}$ the relation (1) can be formulated in terms of tropical geometry. As in the preceding example, we work projectively, writing $\lambda_{ij} = \eta_{ij} - \eta_{p+1,j}$ with $\eta_i = (\eta_{i0}, \ldots, \eta_{in}) \in \mathbb{N}^{n+1}$ for $0 \leq i \leq p+1$. Let $\Phi_i = \prod_{j=0}^{n} \alpha_j^{\eta_{ij}}$, a polynomial, for $0 \leq i \leq p+1$. Then $a_i = \mathrm{d}\log(\Phi_i/\Phi_{p+1})$ for $0 \leq i \leq p$. Consider the rational map

$$\Phi = \left[\Phi_0 \colon \cdots : \Phi_{p+1}\right] : \mathbb{P}^\ell \to \mathbb{P}^{p+1}.$$

Φ is regular on M, and relation (1) holds if and only if the image $\Sigma_0 = \Phi(M)$ is at most p-dimensional. Equivalently, (1) holds if and only if there is a nonzero homogeneous polynomial $P(w_0, \ldots, w_{p+1})$ such that $P(\Phi_0, \ldots, \Phi_{p+1})$ vanishes identically.

The mapping Φ can be factored, $\Phi = \mu \circ \alpha$, where $\alpha = \left[\alpha_0 \colon \cdots : \alpha_n\right] : \mathbb{P}^\ell \to \mathbb{P}^n$ is the canonical linear map associated with $\mathcal{A}$ and $\mu \colon \mathbb{P}^n \to \mathbb{P}^{p+1}$ is the monomial mapping that sends $\left[y_0 \colon \cdots : y_n\right]$ to $\left[y^{\eta_0} \colon \cdots : y^{\eta_{p+1}}\right]$, using the usual vector notation for monomials. According to [DFS07], the factorization $\Phi = \mu \circ \alpha$ tropicalizes faithfully, in the following sense. The tropicalization of the linear variety $\alpha(M) \subseteq (\mathbb{C}^*)^n$ is the *Bergman fan* $B_\mathcal{A}$ of $\mathcal{A}$; the tropicalization of the monomial map μ is the linear map $\mathbb{P}^n \to \mathbb{P}^{p+1}$ with matrix $\Lambda = \left[\eta_{ij}\right]$. By [DFS07], the tropicalization $\tau(\Sigma_0)$ of the image Σ_0 of $\mu \circ \alpha$ is equal to the image of $B_\mathcal{A}$ under the linear map Λ.

The tropical variety $\tau(\Sigma_0)$, determined by $B_{\mathcal{A}}$, which depends only on the underlying matroid of $\mathcal{A}$, and the exponent matrix Λ, carries a lot of geometric information about Σ_0. In particular $\dim(\Sigma_0) = \dim(\tau(\Sigma_0))$. Thus relation (1) holds if and only if $\Lambda(B_{\mathcal{A}})$ has dimension at most p. $B_{\mathcal{A}}$ is a pure ℓ-dimensional fan in $\mathbb{R}^n$. By [AK06, FS05], the linear hulls of the maximal cones of $B_{\mathcal{A}}$ are spanned by the characteristic vectors of flats appearing in maximal *nested sets* in the lattice of flats $L(\mathcal{A})$, relative to the building set of all irreducible flats. These subspaces are flats of the braid arrangement in $\mathbb{R}^n$; the corresponding partitions can be extracted from $L(\mathcal{A})$. To establish the relation (1) one must find pairs $(\mathcal{A}, \Lambda)$ such that the kernel of Λ meets each of these flats nontrivially, so that the image $\tau(\Sigma_0)$ of $B_{\mathcal{A}}$ has codimension at least one.

The rigid combinatorics and geometry of the $p = 1$ case lends special interest to the case where the syzygy $P(w_0, \ldots, w_{p+1})$ is linear. Then the image tropical variety $\tau(\Sigma_0)$ is the Bergman fan of an arrangement $\mathcal{B}$ of rank at most $p + 1$. This raises an interesting matroid-theoretic question, to characterize the triples $(\mathcal{A}, \mathcal{B}, \Lambda)$ where $\mathcal{A}$ and $\mathcal{B}$ are arrangements with $\mathrm{rk}(\mathcal{B}) < \mathrm{rk}(\mathcal{A})$, and Λ is a linear projection carrying $B_{\mathcal{A}}$ to $B_{\mathcal{B}}$.

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An introduction to Tropical Geometry II

Eva-Maria Feichtner

Exemplifying the concepts presented in the first part of this informal introduction to Tropical Geometry, we give a detailed discussion of the tropicalization of linear spaces. In fact, we define the *Bergman fan* as a matroid invariant regardless of the realizability of the matroid. Then we show how the Bergman fan of a matroid coming from a complex hyperplane arrangement ties in with the nested set fans of the arrangement. This is the combinatorial shadow of a hierarchy of compactifications of the arrangement complement including the wonderful compactifications of De Concini–Procesi and tropical compactifications devised by Tevelev. We discuss an example of 5 lines in $\mathbb{CP}^2$ whose smallest wonderful compactification yet allows for a morphism onto the tropical compactification prescribed by the Bergman fan, where the morphism is *not* an isomorphism. This answers a question asked by Corrado De Concini earlier during the mini-workshop.

Equivariant cohomology, syzygies and orbit structure

Matthias Franz
(joint work with Christopher Allday and Volker Puppe)

Consider an action of the torus $T = (S^1)^r$ on a space X satisfying some mild conditions. Let X_i be the T-equivariant i-skeleton of X, i.e., the union of all orbits of dimension at most i. By work of Atiyah [2] and Bredon [3], the following "Atiyah–Bredon sequence" is exact if the equivariant cohomology $H_T^*(X)$ with rational coefficients is free over the polynomial ring $R = H^*(BT)$:

$$0 \to H_T^*(X) \to H_T^*(X_0) \to H_T^{*+1}(X_1, X_0) \to \cdots \to H_T^{*+r}(X_r, X_{r-1}) \to 0. \tag{1}$$

The part

$$0 \to H_T^*(X) \to H_T^*(X_0) \to H_T^{*+1}(X_1, X_0) \tag{2}$$

is also called the "Chang–Skjelbred sequence" because Chang and Skjelbred [4] proved, roughly at the same time as Atiyah and Bredon, that (2) is exact if $H_T^*(X)$ is free over R. This assumption is known to hold for large classes of spaces, including compact Hamiltonian T-manifolds and rationally smooth, complete complex algebraic varieties with an algebraic action of the complexification of T. In all these cases the sequence (2) provides a powerful way to compute $H_T^*(X)$, including the cup product, out of data related only to the fixed points and the one-dimensional orbits. In the important special case where X_1 is a finite union of 2-spheres, glued together at their poles, this is often referred to as the "GKM method", following work of Goresky–Kottwitz–MacPherson [8]. It should be noted that one only needs exactness of a very small part of the Atiyah–Bredon sequence in order to apply this method. This suggests that the sequence (2) might be exact under much weaker assumptions than the freeness of $H_T^*(X)$.

One can define an *equivariant homology* $H_*^T(X)$ of X (which is *not* the homology of the Borel construction X_T) that is related to equivariant cohomology via universal coefficient spectral sequences. Moreover, equivariant Poincaré duality holds in the sense that for a rational Poincaré duality space X capping with the equivariant fundamental class gives an isomorphism of R-modules $H_T^*(X) \to H_*^T(X)$. Equivariant homology also behaves very well with respect to the orbit filtration:

Proposition 1. *For any $i \geq 0$ there is a short exact sequence*

$$0 \to H_*^T(X_i) \to H_*^T(X) \to H_*^T(X, X_i) \to 0.$$

If X is a manifold, one can translate this into a result of Duflot [5] about equivariant cohomology by using equivariant Poincaré–Alexander–Lefschetz duality.

Drop the first term $H_T^*(X)$ from the Atiyah–Bredon sequence (1) and write $AB^*(X)$ for the resulting complex of R-modules. Our main result implies particular that the cohomology of this complex is completely determined by $H_*^T(X)$.

Theorem 1. *For any $i \geq 0$,*

$$H^i(AB^*(X)) = \operatorname{Ext}_R^i(H_*^T(X), R).$$

Recall that a finitely generated R-module M is called a *j-th syzygy* if there is an exact sequence

$$0 \to M \to F^1 \to \cdots \to F^j \tag{3}$$

with finitely generated free R-modules $F^1, \ldots, F^j$. First syzygies are exactly the torsion-free modules, and r-th syzygies the free modules.

Theorem 2. *Let $1 \leq j \leq r$. The Atiyah–Bredon sequence* (1) *is exact at $H_T^*(X_i, X_{i-1})$ for all $-1 \leq i \leq j-2$ if and only if $H_T^*(X)$ is a j-th syzygy.*

For $i = -1$, exactness at $H_T^*(X_i, X_{i-1})$ has to be interpreted as exactness at $H_T^*(X)$.

Recall that an R-module M is called *reflexive* if the canonical map

$$M \to \operatorname{Hom}_R(\operatorname{Hom}_R(M, R), R) \tag{4}$$

is an isomorphism. This is equivalent to M being a second syzygy. Moreover, a (graded symmetric) R-bilinear pairing $M \times M \to R$ is called *perfect* if it induces an isomorphism $M \to \operatorname{Hom}_R(M, R)$.

Let X be a rational Poincaré duality space. Since the equivariant coefficient ring R is not a field (unless $r = 0$), the duality isomorphism between $H_T^*(X)$ and $H_*^T(X)$ does not imply that the corresponding equivariant Poincaré pairing

$$H_T^*(X) \times H_T^*(X) \to R \tag{5}$$

is non-degenerate, let alone perfect. For instance, one has $H_T^*(X) = \mathbb{Q}$ for $X = T$, so that the map (5) is trivial in this case.

The following result is an immediate consequence of Theorems 1 and 2. It essentially answers an open point raised by Guillemin–Ginzburg–Karshon [9].

Corollary 1. *Let X be a rational Poincaré duality space. Then the following are equivalent:*

1. *The Chang–Skjelbred sequence* (2) *is exact.*
2. *The R-module $H_T^*(X)$ is reflexive.*
3. *The equivariant Poincaré pairing* (5) *is perfect.*

For any $j \geq -1$ there are T-spaces (in fact, smooth toric varieties) such that the sequence (1) is exact at all positions $i < j$, but not at position j. The situation changes if one restricts to rational Poincaré duality spaces. The following result says roughly that if the first half of the Atiyah–Bredon sequence is exact, then so is the rest in this case:

Corollary 2. *Let X be a rational Poincaré duality space and set $j = \lfloor \frac{n+1}{2} \rfloor$. If $H_T^*(X)$ is a j-th syzygy, then it is free over R.*

In other words, we have the following situation for a rational Poincaré duality space X: if $H^*_T(X)$ not free over R, then it is at most a syzygy of order $\lfloor \frac{n-1}{2} \rfloor$. For $r \leq 4$ this bound is known to be sharp; for $r \geq 5$ this remains an open question. An example of an orientable compact $(S^1)^3$-manifold whose equivariant cohomology is torsion-free, but not free can be found in [7].

The following is a consequence of the "geometric criterion" for syzygies in equivariant cohomology established in [6].

Theorem 3. *Let X be a smooth manifold. Whether or not $H^*_T(X)$ is a certain syzygy depends only on the orbit space X/T with its stratification by infinitesimal orbit type. In particular, whether or not $H^*_T(X)$ is free depends only on X/T as a stratified space.*

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Families of building sets and wonderful models

Giovanni Gaiffi

In [2] and [3], De Concini and Procesi constructed *wonderful models* for the complement of a subspace arrangement in a vector space. These are smooth varieties, proper over the given space, in which the union of the subspaces is replaced by a divisor with normal crossings.

The interest in these varieties was at first motivated by an approach to Drinfeld construction of special solutions for Khniznik-Zamolodchikov equation (see [7]). Moreover, in [3] it was shown, using the cohomology description of these models to give an explicit presentation of a Morgan algebra, that the mixed Hodge structure and the rational homotopy type of the complement of a complex subspace arrangement depend only on the intersection lattice (viewed as a ranked poset).

Real and complex De Concini–Procesi models turned out to play a relevant role in several fields of mathematical research: subspace and toric arrangements, toric varieties and tropical geometry, moduli spaces of curves, configuration spaces, box splines, index theory, discrete geometry (see for instance [4], [5], [8], [9], [10], [11], [15], [16] and [19]).

In general, given a subspace arrangement, there are several De Concini–Procesi models associated to it, depending on distinct sets of initial combinatorial data (*building sets*). Among these building sets there are always a minimal one and a maximal one with respect to inclusion: as a consequence there are always a minimal and a maximal De Concini–Procesi model.

The importance of the minimal construction was immediately pointed out, but real and complex non minimal models (in particular maximal models) appeared in various contexts (see [6], [14], [17], [1]). For instance it is well known that the toric variety of type A_{n-1} is isomorphic to the maximal building set associated to the boolean arrangement (see [13] for further references).

In this talk we will deal with the De Concini–Procesi models constructed starting from the root arrangement A_{n-1}. Our first goal is to describe the poset of all the associated building sets (ordered by inclusion) which are invariant with respect to the symmetric group group action: we will therefore classify all the wonderful models which are obtained by adding to the complement of the arrangement an equivariant divisor.

Let us describe our results more in detail: we will introduce a partial order on the set Λ_n of all the partitions of n, and we will define a family of S_n-equivariant building sets $\mathcal{G}_\lambda$, where $\lambda \in \Lambda_n$ is a *building partition*, i.e. it is (n) or a partition with at least two parts greater than or equal to 2.

Then, given any subset $\{\lambda^1, \lambda^2, ..., \lambda^k\}$ of pairwise not comparable building partitions, the union $\{\mathcal{G}_{\lambda^1} \cup \mathcal{G}_{\lambda^2} \cup \cdots \cup \mathcal{G}_{\lambda^k}\}$ is an S_n-equivariant building set, and all the S_n-equivariant building sets can be obtained in this way.

Some particularly regular objects come out of this picture, i.e. the building sets $\mathcal{G}_s(A_{n-1})$ obtained as the union of the building partitions $\mathcal{G}_\lambda$ such that λ has exactly s parts. Therefore, for every $n \geq 2$ we have a family of $n-2$ *regular building sets*:

$$\mathcal{G}_1(A_{n-1}) \subset \mathcal{G}_2(A_{n-1}) \subset \cdots \subset \mathcal{G}_{n-2}(A_{n-1})$$

where $\mathcal{G}_1(A_{n-1})$ coincides with the minimal building set and $\mathcal{G}_{n-2}(A_{n-1})$ with the maximal one. We will give formulas for the Poincaré series of all the *regular models* $Y_{\mathcal{G}_s(A_{n-1})}$. For $s = 1$ this series is the well known series for the moduli spaces of stable $(n+1)$-pointed curves of genus zero, while in the case of maximal models the formulas we obtain are explicit sums and products of polynomials whose coefficients involve the Stirling numbers of the second kind (different formulas for these Poincaré polynomials were described in [12]). The formulas for the intermediate models are "interpolations" between the formulas for the maximal and the minimal cases.

Furthermore we will provide an inductive formula for the character series associated to the S_n action on the cohomology of some of the regular models $Y_{\mathcal{G}_s(A_{n-1})}$, and explicit computations in the small dimensional cases.

We will also point out the connection of our formulas with the rich combinatorics of the corresponding real De Concini–Procesi models. The real models can be constructed, as it is well known, by gluing nestohedra, and from this one obtains formulas for their Euler characteristics. Different formulas for these Euler characteristics can also be obtained by evaluating in $q = -1$ the Poincaré polynomials of the corresponding complex models. From the comparison of these two different computations of the Euler characteristics one obtains nice combinatorial equivalences.

A paper containing all the details is available and will be posted on the arXiv. The above described results can be generalized to the root arrangements of types B_n and D_n (this generalization is joint work with Matteo Serventi).

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The free and ℓ^2 cohomology of hyperplane complements

Ian Leary
(joint work with Mike Davis, Tadeusz Januszkiewicz, Boris Okun)

Let $\mathcal{A}$ denote an affine hyperplane arrangement in $\mathbb{C}^n$, and let $\overline{\mathcal{L}}(\mathcal{A})$ denote the poset whose elements are $\mathbb{C}^n$ and the intersections of elements of $\mathcal{A}$. The rank of $\mathcal{A}$ is defined to be the codimension of a minimal element of $\overline{\mathcal{L}}(\mathcal{A})$, and will be denoted by l. As usual, the union of the hyperplanes in $\mathcal{A}$ is denoted $\Sigma(\mathcal{A})$, and the complement of $\Sigma(\mathcal{A})$ is denoted by $M(\mathcal{A})$. These are the singular set and the hyperplane complement for the arrangement $\mathcal{A}$. Let $\pi = \pi_1(M(\mathcal{A}))$ be the fundamental group of $M(\mathcal{A})$. Recall that $\Sigma(\mathcal{A})$ is homotopy equivalent to a wedge of a number of copies of the $(l-1)$-sphere; call this number $\beta(\mathcal{A})$.

We describe the reduced ℓ^2-cohomology of $M(\mathcal{A})$, and the cohomology of $M(\mathcal{A})$ with coefficients in the free module $\mathbb{Z}\pi$. Firstly we recall the definition of these. Let S denote a finite CW-complex homotopy equivalent to $M(\mathcal{A})$, and let C_* be the cellular chain complex on the universal cover of S. Note that C_* is a chain complex of finitely-generated free $\mathbb{Z}\pi$-modules. By definition, $H^*(M(\mathcal{A});\mathbb{Z}\pi)$ is the cohomology of the cochain complex $\mathrm{Hom}_\pi(C_*,\mathbb{Z}\pi)$. Equivalently, this is the cohomology with compact supports of the universal cover of S. Now let $\ell^2(\pi)$ denote the Hilbert space of square-summable complex-valued functions on π. The unreduced ℓ^2-cohomology is the cohomology of the cochain complex $\mathrm{Hom}_\pi(C_*,\ell^2(\pi))$. The definition of the reduced ℓ^2-cohomology $\mathcal{H}^*_{(2)}(M(\mathcal{A}))$ takes in to account the fact that $\mathrm{Hom}_\pi(C_*,\ell^2(\pi))$ is a cochain complex of Hilbert spaces; instead of taking kernels modulo images one takes kernels modulo the closure of images. Equivalently, the reduced ℓ^2-cohomology group can be defined as the orthogonal complement of the image inside the kernel, which shows that each ℓ^2-cohomology group is isomorphic to a direct summand of the corresponding chain group.

Theorem 1. *The ℓ^2-cohomology groups of $M(\mathcal{A})$ are trivial, except $\mathcal{H}^l_{(2)}(M(\mathcal{A}))$, which is isomorphic to a direct sum of $\beta(\mathcal{A})$ copies of $\ell^2(\pi)$.*

Theorem 2. *Each $H^i(M(\mathcal{A});\mathbb{Z}\pi)$ is trivial, except that $H^l(M(\mathcal{A});\mathbb{Z}\pi)$ is free abelian of infinite rank.*

Corollary 3. *As a $\mathbb{Z}\pi$-module, $H^l(M(\mathcal{A});\mathbb{Z}\pi)$ is of type FL.*

Recall that an R-module is said to be of type FL if it admits a finite-length resolution by finitely-generated free modules. The Corollary stated above follows easily from the previous Theorem, since the cochain complex $\mathrm{Hom}_\pi(C_*, \mathbb{Z}\pi)$ is a resolution of the required type for $H^l(M(\mathcal{A}); \mathbb{Z}\pi)$.

The proofs of the theorems are similar [1, 2], with the ℓ^2-cohomology theorem slightly easier. As a $\mathbb{Z}\pi$-module, $H^l(M(\mathcal{A}); \mathbb{Z}\pi)$ is not usually free, although it contains a free summand of rank $\beta(\mathcal{A})$. This summand maps injectively to $\mathcal{H}^l_{(2)}(M(\mathcal{A}))$ under the map induced by $\mathbb{Z}\pi \to \ell^2(\pi)$. There is a filtration of $H^l(M(\mathcal{A}); \mathbb{Z}\pi)$ indexed by the subspaces of $\overline{\mathcal{L}}(\mathcal{A})$. The subspaces other than $\mathbb{C}^n$ give rise to non-free modules in this filtration. The free term appears 'at the top' of the filtration and so it splits off.

We use the following tools in the proofs of the theorems:

- Lück's algebraic reformulation of ℓ^2-cohomology [3], which reduces the computation of reduced ℓ^2-cohomology to the computation of ordinary cohomology with coefficients in the ring $\mathcal{N}(\pi)$ of bounded π-equivariant self-maps of the Hilbert space $\ell^2(\pi)$;
- The Mayer-Vietoris spectral sequence for computing the cohomology of a space broken up into more than two pieces [1];
- An induction on the rank l of the arrangement;
- A careful choice of covering of the universal cover of $M(\mathcal{A})$, coming from a covering of $\mathbb{C}^n$.

Concerning the choice of covering that we use: A convex open set U in $\mathbb{C}^n$ is *small for* $\mathcal{A}$ if firstly the set of planes $G \in \overline{\mathcal{L}}(\mathcal{A})$ that meet U has a minimal element $\mathrm{Min}(U)$, and secondly a hyperplane $H \in \mathcal{A}$ meets U if and only if it contains $\mathrm{Min}(U)$. A finite covering of $\mathbb{C}^n$ by sets that are small for $\mathcal{A}$ lifts to a finite π-equivariant covering of the universal cover of $M(\mathcal{A})$. The free and ℓ^2 cohomology of the pieces making up this covering can be computed, either by induction on l in the free case or using the fact that the ℓ^2-cohomology of a central arrangement vanishes in the other case.

We use the Mayer-Vietoris spectral sequence for such a cover to make the computations. For ℓ^2-coefficients, the terms making up the E_2-page of the Mayer-Vietoris spectral sequence all have dimension zero, except for one term that contributes all of $\mathcal{H}^l_{(2)}(M(\mathcal{A}))$. For free coefficients, the non-zero terms in the E_2-page of the Mayer-Vietoris spectral sequence all lie on the diagonal line that contributes to $H^l(M(\mathcal{A}); \mathbb{Z}\pi)$.

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Towards the effective cone of a wonderful compactification

Diane Maclagan

Let $\mathcal{A}$ be an arrangement of $n+1$ hyperplanes $\{H_0, \dots, H_n\}$ in $\mathbb{P}^d$. Write $H_i = \{x \in \mathbb{P}^d : a_i \cdot x = 0\}$ for $a_i \in K^{d+1}$. We regard the complement $Y = \mathbb{P}^d \setminus \mathcal{A}$ as a closed subvariety of $(K^*)^n \cong (K^*)^{n+1}/K^*$ via the embedding $y \mapsto (a_0 \cdot y : \dots : a_n \cdot y)$. The closure of Y in $\mathbb{P}^n$ is a copy of $\mathbb{P}^d$.

The tropical variety of Y is the support of a rational d-dimensional polyhedral fan in $\mathbb{R}^n$ (see the abstract of Cueto, p. 10). As described in Feichtner's talk (p. 15), each choice of a wonderful compactification of Y (ie a choice of a building set $\mathcal{G}$) gives a choice of fan structure Σ on $\mathrm{trop}(Y)$. The underlying set of $\mathrm{trop}(Y)$ depends only on the underlying matroid of the arrangement (ie on the lattice of flats). Given such a choice of fan structure Σ, let X_Σ be the n-dimensional toric variety corresponding to Σ. We have $Y \subset (K^*)^n \subset X_\Sigma$. The insight of Tevelev [5] was that (for any $Y \subset (K^*)^n$) the closure $\overline{Y}$ of Y in X_Σ is a good choice of compactification of Y. The connection to wonderful models is the following, which was first observed by Tevelev, and further clarified by Feichtner and Sturmfels.

Theorem 1 ([3], [5]). *The closure $\overline{Y}$ is the wonderful model of $Y = \mathbb{P}^d \setminus \mathcal{A}$ corresponding to the building set $\mathcal{G}$.*

In the talk I discussed work-in-progress, joint with Florian Block (UC Berkeley), to determine the effective cone of a wonderful compactification of Y. The effective cone is a convex cone in the Néron-Severi space $N^1(\overline{Y})_{\mathbb{R}}$ of divisors modulo numerical equivalence. It is the closure of the classes of effective divisors. In the case of wonderful compactifications defined over $\mathbb{C}$, it can be regarded as the closure in $H^2(\overline{Y}, \mathbb{R})$ of the cones spanned by classes of codimension-one subvarieties of $\overline{Y}$. The effective cone of a variety is an important invariant in birational geometry; for example, its interior describes divisors that give rise to birational maps from the variety. Following the breakthroughs in the minimal model program in the last decade [1], there is increased interest in explicitly understanding these and related cones. In joint work with Block we focus on using tropical methods to understand the effective cones of wonderful models computationally.

Theorem 2. *There is an iterative procedure to determine* $\mathrm{Eff}(\overline{Y})$ *for a wonderful compactification of a hyperplane arrangement complement. This uses tropical geometry.*

The key idea is that an irreducible effective divisor that is not an exceptional divisor of one of the blow-ups or a strict transform of one of the hyperplanes must intersect the arrangement complement Y in a codimension one subvariety. Since Y is defined by linear equations, this codimension-one subvariety is defined by a single (Laurent) polynomial. The class of the divisor in $N^1(\overline{Y})_{\mathbb{R}}$ is determined by the tropicalization of this polynomial.

We note that this iterative procedure gives a sequence of polyhedral subcones of the effective cone representing divisors up to a certain "degree". These cones limit to the true effective cone, and may equal it if the effective cone is polyhedral.

Note all effective cones are polyhedral, even for wonderful compactifications; the blow-up of $\mathbb{P}^2$ at at least nine points in sufficiently general position has a nonpolyhedral effective cone, and these blow-ups are wonderful compactifications. For these surfaces the tropical approach seems to recover one approach to the SGHH conjecture describing the effective cones; see [4] for an overview of this. This iterative procedure also gives an approach to understand the effective cone of the moduli space $\overline{M}_{0,n}$, and to study the conjecture of Castravet and Tevelev [2].

A consequence of the iterative procedure is the following.

Corollary 4. *If the realization space of the matroid of $\mathcal{A}$ is irreducible, then for a very general realization the effective cone of the wonderful compactification $\overline{Y}$ is constant.*

The effective cone is not, however, constant on the entire realization space. One example is given by the case of a line arrangement of all lines through six points in the plane in linearly general position, where the effective cone depends on whether the six points lie on a conic. This contrasts with the case of the cohomology ring of the arrangement complement, which famously depends only on the matroid of the arrangement, and with the fundamental group of the arrangement complement, which depends only on the connected component of the realization space of the matroid. This prompts the following question.

Question 2. Which invariants of hyperplane arrangement complements hold on an open set or a countable intersection of open sets of the realization space of the associated matroid?

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L^2–Betti numbers of hypersurface complements

Laurentiu G. Maxim

Let Γ be a countable group, M a CW-complex, and $\alpha : \pi_1(M) \to \Gamma$ epimorphism. Let M_Γ be the regular cover of M defined by α. Denote by $\mathcal{N}(\mathcal{G})$ the von Neumann algebra of Γ. To the pair (M, α) one can associate L^2-Betti numbers (e.g., see [4]):

$$b_i^{(2)}(M, \alpha) := \dim_{\mathcal{N}(\Gamma)} H_i \left(C_*(M_\Gamma) \otimes_{\mathbb{Z}[\mathcal{G}]} \mathcal{N}(\Gamma) \right) \in [0, \infty],$$

where $C_*(M_\Gamma)$ is the cellular (or singular) chain complex of M_Γ, with right Γ-action by deck transformations. These L^2-Betti numbers are homotopy invariants of the pair (M, α), and for a finite CW complex M the following holds:

$$\sum_i (-1)^i b_i^{(2)}(M, \alpha) = \chi(M),$$

where $\chi(M)$ is the topological Euler characteristic of M.

In [1] it was shown that if $\mathcal{A}$ is an affine hyperplane arrangement in $\mathbb{C}^n$, then at most one of the L^2–Betti numbers $b_i^{(2)}(\mathbb{C}^n \setminus \mathcal{A}, \mathrm{id})$ is non–zero. Here we present an analogous statement for complements of complex affine hypersurfaces in general position at infinity.

Let $X \subset \mathbb{C}^n$ $(n \geq 2)$ be a reduced affine hypersurface defined by a polynomial equation $f = 0$. Let $M := \mathbb{C}^n \setminus X$ be the hypersurface complement. Then M has the homotopy type of a finite CW complex of real dimension n. Denote by $\phi : \pi_1(M) \to \mathbb{Z}$ the total linking number homomorphism: $\gamma \mapsto lk\#(\gamma, X)$, and let $\widetilde{M}$ be the infinite cyclic cover of M defined by $\ker(\phi)$. We call an epimorphism $\alpha : \pi_1(M_X) \to \Gamma$ admissible if the total linking number homomorphism ϕ factors through α, i.e.,

$$\phi = \widetilde{\phi} \circ \alpha : \pi_1(M) \overset{\alpha}{\to} \Gamma \overset{\widetilde{\phi}}{\to} \mathbb{Z}.$$

Set $\widetilde{\Gamma} := \ker(\widetilde{\phi})$. We get an induced epimorphism $\widetilde{\alpha} : \pi_1(\widetilde{M}) \to \widetilde{\Gamma}$.

We are interested to study the L^2-Betti numbers $b_i^{(2)}(M, \alpha)$ and $b_i^{(2)}(\widetilde{M}, \widetilde{\alpha})$, respectively. Note that since $M = \mathbb{C}^n \setminus X$ has the homotopy type of a finite CW complex of dimension n, it follows by definition that

$$b_i^{(2)}(M, \alpha) = 0 \;, \quad i > n.$$

However, the infinite cyclic cover $\widetilde{M}$ is an infinite CW complex, so its L^2-Betti numbers could as well be infinite.

In [5], we proved the following "nonresonance-type" theorem.

Theorem 1. *Assume that the affine hypersurface $X \subset \mathbb{C}^n$ is in general position at infinity, i.e., the hyperplane at infinity is transversal in the stratified sense to the projective completion of X. Then:*

(a) *The L^2–Betti numbers $b_i^{(2)}(\widetilde{M}, \widetilde{\alpha})$ of the infinite cyclic cover are finite for all $0 \leq i \leq n - 1$.*

(b) *The L^2–Betti numbers of the complement M are computed by*

$$b_i^{(2)}(M, \alpha) = \begin{cases} 0, & \text{for } i \neq n, \\ (-1)^n \chi(M), & \text{for } i = n. \end{cases}$$

In particular,

$$(-1)^n \cdot \chi(M) \geq 0.$$

The proof uses the Lefschetz hyperplane theorem to reduce the problem to the study of L^2-Betti numbers of the link of X *at infinity*. The latter vanish by the assumption on the general position of X at infinity. For plane curves (i.e., $n = 2$), the result was proved in [2].

Our result in Theorem 1(b) is reminiscent of a similar calculation by Jost-Zuo [3] of L^2–Betti numbers of a compact Kähler manifold of non-positive sectional curvature. This was considered in relation to an old question of Hopf whether the Euler characteristic of a compact manifold M of even real dimension $2n$ has sign equal to $(-1)^n$, provided M admits a metric of negative sectional curvature. However, the statement of our Theorem 1 is metric independent. Finally, one should not be misled by these calculations into thinking that the L^2–Betti numbers of finite CW-complexes are always integers, or that most of them usually vanish. In fact, the Atiyah conjecture asserts that these L^2–Betti numbers are always rational; see [4] for more details on this conjecture and related matters.

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Artin groups of euclidean type

Jon McCammond
(joint work with Robert Sulway)

Coxeter groups were introduced by Jacques Tits in the 1960s as a natural generalization of the groups generated by reflections which act geometrically (which means properly discontinuously cocompactly by isometries) on spheres and euclidean spaces. And ever since their introduction their basic structure has been reasonably well understood [BB05, Bou02, Dav08]. More precisely, every Coxeter group has a faithful linear representation which preserves a symmetric bilinear form and has an algorithmic solution to its word problem. Moreover, the signature of the quadratic form can be used to coarsely classify Coxeter groups by the type of Riemannian symmetric space on which they naturally act: spherical, euclidean, hyperbolic and higher-rank. For the motivating examples, the spherical Coxeter groups are enumerated by the Dynkin diagrams and the euclidean Coxeter groups are enumerated by the extended Dynkin diagrams.

Artin groups were introduced in the 1970s as a natural class of groups associated to Coxeter groups and are related to them as the braid groups are related to the symmetric groups. More precisely, Artin groups try to capture information about the fundamental group of the quotient of the complexified hyperplane complement by the action of the Coxeter group. To illustrate, the symmetric group acts on $\mathbb{R}^n$

by permuting coordinates, the complexified hyperplane complement is the braid arrangement, its fundamental group is the pure braid group and the fundamental group of the quotient of the complement by the free action by the symmetric group is the full braid group.

The spherical Artin groups (i.e., the ones corresponding to the spherical Coxeter groups) have been well understood since they were introduced [BS72, Del72]. Somewhat surprisingly, the euclidean Artin groups have remained relatively mysterious outside of a few simple cases investigated by Craig Squier and Fran Digne [Dig, Dig06, Squ87]: it was not known whether they have a solvable word problem, whether they are torsion-free, whether they have trivial center, and whether they have a finite classifying space. These were recently highlighted as four main questions that are open about Artin groups in general and the euclidean Artin groups in particular [GP].

In my talk I discussed progress on these questions with my recent graduate student Robert Sulway. In particular, we prove the following result.

Theorem 1. *Every irreducible euclidean Artin group is a torsion-free centerless group with a solvable word problem and a finite dimensional classifying space.*

Two notes about the statement of the theorem itself. The classifying space we construct is merely finite dimensional but not finite. It is, in fact, locally infinite. And second, we do not know whether or not the classifying space we construct is homotopy equivalent to the usual space associated to these groups constructed from the complexified hyperplane complement. In particular, we do not resolve the classical $K(\pi, 1)$ conjecture for these groups.

The proof proceeds by showing that every irreducible euclidean Artin group has a dual presentation (with infinitely many generators and infinitely many relations) which canonically embeds as a subgroup of a Garside group. The theory of Garside groups is well-developed and our main results are deduced by working within this larger group.

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Fundamental group and E_∞-coalgebra structure on homology for complements of complex hyperplane arrangements

Grigory Rybnikov

About 20 years ago there was a problem whether the complements of any two combinatorially equivalent complex hyperplane arrangements have isomorphic fundamental groups. The answer is 'no' (see [1]). The crucial role in the proof is played by an obstruction to the existence of an isomorphism $\pi_1(M) \to \pi_1(M')$ extending the canonical isomorphism $H_1(M,\mathbb{Z}) \to H_1(M',\mathbb{Z})$, where M and M' are complements of two combinatorially equivalent arrangements. This obstruction is an element of a certain abelian group T, which can be constructed via the standard coalgebra structure on homology in the following way.

Let H_1 and H_2 be the first and second integer homology groups of M (they are canonically determined by the combinatorial structure of the arrangement). As a part of the standard coalgebra structure on homology, we have the comultiplication map $\mu : H_2 \to H_1 \otimes H_1$. Let L be the free Lie algebra over $\mathbb{Z}$ generated by H_1. It is graded by degree: $L = \bigoplus L_k$. Thus $L_1 = H_1$, and we identify L_2 with the subgroup of $H_1 \otimes H_1$ consisting of skew-symmetric tensors. Clearly, $\mu(H_2) \subseteq L_2$. Denote by $P = \bigoplus P_k$ the quotient Lie algebra L/I, where I is the ideal in L generated by $\mu(H_2)$. The Lie algebra P is called the *holonomy Lie algebra* (the complex holonomy Lie algebra of an arrangement and its relation to fundamental group were studied in [2]). Let $\nu\colon H_1 \otimes L_2 \to L_3$ be the commutator map, and let $\delta\colon \operatorname{Hom}(H_1, L_2) \to \operatorname{Hom}(H_2, L_3)$ be the homomorphism of abelian groups defined as follows: $(\delta f)(c) = \nu \circ (\operatorname{id} \otimes f)(\mu(c))$ for $c \in H_2$. Clearly, the projection of δf to $\operatorname{Hom}(H_2, P_3)$ depends only on projection of f to $\operatorname{Hom}(H_1, P_2)$. Thus we have a map $\bar{\delta}\colon \operatorname{Hom}(H_1, P_2) \to \operatorname{Hom}(H_2, P_3)$. Now we can write the definition of our abelian group: $T = \operatorname{Hom}(H_2, P_3)/\bar{\delta}\operatorname{Hom}(H_1, P_2)$.

To any arrangement $\mathcal{A}$ of the given combinatorial type, we associate an element $\kappa(\mathcal{A}) \in T$ (more precisely, it is an element of a certain principal homogeneous space over T, which can be identified with T). Then $\kappa(\mathcal{A}) - \kappa(\mathcal{A}')$ is an obstruction to the existence of an isomorphism $\pi_1(M) \to \pi_1(M')$ extending the canonical isomorphism $H_1(M,\mathbb{Z}) \to H_1(M',\mathbb{Z})$. The aim of this talk is to explain how to compute $\kappa(\mathcal{A})$ using the natural E_∞-coalgebra structure on homology.

To define the natural E_∞-coalgebra structure on homology, we need the language of operads (see [3]). The notion of operad is in the same relation to multilinear operations on a vector space (or $\mathbb{Z}$-module) as the notion of associative algebra to linear automorphisms. Since we work over $\mathbb{Z}$, we define an operad A in the category of $\mathbb{Z}$-modules as a family of $\mathbb{Z}$-modules $A(n)$ endowed with composition maps

$$A(m) \otimes (A(n_1) \otimes A(n_2) \otimes \cdots \otimes A(n_m)) \to A(n_1 + \cdots + n_m).$$

These composition maps must satisfy an obvious associativity condition.

The notion of operad splits into symmetric and nonsymmetric ones. In the symmetric case each $A(n)$ is acted on by the symmetric group S_n, and the composition maps are consistent with these actions. For any $\mathbb{Z}$-module V, we define families $\mathcal{E}_V$ and $\mathcal{E}^V$ by setting $(\mathcal{E}_V)(n) = \mathrm{Hom}(V^{\otimes n}, V)$ and $(\mathcal{E}^V)(n) = \mathrm{Hom}(V, V^{\otimes n})$. The composition maps are obvious. If we are given a morphism of operads $A \to \mathcal{E}_V$, then we say that V is an algebra over A, and if we are given a morphism of operads $A \to \mathcal{E}^V$, then we say that V is a coalgebra over A. For example, an associative algebra is an algebra over nonsymmetric operad As, where $As(n) = \mathbb{Z}$ for each n, and a commutative algebra is an algebra over symmetric operad Com, where $\mathsf{Com}(n) = \mathbb{Z}$ with the trivial action of S_n for each n.

We say that a family B of $\mathbb{Z}$-modules (acted on by the symmetric groups, if we are interested in the symmetric case) is a left module over an operad A if we are given composition maps

$$A(m) \otimes (B(n_1) \otimes B(n_2) \otimes \cdots \otimes B(n_m)) \to B(n_1 + \cdots + n_m)$$

which, taken together with composition maps of A, satisfy the associativity condition. In the same way we define right modules over A. For example, if we set $\mathcal{F}_{V,W}(n) = Hom(V^{\otimes n}, W)$ and $\mathcal{F}^{V,W}(n) = Hom(V, W^{\otimes n})$, then $\mathcal{F}_{V,W}$ is a left $\mathcal{E}_V$-module and a right $\mathcal{E}_W$-module, while $\mathcal{F}^{V,W}$ is a right $\mathcal{E}^V$-module and a left $\mathcal{E}^W$-module.

In the same way we define operads, algebras, and modules over them in the category of graded $\mathbb{Z}$-modules and in the category of differential graded $\mathbb{Z}$-modules (the grading is denoted by subscript, and the differential is of degree -1). We say that an operad in the category A of graded $\mathbb{Z}$-modules is free with the set of generators $G = \bigcup G(n)$ if there is a family of maps $G(n) \to A(n)$ and for any operad B and any family of maps $G(n) \to B(n)$ there is a morphism of operads $A \to B$ making the diagrams commutative. A differential graded operad is said to be quasi-free if it is free as a graded operad. Any quasi-free operad has a basis consisting of trees whose vertices are labeled by elements of G. Quasi-free modules are defined in a similar way.

We say that a nonsymmetric operad A in the category of dg-$\mathbb{Z}$-modules is an A_∞-operad if it is quasi-free and there is a quasi-isomorphism $A \to As$, and a symmetric operad E in the category of dg-$\mathbb{Z}$-modules is an E_∞-operad if it is quasi-free and there is a quasi-isomorphism $A \to \mathsf{Com}$.

Let us choose an E_∞-operad E. We shall now state several theorems. Most of the statements are well known to the specialists (at least in the simply-connected case or over a field of characteristic 0).

Theorem 1. *The chain complex of any simplicial set (and, in particular, the singular chain complex of any topological space) has a natural structure of E-coalgebra.*

Let F be a quasi-free E-bimodule such that there is a quasi-isomorphism of E-bimodules $F \to \mathsf{Com}$.

Theorem 2. *Let X and Y be arbitrary E-coalgebras. We define morphisms $X \to Y$ as classes of homomorphisms of E-bimodules $F \to \mathcal{F}^{X,Y}$ modulo chain homotopy. Then there is a natural composition law for such morphisms, which makes the class of E-coalgebras a category. Different choices of E_∞-operads and bimodules over them give equivalent categories of E-coalgebras. Assigning the singular chain complex to a topological space defines a functor from the homotopy category of topological spaces to the category of E-coalgebras.*

Theorem 3. *Let X be an E-coalgebra, and let HX be its homology. If HX is free as a $\mathbb{Z}$-module, then there is a natural structure of E-coalgebra on HX such that X and HX are isomorphic in the category of E-coalgebras.*

Theorem 4. *The cobar-construction can be regarded as a functor from the category of E-coalgebras to itself (cf.* [4]*).*

Theorem 5. *For the singular chain complex of an arcwise connected topological space T, the E-coalgebra structure on the cobar construction induces a Hopf algebra structure on its 0-homology. This Hopf algebra is the completion of the group algebra of $\pi_1(T)$ with respect to the powers of the augmentation ideal. Thus the E-coalgebra structure on the chain complex determines the dimension completion of the fundamental group (over $\mathbb{Z}$, see* [5]*).*

Since in the case of the complement to a complex hyperplane arrangement the integer homology is free as a $\mathbb{Z}$-module, we see that the dimension completion of the fundamental group is determined by the E-coalgebra structure on the homology. Moreover, obstructions to the existence of an isomorphism $\pi_1(M) \to \pi_1(M')$ extending the canonical isomorphism $H_1(M, \mathbb{Z}) \to H_1(M', \mathbb{Z})$, where M and M' are complements of two combinatorially equivalent arrangements, can be computed as obstructions to extending the canonical map $H_*(M, \mathbb{Z}) \to H_*(M', \mathbb{Z})$ to an isomorphism in the category of E-coalgebras. The first obstruction of this type is just the one described in the beginning of this talk. It is closely related to the triple Massey product on cohomology. Hopefully, it will be possible to compute next invariants of this type (related to the Massey products of higher order) and apply them to find other examples of combinatorially equivalent arrangements whose complements have non-isomorphic fundamental groups. The work is in progress.

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Equivariant Chow cohomology of nonsimplicial toric varieties

Hal Schenck

For a toric variety X_Σ determined by a polyhedral fan $\Sigma \subseteq N \simeq \mathbb{Z}^n$, Payne shows that the equivariant Chow cohomology is the $\mathrm{Sym}(N)$-algebra $C^0(\Sigma)$ of integral piecewise polynomial functions on Σ. We use the Cartan-Eilenberg spectral sequence to analyze the associated reflexive sheaf $\mathcal{C}^0(\Sigma)$ on $\mathrm{Proj}_{\mathbb{Q}}(N)$, showing that the Chern classes depend on subtle geometry of Σ and giving a criterion for the splitting of $\mathcal{C}^0(\Sigma)$ as a sum of line bundles. For certain fans associated to the reflection arrangement A_n, we describe a connection between $C^0(\Sigma)$ and logarithmic vector fields tangent to A_n.

In [1], Bifet–De Concini–Procesi prove that the integral equivariant cohomology ring $H^*_T(X_\Sigma)$ of a smooth toric variety X_Σ is isomorphic to the integral Stanley–Reisner ring A_Σ of the unimodular fan Σ, and in [5], Brion proves that for Σ simplicial, the rational equivariant Chow ring $A^*_T(X_\Sigma)_{\mathbb{Q}}$ is isomorphic to the ring of rational piecewise polynomial functions $C^0(\Sigma)_{\mathbb{Q}}$. A result of Billera [2] shows that for a simplicial fan, $C^0(\Sigma)_{\mathbb{Q}}$ is isomorphic to the rational Stanley–Reisner ring of the fan, so Brion's result is similar in spirit to [1]. Brion and Vergne completed the picture for the simplicial case by showing in [6] that

$$A^*_T(X_\Sigma)_{\mathbb{Q}} \simeq H^*_T(X_\Sigma)_{\mathbb{Q}}.$$

Our main results are the following. First, $C^0(\Sigma)$ is the top homology module H_n of a certain chain complex C, similar to a chain complex introduced by Billera in [2], but with subtle differences. We prove that for all $i \geq 1$, the modules $H_{n-i}(C)$ are supported in codimension at least $i+1$, and that the associated primes of codimension exactly $i+1$ are linear. Using the Cartan-Eilenberg spectral sequence, we show that $C^0(\Sigma)$ is a free $\mathrm{Sym}(N)_{\mathbb{Q}}$ module if $H_{n-i}(C) = 0$ or is Cohen-Macaulay of codimension $i+1$ for all $i \geq 1$. Several natural questions arise. First, does the converse implication hold, that is, does freeness of $C^0(\Sigma)$ imply the conditions on the lower homology modules. This is the case when Σ is simplicial, see [11]. Second, does there exist a simple combinatorial condition for freeness? In the simplicial case for $r = 0$, shellability is sufficient, but this is not so in the polyhedral case, due to a recent result of DiPasquale [8].

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Resonance varieties

Alexander I. Suciu

Introduction. One of the most fruitful ideas to arise from the theory of hyperplanes arrangements is that of turning the cohomology ring of a space into a family of cochain complexes, parametrized by the cohomology group in degree one, and extracting certain varieties from these data, as the loci where the cohomology of those cochain complexes jumps.

What makes these "resonance" varieties especially useful is their close connection with a different kind of jumping loci: the "characteristic" varieties, which record the jumps in homology with coefficients in rank 1 local systems. The geometry of these varieties is intimately related to the formality, (quasi-) projectivity, and homological finiteness properties of the fundamental group, and controls to a large extent the Betti numbers of finite abelian covers. For more on this, we refer to [8, 9, 10], and references therein.

I will present here an abstraction of the first resonance variety of a group, based on recent work with Stefan Papadima [6, 7]. This point of view leads to a new stratification of the Grassmannian, and a host of new questions.

Resonance schemes and Koszul modules. Let V be a finite-dimensional complex vector space, and let $K \subset V \wedge V$ be a subspace. The *resonance variety* associated to these data, $\mathcal{R} = \mathcal{R}(V, K)$, is the set of elements a in the dual vector space V^* for which there is an element $b \in V^*$, not proportional to a, such that $a \wedge b$ belongs to the orthogonal complement $K^{\perp} \subseteq V^* \wedge V^*$; we also declare that $0 \in \mathcal{R}$. It is readily seen that $\mathcal{R}$ is a conical, Zariski-closed subset of the affine space V^*. For instance, if $K = 0$ and if $\dim V > 1$, then $\mathcal{R} = V^*$; at the other extreme, if $K = V \wedge V$, then $\mathcal{R} = 0$.

The resonance variety comes endowed with a natural scheme structure: its defining ideal is the annihilator of the *Koszul module*, $\mathcal{B} = \mathcal{B}(V, K)$. This is a graded module over the symmetric algebra $S = \operatorname{Sym}(V)$, with presentation matrix $\delta_3 \oplus (\mathrm{id}_S \otimes \iota)$, where $\delta_3 \colon S \otimes \bigwedge^3 V \to S \otimes \bigwedge^2 V$ is the third Koszul differential, and $\iota \colon K \to V \wedge V$ is the inclusion map.

Here is an alternate point of view. Let $A = A(V, K)$ be the quadratic algebra defined as the quotient of the exterior algebra $E = \bigwedge V^*$ by the ideal generated by $K^\perp$. Then $\mathcal{R}$ is the set of points $a \in A^1$ where the first Betti number of the cochain complex $A^0 \xrightarrow{a} A^1 \xrightarrow{a} A^2$ jumps. Using results from [2, 4, 5], we may reinterpret the graded pieces of the Koszul module in terms of the linear strand in an appropriate Tor module: $\mathcal{B}_q^* \cong \operatorname{Tor}^E_{q+1}(A, \mathbb{C})_{q+2}$.

Groups and resonance. The main example I have in mind is as follows. Let G be a finitely generated group. The resonance variety of G is then defined as $\mathcal{R}(G) = \mathcal{R}(V, K)$, where $V^* = H^1(G, \mathbb{C})$ and $K^\perp$ is the kernel of the cup-product map $\cup_G \colon V^* \wedge V^* \to H^2(G, \mathbb{C})$.

Rationally, every resonance variety arises in this fashion. More precisely, let V be an n-dimensional $\mathbb{C}$-vector space, and suppose $K \subseteq V \wedge V$ is a linear subspace, defined over $\mathbb{Q}$. Then, as shown in [7], there is a finitely presented, commutator-relators group G with $V^* = H^1(G, \mathbb{C})$ and $K^\perp = \ker(\cup_G)$.

For instance, suppose G_Γ is a right-angled Artin group associated to a finite simple graph Γ on vertex set V. As shown in [5], the resonance variety $\mathcal{R}(G_\Gamma) \subset \mathbb{C}^{\mathsf{V}}$ is the union of all coordinate subspaces $\mathbb{C}^{\mathsf{W}}$ corresponding to subsets $\mathsf{W} \subset \mathsf{V}$ for which the induced graph Γ_{W} is disconnected. Moreover, the Hilbert series $\sum_{q \geq 0} \dim \mathcal{B}_q t^{q+2}$ equals $Q_\Gamma(t/(1-t))$, where $Q_\Gamma(t)$ is the "cut polynomial" of Γ, with coefficient of t^k equal to $\sum_{\mathsf{W} \subset \mathsf{V}\colon |\mathsf{W}| = k} \tilde{b}_0(\Gamma_{\mathsf{W}})$, where $\tilde{b}_0(\Gamma_{\mathsf{W}})$ is one less than the number of components of the induced subgraph on W.

A stratification of the Grassmannian. Now fix $n = \dim V$ and $m = \dim K$. Then K can be viewed as a point in the Grassmannian of m-planes in $V \wedge V$. Moving about this Grassmannian and recording the way the resonance scheme $\mathcal{R}(V, K)$ varies defines a stratification of $\mathbb{G} = \operatorname{Gr}_m(V \wedge V)$.

For instance, consider the "generic" stratum $U = U(n, m)$, consisting of those planes $K \in \mathbb{G}$ for which $\mathcal{R}(V, K) = 0$. Clearly, K belongs to U if and only if the plane $\mathbb{P}(K^\perp) \subset \mathbb{P}(V^* \wedge V^*)$ misses the image of $\operatorname{Gr}_2(V^*)$ under the Plücker embedding. Thus, U is a Zariski open subset of $\mathbb{G}$. Moreover, as noted in [7], this set is non-empty if and only if $m \geq 2n - 3$, in which case there is an integer $q = q(n, m)$ such that $\mathcal{B}_q(V, K) = 0$, for every $K \in U$.

The geometry of the non-generic strata is being studied in joint work with Eric Babson [1]. A key ingredient in this study is the Fulton–MacPherson compactification [3] of the configuration space of $\binom{n}{2} - m$ distinct points in $\operatorname{Gr}_2(\binom{n}{2})$.

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The rational homology of real toric manifolds

Alexander I. Suciu

Toric manifolds. In a seminal paper [7] that appeared some twenty years ago, Michael Davis and Tadeusz Januszkiewicz introduced a topological version of smooth toric varieties, and showed that many properties previously discovered by means of algebro-geometric techniques are, in fact, topological in nature.

Let P be an n-dimensional simple polytope with facets $F_1, \dots, F_m$, and let χ be an integral $n \times m$ matrix such that, for each vertex $v = F_{i_1} \cap \dots \cap F_{i_n}$, the minor of columns $i_1, \dots, i_n$ has determinant ± 1. To such data, there is associated a $2n$-dimensional toric manifold, $M_P(\chi) = T^n \times P/\sim$, where $(t, p) \sim (u, q)$ if $p = q$, and tu^{-1} belongs to the image under $\chi\colon T^m \to T^n$ of the coordinate subtorus corresponding to the smallest face of P containing q in its interior.

Here is an alternate description, using the moment-angle complex construction (see for instance [10] and references therein). Given a simplicial complex K on vertex set $[n] = \{1, \dots, n\}$, and a pair of spaces (X, A), let $\mathcal{Z}_K(X, A)$ be the subspace of the cartesian product $X^{\times n}$, defined as the union $\bigcup_{\sigma \in K} (X, A)^\sigma$, where $(X, A)^\sigma$ is the set of points for which the i-th coordinate belongs to A, whenever $i \notin \sigma$. It turns out that the quasi-toric manifold $M_P(\chi)$ is obtained from the moment angle manifold $\mathcal{Z}_K(D^2, S^1)$, where K is the dual to ∂P, by taking the quotient by the relevant free action of the torus $T^{m-n} = \ker(\chi)$.

Real toric manifolds. An analogous theory works for real quasi-toric manifolds, also known as small covers. Given a homomorphism $\chi\colon \mathbb{Z}_2^m \to \mathbb{Z}_2^n$ satisfying a minors condition as above, the resulting n-dimensional manifold, $N_P(\chi)$, is the quotient of the real moment angle manifold $\mathcal{Z}_K(D^1, S^0)$ by a free action of the group $\mathbb{Z}_2^{m-n} = \ker(\chi)$. The manifold $N_P(\chi)$ comes equipped with an action of $\mathbb{Z}_2^n$; the associated Borel construction is homotopy equivalent to $\mathcal{Z}_K(\mathbb{RP}^\infty, *)$.

If X is a smooth, projective toric variety, then $X(\mathbb{C}) = M_P(\chi)$, for some simple polytope P and characteristic matrix χ, and $X(\mathbb{R}) = N_P(\chi \bmod 2\mathbb{Z})$. Not all toric manifolds arise in this manner. For instance, $M = \mathbb{CP}^2 \sharp \mathbb{CP}^2$ is a toric manifold over the square, but it does not admit any (almost) complex structure; thus, $M \not\cong X(\mathbb{C})$.

The same goes for real toric manifolds. For instance, take P to be the dodecahedron, and use one of the characteristic matrices χ listed in [12]. Then, by a theorem of Andreev [1], the small cover $N_P(\chi)$ is a hyperbolic 3-manifold; thus, by a theorem of Delaunay [8], $N_P(\chi) \not\cong X(\mathbb{R})$.

The Betti numbers of real toric manifolds. In [7], Davis and Januszkiewicz showed that the sequence of mod 2 Betti numbers of $N_P(\chi)$ coincides with the h-vector of P. In joint work with Alvise Trevisan [18], we compute the rational cohomology groups (together with their cup-product structure) for real, quasi-toric manifolds. It turns out that the rational Betti numbers are much more subtle, depending also on the characteristic matrix χ.

More precisely, for each subset $S \subseteq [n]$, let $\chi_S = \sum_{i \in S} \chi_i$, where χ_i is the i-th row of χ, and let $K_{\chi,S}$ be the induced subcomplex of K on the set of vertices $j \in [m]$ for which the j-th entry of χ_S is non-zero. Then

$$\dim H_q(N_P(\chi), \mathbb{Q}) = \sum_{S \subseteq [n]} \dim \widetilde{H}_{q-1}(K_{\chi,S}, \mathbb{Q}). \tag{*}$$

The proof of formula (*), given in [18], relies on two fibrations relating the real toric manifold $N_P(\chi)$ to some of the aforementioned moment-angle complexes,

$$\begin{array}{ccccc} & & \mathbb{Z}_2^{m-n} & & \\ & & \downarrow & & \\ & & \mathcal{Z}_K(D^1, S^0) & & \\ & & \downarrow & & \\ \mathbb{Z}_2^n & \longrightarrow & N_P(\chi) & \longrightarrow & \mathcal{Z}_K(\mathbb{RP}^\infty, *)\,. \end{array}$$

The proof entails a detailed analysis of homology in rank 1 local systems on the space $\mathcal{Z}_K(\mathbb{RP}^\infty, *)$, exploiting at some point the stable splitting of moment-angle complexes due to Bahri, Bendersky, Cohen, and Gitler [2]. Some of the details of the proof appear in Trevisan's Ph.D. thesis [19].

As an easy application of formula (*), one can readily recover a result of Nakayama and Nishimura [14]: A real, n-dimensional toric manifold $N_P(\chi)$ is orientable if and only if there is a subset $S \subseteq [n]$ such that $K_{\chi,S} = K$.

The Hessenberg varieties. A classical construction associates to each Weyl group W a smooth, complex projective toric variety $\mathcal{T}_W$, whose fan corresponds to the reflecting hyperplanes of W and its weight lattice.

In the case when W is the symmetric group S_n, the manifold $\mathcal{T}_n = \mathcal{T}_{S_n}$ is the well-known Hessenberg variety, see [9]. Moreover, $\mathcal{T}_n$ is isomorphic to the De Concini–Procesi wonderful model $\overline{Y_{\mathcal{G}}}$, where $\mathcal{G}$ is the Boolean building set in

$(\mathbb{C}^n)^*$. Thus, $\mathcal{T}_n$ can be obtained by iterated blow-ups: first blow up $\mathbb{CP}^{n-1}$ at the n coordinate points, then blow up along the proper transforms of the $\binom{n}{2}$ coordinate lines, etc.

The real locus, $\mathcal{T}_n(\mathbb{R})$, is a smooth, real toric variety of dimension $n-1$; its rational cohomology was recently computed by Henderson [13], who showed that

$$\dim H_i(\mathcal{T}_n(\mathbb{R}), \mathbb{Q}) = A_{2i}\binom{n}{2i},$$

where A_{2i} is the Euler secant number, defined as the coefficient of $x^{2i}/(2i)!$ in the Maclaurin expansion of $\sec(x)$. As announced in [17], we can recover this computation, using formula (*).

To start with, note that the $(n-1)$-dimensional polytope associated to $\mathcal{T}_n(\mathbb{R})$ is the permutahedron P_n. Its vertices are obtained by permuting the coordinates of the vector $(1, \dots, n) \in \mathbb{R}^n$, while its facets are indexed by the non-empty, proper subsets $Q \subset [n]$. The characteristic matrix $\chi = (\chi^Q)$ for $\mathcal{T}_n(\mathbb{R})$ can be described as follows: χ^i is the i-th standard basis vector of $\mathbb{R}^{n-1}$ for $1 \le i < n$, while $\chi^n = \sum_{i<n} \chi^i$ and $\chi^Q = \sum_{i \in Q} \chi^i$.

The simplicial complex K_n dual to ∂P_n is the barycentric subdivision of the boundary of the $(n-1)$-simplex. Given a subset $S \subset [n-1]$, the induced subcomplex $(K_n)_{\chi,S}$ depends only on the cardinality $r = |S|$; denote any one of these $\binom{n-1}{r}$ subcomplexes by $K_{n,r}$. It turns out that $K_{n,r}$ is the order complex associated to a rank-selected poset of a certain subposet of the Boolean lattice B_n. A result of Björner and Wachs [5] insures that such simplicial complexes are Cohen–Macaulay, and thus have the homotopy type of a wedge of spheres (of a fixed dimension); in fact, $K_{n,2r-1} \simeq K_{n,2r} \simeq \bigvee^{A_{2r}} S^{r-1}$. Hence,

$$\begin{aligned}
\dim H_i(\mathcal{T}_n(\mathbb{R}), \mathbb{Q}) &= \sum_{S \subseteq [n-1]} \dim \widetilde{H}_{i-1}((K_n)_{\chi,S}, \mathbb{Q}) \\
&= \sum_{r=1}^{n-1} \binom{n-1}{r} \dim \widetilde{H}_{i-1}(K_{n,r}, \mathbb{Q}) \\
&= \left(\binom{n-1}{2i-1} + \binom{n-1}{2i}\right) A_{2i} = \binom{n}{2i} A_{2i}.
\end{aligned}$$

Recently, Choi and Park [6] have extended this computation to a much wider class of real toric manifolds. Given a finite simple graph Γ, let $\mathcal{B}(\Gamma)$ be the building set obtained from the connected induced subgraphs of Γ, and let $P_{\mathcal{B}(\Gamma)}$ be the corresponding graph associahedron. Using formula (*), these authors compute the Betti numbers of the smooth, real toric variety $X_\Gamma(\mathbb{R})$ defined by $P_{\mathcal{B}(\Gamma)}$. When $\Gamma = K_n$ is a complete graph, $X_{K_n} = \mathcal{T}_n$, and one recovers the above calculation.

The formality question. A finite-type CW-complex X is said to be *formal* if its Sullivan minimal model is quasi-isomorphic to the rational cohomology ring of X, endowed with the 0 differential. Under a nilpotency assumption, this means that $H^*(X, \mathbb{Q})$ determines the rational homotopy type of X.

As shown by Notbohm and Ray [15], if X is formal, then $\mathcal{Z}_K(X,*)$ is formal; in particular, $\mathcal{Z}_K(S^1,*)$ and $\mathcal{Z}_K(\mathbb{CP}^\infty,*)$ are always formal. More generally, as shown by Félix and Tanré [11], if both X and A are formal, and the inclusion $A \hookrightarrow X$ induces a surjection in rational cohomology, then $\mathcal{Z}_K(X,A)$ is formal.

On the other hand, as sketched in [4], and proved with full details in [10], the spaces $\mathcal{Z}_K(D^2,S^1)$ can have non-trivial triple Massey products, and thus are not always formal. In fact, as shown in [10], there exist polytopes P and dual triangulations $K = K_{\partial P}$ for which the moment-angle manifold $\mathcal{Z}_K(D^2,S^1)$ is not formal. Using these results, as well as a construction from [3], we can exhibit real moment-angle manifolds $\mathcal{Z}_L(D^1,S^0)$ that are not formal.

In view of this discussion, the following natural question arises: are toric manifolds formal? Of course, smooth (complex) toric varieties are formal, by a classical result of Deligne, Griffith, Morgan, and Sullivan. More generally, Panov and Ray showed in [16] that all toric manifolds are formal. So we are left with the question whether real toric manifolds are always formal.

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DGAs for subspace arrangement complements

Sergey Yuzvinsky

First in this talk we recall certain rational differential graded algebras for the complement M of an arrangement $\mathcal{A}$ of linear subspaces in a complex space $\mathbb{C}^n$. In particular we denote by D_1 the Morgan algebra that is constructively defined for an arbitrary quasi-projective complex variety represented as the complement to normal crossing divisor in a projective space (see, for instance, [1]). For such a representation of M, we use the De Concini–Procesi wonderful model corresponding to the minimal building set, [1].

The model D_1 has been simplified by the speaker. Here we recall a simple model of M from [5, 6]. Denote by L the lattice of all intersections of subspaces from $\mathcal{A}$. Without any loss of generality we can assume that $\mathcal{A}$ is free of inclusions whence it can be identified with the set of atoms of L. We can also assume that $\mathcal{A}$ is essential (i.e., the intersection of all of its elements is 0) and mark the elements of L by their codimensions $\operatorname{cd} X$ for $X \in L$.

Now let us define the cochain complex D spent by the basis consisting of all subsets $\sigma \subset \mathcal{A}$. We define the dimension $\dim(\sigma) = 2 \operatorname{cd} \bigvee(\sigma) - |\sigma|$. For instance, $\dim(\{H\}) = 2 \operatorname{cd} H - 1$ for every $H \in \mathcal{A}$. Notice that $\dim(\sigma)$ may be negative.

On $\sigma = \{H_1, \ldots, H_k\} \in A$ the differential d is defined by

$$d\sigma = \sum_{j:\bigvee(\sigma_j = \bigvee(\sigma)} (-1)^j \sigma_j$$

where $\sigma_j = \sigma \setminus \{H_j\}$ for $j = 1, \ldots, k$, and the indexing of elements in σ follows some linear order imposed on $\mathcal{A}$. It is a straightforward check that (D, d) is a cochain complex.

We define a multiplication on this complex as follows. For subsets σ and τ of $\mathcal{A}$, we put

$$\sigma\tau = \sum (-1)^{\operatorname{sgn} \epsilon(\sigma,\tau)} \sigma \cup \tau$$

if $\operatorname{cd} \bigvee(\sigma) + \operatorname{cd} \bigvee(\tau) = \operatorname{cd}(\sigma \cup \tau)$ and it is 0 otherwise. Here $\epsilon(\sigma, \tau)$ is the shuffle of the two sets and sgn is the parity of a permutation.

Theorem 1 (cf. [5], Prop. 3.1)**.** *For any subspace arrangement $\mathcal{A}$, the following hold.*

(i) *The complex D with chain complex the multiplication defined above is a differential graded algebra.*
(ii) *This DGA is quasi-isomorphic to the Morgan DGA D_1.*
(iii) *For every $X \in L$ denote by $D(X)$ the sub complex of D generated by σ with $\bigvee(\sigma) = X$. Then $D = \bigoplus_{X\in L} D(X)$ where the sum is in the category of rational algebras. Each subalgebra $D(X)$ has 0 multiplication and $D(X)D(Y) \subset D(X \vee Y)$.*

Remark 1. A differential graded algebra similar to the one discussed here can be defined for arbitrary lattices with a labeling of elements that satisfies certain rank-like conditions.

Remark 2. Recall that there is a cochain complex associated with any finite lattice L: the atomic complex $C(L)$, whose cohomology is that of L. The complex is generated by all subsets σ of $\mathcal{A}$ (the set of atoms) with $\bigvee(\sigma) < 1$. It is clear that the complex $D(X)$ is the factor complex of the cochain complex of the simplex on $\mathcal{A}$ over the atomic complex (up-to slight shift of dimension). More precisely, we may reconstruct a rational version of a formula by Goresky and MacPherson, as follows:

$$H^p(M, \mathbb{Q}) = H^p(D(X)) = H_{2\,\mathrm{cd}(X)-p-2}(L_{\leq X}).$$

In the rest of the talk, we reduce our consideration to the special case where L is a geometric lattice (equivalently, the lattice of a simple matroid). Then independent sets σ of atoms can be characterized by the property that σ is a cocycle in D. Moreover these cocycles have the following two properties:

(1) Their cohomology classes generate the cohomology groups;
(2) For arbitrary $\sigma = \{H_1, \dots, H_k\} \in \mathcal{A}$ either all the sets σ_j are cocycles or all of them are not.

Using these two properties, one can prove the following result from [3].

Theorem 2. *If the intersection lattice L of a subspace arrangement is geometric then the DGA D is formal. More precisely the assignment $\sigma \mapsto [\sigma]$ if σ is independent and $\sigma \mapsto 0$ otherwise generates a DGA homomorphism $D \to H^*(D)$ which produces the identical map on cohomology.*

Using Theorem 2 we obtain the corollary.

Corollary 5. *If the lattice L of a subspace arrangement $\mathcal{A}$ is geometric then the complement M is formal over $\mathbb{Q}$.*

There are simple examples in [3] showing that the condition in the Corollary is not necessary. There are more recent examples in [2] even of coordinate subspaces with a non-formal complement.

This leads to our first problem posed in the problem session, described below.

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Problem Session

All Participants

Problem 1 (S. Yuzvinsky)**.** Given a subspace arrangement, find a necessary and sufficient condition for the formality of its complement.

Problem 2 (S. Yuzvinsky)**.** Find more examples of multinets in $\mathbb{P}^2$. Prove or disprove the conjecture that every multinet is a limit of nets. See for instance [4] in Yuzvinsky's references.

Problem 3 (G. Rybnikov)**.** Let A, B, C, D, E be the vertices of a regular pentagon. Consider the real affine arrangement consisting of lines AB, CE, AC, DE, AD, BC, AE, BD (four pairs of parallel lines). The combinatorial structure of this arrangement admits an isomorphism induced by the permutation of vertices $B \to C \to E \to D \to B$. Thus we get an automorphism of the homology of the complement of the complexified arrangement. This automorphism preserves the ordinary coalgebra structure. Can it be extended to an automorphism of E_∞-coalgebra? Can it be extended to an automorphism of the fundamental group?

Problem 4 (G. Rybnikov)**.** Every realization space—the set of arrangements with a given combinatorial structure—is an algebraic variety defined over $\mathbb{Z}$. In the previous example, the defining equation of this variety is essentially $x^2 - x - 1 = 0$, and for the MacLane arrangement, the equation is $x^2 + x + 1 = 0$. Relate the invariants of E_∞-coalgebra structure on homology to some properties of this algebraic variety. For example, in the case of the MacLane arrangement, the invariant takes values in 3-torsion, and the variety has fewer points over a field of characteristic 3.

Problem 5 (A. Suciu)**.** Given a simple polytope P and a characteristic matrix χ, determine whether the corresponding real toric manifold, $N_P(\chi)$, is formal.

Problem 6 (A. Suciu)**.** Let V be a complex vector space of dimension n, let $K \subseteq V \wedge V$ be a linear subspace of dimension m, and let $\mathcal{B}(V, K)$ be the corresponding "Koszul" module over the polynomial ring $S = \mathrm{Sym}(V)$.

Let $U = U(n, m)$ be the set of planes K for which the support of $\mathcal{B}(V, K)$ vanishes. As shown in reference [7] from Suciu's first abstract, the set U is a

Zariski open subset of the Grassmannian $\mathrm{Gr}_m(V \wedge V)$. Moreover, this set is non-empty if and only if $m \geq 2n-3$, in which case there is a (smallest) integer $q = q(n,m)$ such that $\mathcal{B}_q(V,K) = 0$, for every $K \in U$.

For instance, if $\binom{n}{2} - m \leq 1$, then $q(n,m) = \binom{n}{2} - m$. In general, though, the determination of the integer $q(n,m)$ is a challenging, yet interesting problem.

In the special case when V is an irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$ and $\mathcal{B} = \mathcal{B}(V,K)$ is the corresponding Weyman module, a solution to this problem would offer new insights on an old conjecture of Mark Green, regarding the syzygies of canonically embedded curves.

Problem 7 (Questions about tropicalization and arrangements by M. Falk)**.** Given an arrangement $\mathcal{A} = \{H_0, \dots, H_n\}$ of hyperplanes in complex projective space $\mathbb{P}^\ell$, let $\mathcal{S} = \mathcal{S}(\mathcal{A})$ be the arrangement of subspaces of $\mathbb{T}^n := \mathbb{R}^{n+1}/\mathbb{R}(1,\dots,1)$ spanned by maximal cones in the Bergman fan $B_\mathcal{A}$ of $\mathcal{A}$. For given p satisfying $1 \leq p < \ell$, describe the family of rational linear subspaces K of codimension $p+1$ in $\mathbb{T}^n$ satisfying $\mathrm{codim}_S(K \cap S) \leq p$ for all $S \in \mathcal{S}$.

The subspace arrangement $\mathcal{S}$ consists of certain flats of dimension ℓ in the braid arrangement $\mathcal{A}_n$ in V. Indeed, the maximal cones of $B_\mathcal{A}$ are subdivided by cones in the nested set fan corresponding to the maximal building set; these cones are generated by characteristic vectors of elements of chains in the intersection lattice. The linear hull of the cone corresponding to a maximal chain $X_0 < \cdots < X_\ell$ is the flat of $\mathcal{A}_n$ corresponding to the partition $\{X_1 - X_0, \dots, X_\ell - X_{\ell-1}\}$ of $\mathcal{A}$.

If K is a linear subspace satisfying the stated condition, then the projection $\mathbb{T}^n \to \mathbb{T}^n/K \cong \mathbb{T}^{p+1}$ maps $B_\mathcal{A}$ to a tropical variety of dimension at most p, giving rise to a a family of syzygies of $\mathcal{A}$-master functions, along with all the concomitant implications for linear systems and fibrations, decomposable cocycles, and resonance varieties. See my abstract in this volume for details and references.

The description of degree-one resonance varieties in terms of multinets can be viewed as a solution to this problem in case $\ell = 2$, $p = 1$. Of special interest is the case where the image of $B_\mathcal{A}$ in $\mathbb{T}^n/K$ is itself a Bergman fan; in this case K gives rise to a linear syzygy of master functions.

Problem 8 (M. Falk)**.** Let $\mathcal{A}$ be an arrangement of hyperplanes in $\mathbb{P}^\ell$. Let $\pi\colon X \to \mathbb{P}^\ell$ be the De Concini–Procesi wonderful model of $\mathcal{A}$ corresponding to the minimal building set $\mathcal{G}$. For $Y \in \mathcal{G}$ let $D_Y \subseteq X$ be the corresponding boundary divisor. Let $\mathcal{G}_1 \subset \mathcal{G}$ with $\mathcal{A} \subseteq \mathcal{G}_1$, and let $U = X \setminus \bigcup_{Y \in \mathcal{G}_1} D_Y$. Use toric methods to compute $H^*(U,\mathbb{C})$, and determine the pairs $(\mathcal{A},\mathcal{G}_1)$ for which $H^p(U,\mathbb{C}) = 0$ for $p > \ell$.

The space U arises in Varchenko's theory of integrable models of arrangements. Write $H = \ker(\alpha_H\colon \mathbb{C}^{\ell+1} \to \mathbb{C})$, and let $M = \mathbb{P}^\ell \setminus \bigcup_{H \in \mathcal{A}} H$. Given a system of nonzero complex weights $\lambda = (\lambda_H \mid H \in \mathcal{A})$ satisfying $\sum_{H \in \mathcal{A}} \lambda_H = 0$, the corresponding master function is $\Phi_\lambda = \prod_{H \in \mathcal{A}} \alpha_H^{\lambda_H}$, a multi-valued function on M. One sets

$$\mathcal{G}_1 = \{X \in \mathcal{G} \mid \sum_{H \supseteq X} \lambda_H \neq 0\}.$$

Then Φ_λ extends to U. Its critical points on U are in many situations related to the cohomology of the associated rank-one local system on U and to the complex of flag forms and related Lie algebra homology. The requirement $H^p(U,\mathbb{C}) = 0$ for $p > \ell$ is necessary for these relationships to hold.

In joint work with A. Varchenko and H. Terao, we have derived a spectral sequence converging to $H^*(U,\mathbb{C})$ or, more generally, to $H^*(U,\mathcal{L})$ for any rank-one local system $\mathcal{L}$. In case $\mathcal{A}$ is the projectivized rank-three braid arrangement and $\mathcal{G}_1 = \mathcal{A}$, one has $H^3(U,\mathbb{C}) \neq 0$. There is a relation here with resonant weights on $\mathcal{A}$.

In general U is an incomplete toric variety; the computation of $H^*(U,\mathbb{C})$ may be susceptible to known methods of toric geometry.

Problem 9 (Questions on the Orlik-Terao algebra, by H. Schenck)**.** For a hyperplane arrangement

$$\mathcal{A} = \bigcup_{i=1}^{d} V(\alpha_i) \subseteq \mathbb{P}^n,$$

the Orlik-Terao algebra is defined as follows:

Definition 1. Let $R = \mathbb{K}[y_1,\dots,y_d]$, and for a relation $\Lambda = \sum_{j=1}^{k} c_{i_j}\alpha_{i_j} = 0$, let

$$f_\Lambda = \sum_{j=1}^{k} c_{i_j}(y_{i_1}\cdots\hat{y}_{i_j}\cdots y_{i_k}).$$

The *Orlik-Terao ideal* $I_\mathcal{A}$ is generated by the f_Λ's, and the quotient $R/I_\mathcal{A}$ is the *Orlik-Terao algebra*.

Strictly speaking, the Orlik-Terao ideal originally introduced in [1] also includes the squares of the variables. In [1], Orlik and Terao prove that the Poincaré series of that algebra is $\pi(\mathcal{A},t)$. In [5], Terao shows the Hilbert series of $R/I_\mathcal{A}$ is $\pi\big(\mathcal{A},\frac{t}{1-t}\big)$. In [2], Proudfoot and Speyer prove that $R/I_\mathcal{A}$ is Cohen-Macaulay, and in [3] Schenck and Tohaneanu show that 2-formality of this algebra is determined by the quadratic component of $I_\mathcal{A}$.

It was recently shown in [4] that for $\mathcal{A} \subseteq \mathbb{P}^2$, the algebra $R/I_\mathcal{A}$ is the homogeneous coordinate ring of the blowup of $\mathbb{P}^2$ at the singular points of $\mathcal{A}$, mapped to $\mathbb{P}^{d-1}$ by a nef but not ample divisor, and that elements of $R^1(\mathcal{A})$ give rise to determinantal equations in $I_\mathcal{A}$, with the result that $V(I_\mathcal{A})$ lies on a scroll.

Among other open questions on these algebras, we can mention:

(1) What can be said in higher dimensions?
(2) What can be said about special classes, such as reflection arrangements?
(3) What can be said about the minimal free resolution, even for $\mathcal{A} \subseteq \mathbb{P}^2$?

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