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Mini-Workshop: Topology of closed one-forms and Cohomology Jumping Loci

Organised by
Michael Farber, Durham
Alexander Suciu, Boston
Sergey Yuzvinsky, Eugene

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Abstract. The purpose of this workshop was to bring together researchers from the two different fields mentioned in the title, and to create more interaction and conections between these fields. Among the topics which appear in both subjects are Lusternik-Schnirelmann category, Bieri-Neumann-Strebel invariants and a spectral sequence introduced by Farber and Novikov.

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Introduction by the Organisers

This Mini-Workshop was organized by M. Farber (Durham), A. Suciu (Boston) and S. Yuzvinsky (Eugene). It brought together researchers working on two distinct, yet related topics:

- The *topology of closed one-forms* is a field of research initiated in 1981 by S. P. Novikov. In this version of Morse theory, one studies closed 1-forms and their zeroes instead of smooth functions and their critical points.
- The *cohomology jumping loci* are the support varieties for cohomology with coefficients in rank one local systems, and the related resonance varieties. In recent years, these varieties have emerged as a central object of study in the theory of hyperplane arrangements and related spaces.

Even though these two fields share some common roots, so far they have developed in parallel, with not much overlap or interaction. Nevertheless, it is becoming increasingly apparent that there are deep connections between the two theories, with potentially fruitful applications going both ways:

- An example is provided by the Lusternik-Schnirelmann category, and the related notions of category weight and topological complexity of robot motion planning. Such notions are amenable to being studied via closed 1-forms, and have applications to dynamical systems and motion planning in robotics. A good understanding of the cohomology ring and resonance varieties yields useful bounds.
- The Bieri-Neumann-Strebel invariants, which generalize the Thurston norm from 3-dimensional topology, are directly related to Novikov - Sikorav homology, Alexander invariants, and the resonance varieties.
- Undergirding some of this theory is a spectral sequence, introduced by Farber and Novikov in the mid 1980s. Recently, this machinery has been extended in a way that connects it to the cohomology jumping loci.

Given the multifaceted nature of these topics, the meeting brought together people with a variety of backgrounds, including topology, algebra, discrete geometry, geometric analysis, and singularity theory. Several participants were recent Ph.D.'s, most of them on their first visit to Oberwolfach. In all, there were 16 people attending the workshop (including the organizers), coming from the United States, Great Britain, France, Romania, Canada, and Germany.

The Mini-Workshop provided a lively forum for discussing a host of questions related to the themes listed above. The day-by-day schedule was kept flexible, and was agreed upon on short notice, making it possible to shape the program on-site, and in response to the interests expressed by the participants. The borderline between problem sessions and formal lectures were often blurred. Spending a concentrated and highly intense week in a relatively small group allowed for in-depth and continuing conversations, in particular with new acquaintances. These opportunities were enhanced by the diversity of backgrounds of the participants.

A basic objective of the Mini-Workshop was to bring together some of the people most actively working in two related fields, and to seek common ground for further advances and collaborations. In the ideally suited research atmosphere at Oberwolfach, participants had the opportunity to explain their respective approaches, and the variety of techniques they use. The lively atmosphere and the free-flow of ideas led to a deeper understanding of the subject, to progress in solving several open problems, and to fruitful insights on how to attack new problems.

Workshop: Mini-Workshop: Topology of closed one-forms and Cohomology Jumping Loci

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Abstracts

Jumping loci and finiteness properties of groups

Alexander I. Suciu
(joint work with Alexandru Dimca, Stefan Papadima)

This is an extended abstract of a talk given on the first day of the Mini-Workshop. In the first part, we give a quick overview of characteristic and resonance varieties. In the second part, we describe recent work [11], relating the cohomology jumping loci of a group to the homological finiteness properties of a related group.

1. Cohomology jumping loci

Characteristic varieties. Let X be a connected CW-complex with finitely many cells in each dimension, and G its fundamental group. The characteristic varieties of X are the jumping loci for cohomology with coefficients in rank 1 local systems:

$$V_k^i(X) = \{\rho \in \operatorname{Hom}(G, \mathbb{C}^*) \mid \dim H^i(X, \mathbb{C}_\rho) \geq k\}.$$

These varieties emerged from the work of Novikov [22] on Morse theory for closed 1-forms on manifolds. It turns out that $V_k^1(X)$ is the zero locus of the annihilator of the k-th exterior power of the complexified Alexander invariant of G; thus, we may write $V_k(G) := V_k^1(X)$. For example, if X is a knot complement, $V_k(G)$ is the set of roots of the Alexander polynomial with multiplicity at least k.

One may compute the first Betti number of a finite abelian regular cover, $Y \to X$, by counting torsion points of a certain order on $\operatorname{Hom}(G, \mathbb{C}^*)$, according to their depth in the filtration $\{V_k(G)\}$, see Libgober [17]. One may also obtain information on the torsion in $H_1(Y, \mathbb{Z})$ by considering characteristic varieties over suitable Galois fields, see [21]. This approach gives a practical algorithm for computing the homology of the Milnor fiber F of a central arrangement in $\mathbb{C}^3$, leading to examples of multi-arrangements with torsion in $H_1(F, \mathbb{Z})$, see [4].

Foundational results on the structure of the cohomology support loci for local systems on smooth, quasi-projective algebraic varieties were obtained by Beauville [2], Green–Lazarsfeld [14], Simpson [28], and ultimately Arapura [1]: if G is the fundamental group of such a variety, then $V_1(G)$ is a union of (possibly translated) subtori of $\operatorname{Hom}(G, \mathbb{C}^*)$. The characteristic varieties of arrangement groups have been studied by, among others, Cohen–Suciu [6], Libgober–Yuzvinsky [20], and Libgober [18]. As noted in [30, 31], translated subtori do occur in this setting; for an in-depth explanation of this phenomenon, see Dimca [7, 8].

Resonance varieties. Consider now the cohomology algebra $H^*(X, \mathbb{C})$. Right-multiplication by a class $a \in H^1(X, \mathbb{C})$ yields a cochain complex $(H^*(X, \mathbb{C}), \cdot a)$. The *resonance varieties* of X are the jumping loci for the homology of this complex:

$$R_k^i(X) = \{a \in H^1(X, \mathbb{C}) \mid \dim H^i(H^*(X, \mathbb{C}), \cdot a) \geq k\}.$$

These varieties were first defined by Falk [12] in the case when X is the complement of a complex hyperplane arrangement. In this setting, a purely combinatorial description of $R^1_k(X)$ was given by Falk [12], Libgober–Yuzvinsky [20], Falk–Yuzvinsky [13], and Pereira–Yuzvinsky [26].

The varieties $R_k(G) := R^1_k(X)$ depend only on $G = \pi_1(X)$. In [30], two conjectures were made, expressing (under some conditions) the lower central series ranks and the Chen ranks of an arrangement group G solely in terms of the dimensions of the components of $R_1(G)$. For recent progress in this direction, see [23, 27].

The tangent cone formula. If G is a finitely presented group G, the tangent cone to $V_k(G)$ at the origin, $TC_1(V_k(G))$, is contained in $R_k(G)$, see Libgober [19]. In general, though, the inclusion is strict, see [21, 9]. Now suppose G is a 1-*formal* group, in the sense of Quillen and Sullivan; that is, the Malcev Lie algebra of G is quadratic. Then, as shown in [9], equality holds:

$$TC_1(V_k(G)) = R_k(G).$$

This extends previous results from [6, 18], valid only for arrangement groups. It is also known that $TC_1(V^i_k(X)) = R^i_k(X)$, for all $i \geq 1$, in the case when X is the complement of a complex hyperplane arrangement, see Cohen–Orlik [5]. A generalization to arbitrary formal spaces is expected.

2. Non-finiteness properties of projective groups

In [29], Stallings constructed the first example of a finitely presented group G with $H_3(G, \mathbb{Z})$ infinitely generated; such a group is of type F_2 but not FP_3. It turns out that Stallings' group is isomorphic to the fundamental group of the complement of a complex hyperplane arrangement, see [23].

More generally, to every finite simple graph Γ, with flag complex $\Delta(\Gamma)$, Bestvina and Brady associate in [3] a group N_Γ and show that N_Γ is finitely presented if and only if $\pi_1(\Delta(\Gamma)) = 0$, while N_Γ is of type FP_{n+1} if and only if $\widetilde{H}_{\leq n}(\Delta(\Gamma), \mathbb{Z}) = 0$.

In joint work with Dimca and Papadima [10], we determine precisely which Bestvina-Brady groups N_Γ occur as fundamental groups of smooth quasi-projective varieties. (The proof uses previous work on the jumping loci of right-angled Artin groups [24, 9] and Bestvina-Brady groups [25].) This classification yields examples of quasi-projective groups which are not commensurable, even up to finite kernels, to the fundamental group of an aspherical, quasi-projective variety.

In [11] we go further, and construct smooth, complex *projective* varieties whose fundamental groups have exotic homological finiteness properties.

Theorem 1 ([11]). *For each $n \geq 2$, there is an n-dimensional, smooth, irreducible, complex projective variety M such that:*

1. *The homotopy groups $\pi_i(M)$ vanish for $2 \leq i \leq n-1$, while $\pi_n(M) \neq 0$.*
2. *The universal cover $\widetilde{M}$ is a Stein manifold.*
3. *The group $\pi_1(M)$ is of type F_n, but not of type FP_{n+1}.*
4. *The group $\pi_1(M)$ is not commensurable (up to finite kernels) to any group having a classifying space of finite type.*

Theorem 1 provides a negative answer to the following question raised by Kollár in [16]: Is a projective group G commensurable (up to finite kernels) with another group G', admitting a $K(G',1)$ which is a quasi-projective variety?

Theorem 1 also sheds light on the following question of Johnson and Rees [15]: Are fundamental groups of compact Kähler manifolds Poincaré duality groups of even cohomological dimension? In [32], Toledo answered this question, by producing examples of smooth projective varieties M with $\pi_1(M)$ of *odd* cohomological dimension. Our results show that fundamental groups of smooth projective varieties need not be Poincaré duality groups of *any* cohomological dimension: their Betti numbers need not be finite.

A key point in our approach is a theorem connecting the characteristic varieties of a group G to the homological finiteness properties of some of its normal subgroups N.

Theorem 2 ([11])**.** *Let G be a finitely generated group. Suppose $\nu\colon G \to \mathbb{Z}^m$ is a non-trivial homomorphism, and set $N = \ker(\nu)$. If $V_1^r(G) = \mathrm{Hom}(G, \mathbb{C}^*)$ for some integer $r \geq 1$, then:*

(1) $\dim_{\mathbb{C}} H_{\leq r}(N, \mathbb{C}) = \infty$.
(2) *N is not commensurable (up to finite kernels) to any group of type FP_r.*

The proof of Theorem 2 depends on the following lemma. Let $\mathbb{T} = \mathrm{Hom}(\mathbb{Z}^m, \mathbb{C}^*)$ be the character torus of $\mathbb{Z}^m$, and let $\Lambda = \mathbb{C}\mathbb{Z}^m$ be its coordinate ring. Let A be a Λ-module which is finite-dimensional as a $\mathbb{C}$-vector space. Then, for each $j \geq 0$, the set $A_j := \{\rho \in \mathbb{T} \mid \mathrm{Tor}_j^{\Lambda}(\mathbb{C}_\rho, A) = 0\}$ is a Zariski open, non-empty subset of the algebraic torus $\mathbb{T}$.

To obtain our examples, we start with an elliptic curve E and take 2-fold branched covers $f_j\colon C_j \to E$ ($1 \leq j \leq r$ and $r \geq 3$), so that each curve C_j has genus at least 2. Setting $X = \prod_{j=1}^r C_j$, we see that X is a smooth, projective variety, whose universal cover is a contractible, Stein manifold. Moreover, $V_1^r(\pi_1(X)) = \mathrm{Hom}(\pi_1(X), \mathbb{C}^*)$.

Using the group law on E, define a map $h\colon X \to E$ by $h = \sum_{j=1}^r f_j$. Let M be the smooth fiber of h. Under certain assumptions on the branched covers f_j, we show that M is connected and h has only isolated singularities. A complex Morse-theoretic argument shows that the induced homomorphism, $\nu = h_\sharp\colon \pi_1(X) \to \pi_1(E)$, is surjective, with kernel N isomorphic to $\pi_1(M)$. Applying Theorem 2 to this situation (with $n = r-1$) finishes the proof of Theorem 1.

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Varieties of flat line bundles, cohomology jumping loci and topology of closed 1-forms

Michael Farber
(joint work with Dirk Schütz)

The aim of the talk was to survey methods of topology of closed one-forms based on studying varieties of flat line bundles and various objects on these varieties depending on topology of the underlying manifold. A classical example is provided by the Novikov theory which was mentioned first. Let X be a finite polyhedron and $\xi \in H^1(X;\mathbb{R})$. The following statement gives an alternative description of the Novikov numbers $b_i(\xi)$. One considers complex flat line bundles L over X having the property that the induced flat bundle over $\tilde{X}$ is trivial; here $\tilde{X} \to X$ denotes the covering corresponding to $\ker \xi$. The set of all such bundles L is an algebraic variety $\mathcal{V}_\xi$ isomorphic to $(\mathbb{C}^*)^r$ where $r = \mathrm{rk}(\xi)$ is the rank of ξ. According to Theorem 1.50 from [1], *there exists a proper algebraic subset $S = S(X,\xi) \in \mathcal{V}_\xi$ such that*

$$\dim H_i(X,L) = b_i(\xi)$$

for all $L \in \mathcal{V}_\xi - S$ and

$$\dim H_i(X,L) \geq b_i(\xi)$$

for $L \in S$.

Next I discussed results of topology of closed 1-forms developed by the author in collaboration with Thomas Kappeler and Dirk Schütz. The classical Lusternik -Schnirelmann category $\mathrm{cat}(X)$ allows several interesting generalizations

$$\mathrm{cat}(X,\xi), \quad \mathrm{cat}^1(X,\xi), \quad \mathrm{Cat}(\mathrm{X},\xi) \tag{1}$$

depending on a finite polyhedron X and on a real cohomology class ξ. Definitions of these invariants are similar to each other and to the definition of the usual category: they are minimal numbers of covers of X having certain topological properties. However, instead of requiring that all sets in the cover are null-homotopic, we allow that one of the sets is *movable* with respect to a closed 1-form representing ξ. Different invariants (1) vary by understanding differently the term *movable.*

Invariants (1) play interesting role in dynamics, they estimate complexity of chain recurrent sets of flows, see [1], [2].

Paper [4] suggests new cohomological lower bounds for $\mathrm{cat}^1(X,\xi)$. These bounds are also based on studying flat line bundles of special kind (not ξ-algebraic integers). Another new tool suggested in [4] is the notion of category weight of a homology classes which is a variation of the notion of category weight of cohomology classes of Faddell and Husseini in 1992. With the help of the notion of category weight of homology classes we establish many examples when the invariants $\mathrm{cat}(X, \xi)$ and $\mathrm{cat}^1(X, \xi)$ are distinct. Moreover, we show that the difference $\mathrm{cat}^1(X, \xi) - \mathrm{cat}(X, \xi)$ can be arbitrarily large.

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Lusternik-Schnirelmann category for closed 1-forms

Dirk Schütz

(joint work with Michael Farber)

For a finite CW-complex X we discuss two versions of a Lusternik-Schnirelmann category for cohomology classes $\xi \in H^1(X; \mathbb{R})$, namely $\mathrm{cat}(X, \xi)$ and $\mathrm{cat}^1(X, \xi)$, which have already appeared in the talk of Michael Farber and which were introduced in [1] and [2], respectively. The difference in their definition is rather subtle, and beside the obvious inequality

$$\mathrm{cat}(X, \xi) \quad \leq \quad \mathrm{cat}^1(X, \xi)$$

there is the natural question whether they are equal. In order to obtain calculations of a Lusternik-Schnirelmann category one usually requires good bounds from below and above. We establish a lower bound by a cup-length estimation which involves coefficients in a complex flat line bundle over X which pulls back trivially to the minimal covering $q : \tilde{X} \to X$ that satisfies $q^*\xi = 0$. This gives rise to our main lower bound for $\mathrm{cat}(X, \xi)$, but in the case of $\mathrm{cat}^1(X, \xi)$ we can improve this slightly: if the resulting non-trivial cup-product evaluates non-trivially on an element coming from the homology of $\tilde{X}$, the estimate improves by the category weight of that homology class, a concept introduced in [4].

This already indicates that $\mathrm{cat}(X, \xi) < \mathrm{cat}^1(X, \xi)$, but in order to get an example, better upper bounds for $\mathrm{cat}(X, \xi)$ are needed. The crucial result here is the following

Theorem 1 ([3]). *Let M be a smooth closed connected manifold of dimension m and $\xi \in H^1(M; \mathbb{R})$ be non-zero. Then*

$$\mathrm{cat}(M, \xi) \quad \leq \quad m - 1$$

It is important here that M is a manifold, as for ordinary CW-complexes X and $\xi \neq 0$ we only get $\mathrm{cat}(X,\xi) \leq \dim X$ and equality is possible. The proof of Theorem 1 does not work for $\mathrm{cat}^1(M,\xi)$ and with the above mentioned cup-length estimations it is easy to see that $\mathrm{cat}^1(M_g,\xi) = 2$ for M_g an orientable surface of genus $g > 1$ and $\xi \neq 0$. As $\mathrm{cat}(M_g,\xi) \leq 1$ by Theorem 1 this gives a first example where $\mathrm{cat}(X,\xi) \neq \mathrm{cat}^1(X,\xi)$.

For direct products of spaces there exist similar estimates as for the classical Lusternik-Schnirelmann category and so we obtain the following theorem.

Theorem 2. *Let $M^{2k} = \Sigma_1 \times \cdots \times \Sigma_k$, where each Σ_i is a closed orientable surface of genus $g_i > 1$. Given $\xi_i \in H^1(\Sigma_i;\mathbb{R})$, one has*

$$\begin{aligned} \mathrm{cat}(M^{2k},\xi) &= 1 + 2r \\ \mathrm{cat}^1(M^{2k},\xi) &= 1 + r + k \end{aligned}$$

where $\xi = p_1^*\xi_1 + \cdots + p_k^*\xi_k$ and $r \leq k$ is the number of indices i such that $\xi_i = 0$.

An immediate consequence is

Corollary 1. *The difference*

$$\mathrm{cat}^1(X,\xi) - \mathrm{cat}(X,\xi)$$

can be arbitrary large.

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Topological complexity of almost-direct products of free groups

Daniel C. Cohen

Let X be a path-connected topological space, and let PX be the space of all continuous paths $\gamma\colon [0,1] \to X$, equipped with the compact-open topology. The map $\pi\colon PX \to X \times X$, $\gamma \mapsto (\gamma(0),\gamma(1))$, defined by sending a path to its endpoints is a fibration.

Definition. The *topological complexity* of X, $\mathsf{TC}(X)$, is the minimal k for which $X \times X = U_1 \cup \cdots \cup U_k$, where U_i is open and there exists a continuous section $s_i\colon U_i \to PX$, $\pi \circ s_i = \mathrm{id}_{U_i}$, for each i, $1 \leq i \leq k$. In other words, the topological complexity of X is the sectional category (or Schwarz genus) of the path space fibration, $\mathsf{TC}(X) = \mathrm{secat}(\pi\colon PX \to X \times X)$.

This notion, introduced by Farber, provides a topological approach to the motion planning problem from robotics, see the survey [6] and the references therein.

Assume that X is a finite-dimensional cell complex. We shall make use of the following properties of topological complexity, which may be found in [6].

(1) $\mathsf{TC}(X)$ is an invariant of the homotopy type of X;
(2) $\mathsf{TC}(X) = 1 \iff X$ is contractible;
(3) $\mathsf{TC}(X) \leq 2\dim(X) + 1$;
(4) $\mathsf{TC}(X) > \mathsf{zcl}(H^*(X)) := \text{cup length}\big(\ker\big(H^*(X) \otimes H^*(X) \xrightarrow{\cup} H^*(X)\big)\big)$.

For the *zero-divisor cup length* $\mathsf{zcl}(H^*(X))$, use cohomology with field coefficients.

Problem (Farber [6])**.** For a discrete group G, define the topological complexity of G to be the topological complexity of an Eilenberg-Mac Lane space of type $K(G,1)$, $\mathsf{TC}(G) := \mathsf{TC}(K(G,1))$. Determine the topological complexity of G in terms of other invariants of G.

Definition. An *almost-direct product of free groups* is an iterated semi-direct product $G = F_{n_\ell} \rtimes \cdots \rtimes F_{n_1}$ of finitely generated free groups F_{n_i}, $n_i < \infty$, such that the action of F_{n_i} on the homology $H^*(F_{n_j};\mathbb{Z})$ is trivial for $1 \leq i < j \leq \ell$.

We pursue the topological complexity of groups of this type. This is motivated by the following results.

Theorem (Farber-Yuzvinsky [8])**.** Let P_ℓ be the Artin pure braid group, the fundamental group of the configuration space of ℓ ordered points in $\mathbb{C}$. Then $\mathsf{TC}(P_\ell) = 2\ell - 2$.

Theorem (Farber-Grant-Yuzvinsky [7])**.** Let $P_{\ell,k} = \ker(P_\ell \to P_k)$ denote the kernel of the map which forgets the last $\ell - k$ strands of an ℓ-strand pure braid, the fundamental group of the configuration space of ℓ ordered points in $\mathbb{C} \setminus \{k \text{ points}\}$. If $k \geq 2$, then $\mathsf{TC}(P_{\ell,k}) = 2(\ell - k) + 1$.

The groups $P_\ell = F_{\ell-1} \rtimes \cdots \rtimes F_1$ and $P_{\ell,k} = F_{\ell-1} \rtimes \cdots \rtimes F_k$ are almost direct products of free groups. For $i < j$, the action of F_i on F_j is given by (the restriction of) the Artin representation.

Let $P\Sigma_\ell$ be the group of basis-conjugating automorphisms of the free group $F_\ell = \langle x_1, \ldots, x_\ell\rangle$. McCool [10], found the following presentation for $P\Sigma_\ell$:

$$P\Sigma_\ell = \langle \beta_{i,j}, 1 \leq i \neq j \leq \ell \mid [\beta_{i,j}, \beta_{k,l}], [\beta_{i,k}, \beta_{j,k}], [\beta_{i,j}, (\beta_{i,k} \cdot \beta j, k)], \rangle,$$

where the indices in the relations are distinct, and the generators $\beta_{i,j}$ are the automorphisms defined by

$$\beta_{i,j}(x_k) = \begin{cases} x_k & \text{if } k \neq j, \\ x_i^{-1} x_j x_i & \text{if } k = j. \end{cases}$$

The subgroup $P\Sigma_\ell^+$ generated by $\beta_{i,j}$ for $i < j$ is an almost-direct product of free groups, see [4]. One has $P\Sigma_\ell^+ = F_{\ell-1} \rtimes P\Sigma_{\ell-1}^+ = F_{\ell-1} \rtimes \cdots \rtimes F_1$, where $F_\ell = \langle \beta_{1,\ell}, \ldots, \beta_{\ell-1,\ell}\rangle$, and the action of $P\Sigma_{\ell-1}^+$ on F_ℓ may be extracted from the above presentation. The *upper triangular McCool group* $P\Sigma_\ell^+$ is *not* isomorphic to the pure braid group P_ℓ.

Theorem (Cohen-Pruidze [2])**.** Let $P\Sigma_\ell^+$ be the upper triangular McCool group. Then $\mathsf{TC}(P\Sigma_\ell^+) = 2\ell - 2$.

Remark. The pure braid group P_ℓ and triangular McCool group $P\Sigma_\ell^+$ each have infinite cyclic center. Denoting the center of a group G by $Z(G)$, and writing $\overline{G} = G/Z(G)$, we have $P_\ell = \overline{P}_\ell \times \mathbb{Z}$ and $P\Sigma_\ell^+ = \overline{P\Sigma_\ell^+} \times \mathbb{Z}$, where $\overline{P}_\ell$ and $\overline{P\Sigma_\ell^+}$ are almost-direct products of free groups, each of which has rank at least two.

The above results are unified by the following:

Theorem. Let $G = F_{n_\ell} \rtimes \cdots \rtimes F_{n_1}$ be an almost-direct product of free groups. If $n_j \geq 2$ for each j and m is a non-negative integer, then $\mathsf{TC}(G \times \mathbb{Z}^m) = 2\ell + m + 1$.

Problem. If $\mathbb{Z}^m$ acts nontrivially on G, what is $\mathsf{TC}(G \rtimes \mathbb{Z}^m)$?

For the sake of brevity, we focus on the case $m = 0$. Let $X = K(G,1)$ be an Eilenberg-Mac Lane space of type $K(G,1)$. The above result may be established using the bounds $\mathsf{zcl}(H^*(G)) < \mathsf{TC}(G) \leq 2\dim(X) + 1$ noted previously. First, it is not difficult to show that the cohomological dimension of G is equal to the geometric dimension of G, which in turn, is equal to ℓ,

$$\mathrm{cd}(G) = \mathrm{geom\ dim}(G) = \ell \implies \mathsf{TC}(G) \leq 2\ell + 1.$$

The lower bound $\mathsf{zcl}(H^*(G)) < \mathsf{TC}(G)$ may be established through analysis of the (integral) cohomology ring $H^*(G)$.

Theorem. The cohomology ring $H^*(G)$ is a quadratic algebra. That is, $H^*(G) \cong E/J$, where E is an exterior algebra generated in degree 1, and J is an ideal generated in degree 2.

The integral homology $H_*(G)$ is torsion-free and the Poincaré polynomial is given by $P(G,t) = \sum_{k=0}^{\ell} b_k(G) \cdot t^k = \prod_{j=1}^{\ell}(1 + n_j t)$, where $b_k(G)$ is the k-th Betti number of G, see [5]. A minimal, free $\mathbb{Z}G$-resolution of $\mathbb{Z}$, which we denote by $C_\bullet(G) \xrightarrow{\epsilon} \mathbb{Z}$, is constructed in [3].

Let $N = b_1(G)$. The abelianization map $G \to \mathbb{Z}^N$ induces a chain map $\Phi_\bullet \colon C_\bullet \to K_\bullet$, where $C_\bullet = C_\bullet(G) \otimes_{\mathbb{Z}G} \mathbb{Z}\mathbb{Z}^N$ and $K_\bullet \to \mathbb{Z}$ is the standard $\mathbb{Z}\mathbb{Z}^N$-resolution of $\mathbb{Z}$. One can show that the induced map in cohomology $\Phi_2^* \colon H^2(\mathbb{Z}^N) \to H^2(G)$ is surjective, and that $H^*(G) \cong E/J$, where $E = H^*(\mathbb{Z}^N)$ and $J = \ker(\Phi_2^*)$.

The identification $H^*(G) \cong E/J$ may be used to show that $\mathsf{zcl}(H^*(G)) = 2\ell$. For each i, $1 \leq i \leq \ell$, let x_i and y_i be classes in $H^1(G)$ corresponding to distinct generators of the free group F_{n_i}. Then one can show that the product

$$\prod_{i=1}^{\ell}(x_i \otimes 1 - 1 \otimes x_i)(y_i \otimes 1 - 1 \otimes y_i)$$

is non-zero. So we have $2\ell = \mathsf{zcl}(H^*(G)) < \mathsf{TC}(G) \leq 2\dim(K(G,1)) + 1 = 2\ell + 1$.

As another consequence of the calculation of the cohomology ring of an almost-direct product of free groups G, one can show that $H^*(G;\mathbb{Q})$ is, in fact, Koszul. If, additionally, the group G is 1-formal, then the space $K(G,1)$ is formal. This is the case for the groups P_ℓ, $P_{\ell,k}$, and $P\Sigma_\ell^+$, see [9] and [1]. When G is 1-formal, the calculation of $\mathsf{TC}(G)$ above may be viewed as evidence in support of the conjecture that, for a formal space X, one has $\mathsf{TC}(X) = 1 + \mathsf{zcl}(H^*(X;\Bbbk))$ for some field $\Bbbk$.

The basis-conjugating automorphism group $P\Sigma_\ell$ is also known to be 1-formal, see [1]. Moreover, we have $\mathsf{TC}(P\Sigma_\ell) = 2\ell - 1 = 1 + \mathsf{zcl}(H^*(P\Sigma_\ell;\mathbb{Q}))$, see [2].

Problem.

(1) Presumably, $P\Sigma_\ell$ is not an almost-direct product of free groups. Prove (or disprove) this.

(2) Determine if the cohomology ring $H^*(P\Sigma_\ell;\mathbb{Q})$ is Koszul.

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Novikov homology and three-manifolds

Jean-Claude Sikorav

Novikov homology is a useful tool to study 1) singularities of closed one-forms and in particular the question of existence of a nonsingular one-form in a given cohomology class 2) finiteness properties in group theory: finite generation, finite presentation, FP_n. In dimension three, there is a close though as yet not fully understood relation with the Thurston norm on $H^1(M;\mathbb{R}) = H_2(M;\mathbb{R})$. This results from a reinterpretation of results of Stallings and Thurston, and is due to

Bieri, Neumann and Strebel in 1987, and independently the author (same year, with some recent improvements). We first describe this relation, and then speculate how it could be interpreted thanks to some hypothetical noncommutative Alexander polynomial. Let M be a closed 3-manifold, with $H^1(M;\mathbb{R}) \neq 0$. We assume that it is oriented and irreducible, thus aspherical. Then $G = \pi_1(M)$ is a 3-dimensional Poincaré duality group, and using a Morse function one can find a resolution

$$0 \longrightarrow \Lambda \xrightarrow{d_3} \Lambda^p \xrightarrow{d_2} \Lambda^p \xrightarrow{d_1} \Lambda \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

By convention, all modules are left-modules, thus d_i is multiplication on the right with a matrix D_i with coefficients in $\Lambda = \mathbb{Z}[G]$. One has $D_1 = (x_1 - 1, \cdots, x_p - 1)$ where $x_1, \cdots, x_p$ are generators of G associated to critical points of index 1, and $D_2 = \left(\dfrac{\partial r_i}{\partial x_j}\right) \in \mathrm{M}_{p,p}(\Lambda)$. Moreover, if the function is "symmetrical" one can assume that $D_3 = D_1^* = (x_1^{-1} \cdots x_p^{-1})$. Ideally, one should think of D_2 as being self-adjoint ($D_2 = D_2^*$), which is a translation of Poincaré duality. Let $\xi \in H^1(M;\mathbb{R}) = \mathrm{Hom}(G,\mathbb{R})$ be nonzero. One defines the Novikov ring $\Lambda_\xi = \mathbb{Z}[G]_\xi = \{\sum_{i=0}^{+\infty} a_i g_i \mid a_i \in \mathbb{Z}, g_i \in G, \lim_{i\to\infty} \xi(g_i) = +\infty\}$ and the Novikov homology $H_*(M;\xi) = H_*(G;\xi) = H_*(C_* \otimes_\Lambda \Lambda_\xi)$, where C_* is the complex $\Lambda \to \Lambda^p \to \Lambda^p \to \Lambda$ above (thus the differentials are the same, but with coefficients in Λ_ξ). Let $N \subset H^1(M;\mathbb{R})$ be the set of classes represented by nonsingular one-forms. Recall that the unit ball $B_T \subset H^1(M;\mathbb{R})$ of the Thurston norm is a polyhedron determined by integral inequations. The relation between Novikov homology and Thurston norm is expressed by the two following facts:

(1) $H_1(M;\xi) = 0 \Leftrightarrow \xi \in N$ (this is also true if one replaces $\mathbb{Z}[G]_\xi$ by $\mathbb{Q}[G]_\xi$)
(2) N is a union of cones over some top-dimensional faces of B_T.

The Thurston norm is up to now a purely geometrical object, we would like to understand it algebraically. The starting point is the following observation, which relies on the fact that Λ_ξ is stably finite (every square matrix which is left invertible is invertible): if $\xi(x_i) \neq 0$ and $(D_2)_{i;i}$ is the matrix obtained by deleting the i-eth line and the i-eth column of D_i, one has

$$H_1(M,\xi) = 0 \Leftrightarrow (D_2)_{i,i} \text{ is invertible in } M_{p-1,p-1}(\Lambda_\xi).$$

The theory of Alexander polynomials is based on a Abelianized version of the above, ie one works over $\overline{\Lambda} = \mathbb{Z}[G_{ab}/\mathrm{Torsion}] = \mathbb{Z}[t_1^{\pm 1}, \cdots, t_m^{\pm 1}]$, $m = b_1(M)$. It is now tempting to postulate a noncommutative version, which would be a positive answer to the following question:

Question 1. Let M be a closed 3-dimensional aspherical manifold. Does there exist an element $\widetilde{\Delta}(M) \in \Lambda$ with the following properties:

(i) $\widetilde{\Delta}(M) = \widetilde{\Delta}(M)^*$
(ii) its image in $\overline{\Lambda}$ is the Alexander polynomial $\Delta(M) = \gcd\det(D_2)_{i,j}$.

(iii) the Thurston norm is
$$||\xi||_T = \max_{g,h\in \mathrm{supp}(\widetilde{\Delta}(M))} |\xi(g)-\xi(h)| - 2 = 2 \max_{g\in \mathrm{supp}(\widetilde{\Delta}(M))} |\xi(g)| - 2$$

(iv) $\xi \in N \Leftrightarrow \widetilde{\Delta}(M)$ is invertible in Λ_ξ (or in $\Lambda_{-\xi}$) $\Leftrightarrow$ is there is only one element in $\mathrm{supp}(\widetilde{\Delta})$ with minimal (or maximal) ξ-value, and its coefficient is ± 1.

A related question is the following:

Question 2. Let G be a finitely generated group. Let A be a matrix in $\mathrm{M}_{p,p}(\mathbb{Z}[G])$, and let $N \subset \mathrm{Hom}(G,\mathbb{R}) \setminus \{0\}$ be the set of ξ such that A is invertible over $\mathbb{Z}[G]_\xi$ (or $\mathbb{Q}[G]_\xi$). Is N defined by a finite number of integral inequations?

Remark. 1) The ring $M_{p,p}(\mathbb{Q}[G])$ is a direct factor of $\mathbb{Q}[G \times S_{p+1}]$, thus one can reduce to the case $p = 1$. 2) The result is clearly true if $p = 1$ and G is left-orderable (which is true in particular if $G = \pi_1(M)$ as above).

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On 1-admissible rank one local systems

Alexandru Dimca

Let M be a connected finite CW-complex. If M is 1-formal, then the first twisted Betti number of M with coefficients in $\mathcal{L}$ may be computed from the cohomology ring of M in low degrees, for rank one complex local systems $\mathcal{L}$ near the trivial local system, see [4], Theorem A. In this paper, assuming moreover that M is a connected smooth quasi-projective variety, our aim is to show that (a version of) the above statement is true globally, with finitely many exceptions. In such a situation the exponential mapping sends the irreducible components E of the first resonance variety $\mathcal{R}_1(M)$ of M onto the non translated irreducible components W of the first characteristic variety $\mathcal{V}_1(M)$ of M.

For $\alpha \in E$, $\alpha \neq 0$ (resp. $\mathcal{L} \in W$, $\mathcal{L} \neq \mathbb{C}_M$), the dimension of the cohomology group $H^1(H^*(M,\mathbb{C}), \alpha\wedge)$ (resp. $H^1(M,\mathcal{L})$) is constant (resp. constant with finitely many exceptions where this dimension may possibly increase). The first result is that the 1-formality assumption implies the inequality
$$\dim H^1(M,\mathcal{L}) \geq \dim H^1(H^*(M,\mathbb{C}), \alpha\wedge)$$
obtained by Libgober and Yuzvinsky when M is a hyperplane arrangement complement, see [5], Proposition 4.2. An *1-admissible local system* is a system for which the equality in the above inequality holds. Various characterization of 1-admissible local systems $\mathcal{L}$ on a non translated component W of the first characteristic variety $\mathcal{V}_1(M)$ are given in the following.

Proposition 1. *If M is 1-formal, then the following three conditions on a local system $\mathcal{L} = f^{-1}\mathcal{L}' \in W$ are equivalent.*

(i) *$\mathcal{L}$ is 1-admissible;*

(ii) $\dim H^1(M, \mathcal{L}) = \min_{\mathcal{L}_1 \in W} \dim H^1(M, \mathcal{L}_1)$. *(This minimum is called the* generic dimension *of $H^1(M, \mathcal{L})$ along W.)*

(iii) *the natural morphism $f^* : H^1(S, \mathcal{L}') \to H^1(M, \mathcal{L})$ is an isomorphism.*

In particular, all local systems, except finitely many, on a non-translated irreducible component W are 1-admissible.

The main novelty is the analysis of local systems belonging to a positive dimensional translated component W' of the first characteristic variety of M.

Theorem 1. *Assume that M is a smooth quasi-projective irreducible complex variety. Let $W = \rho \cdot f^*(\mathbb{T}(S))$ be a translated d-dimensional irreducible component of the first characteristic variety $\mathcal{V}_1(M)$, with $d > 0$. Let $\mathcal{L}_0$ be the rank one local system on M corresponding to ρ, $\mathcal{F} = R^0 f_* \mathcal{L}_0$ and $\Sigma(\mathcal{F})$ the singular support of $\mathcal{F}$. Set $S_0 = S \setminus \Sigma(\mathcal{F})$ and $M_0 = f^{-1}(S_0)$. Assume moreover that M and M_0 are 1-formal.*

Then there is a non-translated irreducible component W_0 of $\mathcal{V}_1(M_0)$, such that $W \subset W_0$ under the obvious inclusion $\mathbb{T}(M) \to \mathbb{T}(M_0)$. In particular, for any local system $\mathcal{L}_1 \in W$, except finitely many, there is a 1-form $\alpha(\mathcal{L}_1) \in H^1(M, \mathbb{C})$ such that $\exp(\alpha(\mathcal{L}_1)) = \mathcal{L}_1$ and $\dim H^1(H^*(M_0, \mathbb{C}), \alpha_0(\mathcal{L}_1)\wedge) = \dim H^1(M, \mathcal{L}_1)$, where $\alpha_0(\mathcal{L}_1) = \iota^*(\alpha(\mathcal{L}_1))$, $\iota : M_0 \to M$ being the inclusion.

Example 1. This is a basic example discovered by A. Suciu, see Example 4.1 in [6], the so called deleted B_3-arrangement. Consider the line arrangement in $\mathbb{P}^2$ given by $xyz(x-y)(x-z)(y-z)(x-y-z)(x-y+z) = 0$. Then there is a 1-dimensional translated component W. In this case the new hypersurface H_W is the line $x + y - z = 0$, and M_0 is exactly the complement of the B_3-arrangement. The characteristic variety $\mathcal{V}_1(M_0)$ has a 2-dimensional component W_0 denoted by Γ in Example 3.3 in [6]. In the notation of loc. cit. one has

$$\Gamma = \{(t, s, (st)^{-2}, s, t, (st)^{-1}, s^2, (st)^{-1}) \mid (s,t) \in (\mathbb{C}^*)^2\}.$$

An easy computation shows that W corresponds to the translated 1-dimensional torus inside $W_0 = \Gamma$ given by $st = -1$.

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Topological complexity and cohomology operations

Mark Grant
(joint work with Michael Farber)

We describe new cohomological lower bounds for the topological complexity $\mathbf{TC}(X)$ of a path-connected topological space X, using cohomology operations. These improve on the standard lower bound in terms of zero-divisors cup-length. Topological complexity is a numerical homotopy invariant, defined by M. Farber to be the Schwarz genus of the free path fibration

$$\pi\colon\ X^I \to X \times X, \quad \pi(\gamma) = \big(\gamma(0), \gamma(1)\big), \tag{1}$$

where X^I denotes the space of all paths $\gamma\colon\ I \to X$ with the compact-open topology. It provides a quantitative measure of the complexity of the task of navigation in X. Knowledge of the value $\mathbf{TC}(X)$ is of practical use in the design of motion planning algorithms for mechanical systems whose configuration space is homotopy equivalent to X. For a recent survey on this invariant and its applications, see [2].

The Schwarz genus of a fibration $p\colon\ E \to B$ was defined and extensively studied by A. S. Schwarz [6], who gave the general cohomological lower bound

$$\mathbf{genus}(p) > \text{cup-length}\big(\ker\big(p^*\colon\ H^*(B) \to H^*(E)\big)\big) \tag{2}$$

(here we may use arbitrary or local coefficients). Inspired by the work of E. Fadell and S. F. Husseini [1], who introduced category weights of cohomology classes to improve the classical cup-length lower bound for the Lusternik-Schnirelmann category, we make the following definition.

Definition 1. Let $p\colon\ E \to B$ be a fibration, and $u \in H^*(B)$ a cohomology class. The *weight of u with respect to p* is

$$\mathrm{wgt}_p(u) = \sup\{k \mid f^*(u) = 0 \text{ for all } f\colon\ A \to B \text{ with } \mathbf{genus}(f^*p) \leq k\}.$$

Here f^*p denotes the pullback fibration of p along f.

One may show that, if $u_1, \ldots, u_\ell \in H^*(B)$ are classes whose product $u_1 \cdots u_\ell$ is non-zero, then

$$\mathbf{genus}(p) > \mathrm{wgt}_p(u_1 \cdots u_\ell) \geq \sum_{i=1}^{\ell} \mathrm{wgt}_p(u_i), \tag{3}$$

and that $\mathrm{wgt}_p(u) \geq 1$ if and only if $p^*(u) = 0$. Hence the lower bound (3) improves (2), providing we can find indecomposable classes u with $\mathrm{wgt}_p(u) \geq 2$. This can be done for the path fibration (1) using stable cohomology operations. Let $\theta = \{\theta_m\colon\ H^m(-;R) \to H^{m+i}(-;S)\}$ be a stable cohomology operation of degree i, where R and S are abelian groups. Define the *excess* of θ, denoted $e(\theta)$, to be the largest n such that $\theta_m \equiv 0$ for all $m < n$.

Theorem 1. *Let $u \in H^n(X;R)$ where $n = e(\theta)$. Then the element*

$$\overline{\theta(u)} = 1 \times \theta(u) - \theta(u) \times 1 \in H^{n+i}(X \times X; S)$$

has $\mathrm{wgt}_\pi(\overline{\theta(u)}) \geq 2$, where $\times$ denotes the cohomology cross product.

This result, inspired by Theorem 3.12 of [1], is proved in [3] and applied to calculate $\mathbf{TC}(L)$ for various lens spaces L. For example for any pair $m, n \in \mathbb{N}$ such that $m \nmid \binom{2n}{n}$, the topological complexity of the $2n+1$-dimensional m-torsion lens space $L^{2n+1}(1,\ldots,1)$ equals $4n+2$.

We can also use Massey products to improve the cup-length lower bound for the Schwarz genus (compare Theorem 4.4 of [5]).

Theorem 2. *Let $p\colon E \to B$ be a fibration, and $\alpha, \beta, \gamma \in H^*(B)$ cohomology classes. Suppose that the Massey product $\langle \alpha, \beta, \gamma \rangle$ is defined and does not contain zero. Then $\mathbf{genus}(p) > \mathrm{wgt}_p(\beta) + \min\{\mathrm{wgt}_p(\alpha), \mathrm{wgt}_p(\gamma)\}$.*

A proof of this result may be found in [4], where it is applied to show that $\mathbf{TC}$ of the Borromean rings link complement in S^3 is at least 4. This is the first known example of a $K(G,1)$ space for which $\mathbf{TC}(G) = \mathbf{TC}(K(G,1))$ is greater than the zero-divisors cup-length plus one.

The next result, also proved in [4], is useful for finding indecomposable zero-divisors z with $\mathrm{wgt}_\pi(z) \geq 2$ in the simply-connected case. Let $\mathrm{swgt}(u)$ denote the strict category weight of $u \in H^*(X)$, in the sense of Y. B. Rudyak [5]. Recall that $\mathrm{swgt}(u) \geq 2$ if u is a Massey product ([5], Corollary 4.6).

Theorem 3. *Let X be an r-connected space, $r \geq 1$. Suppose that $u \in H^\ell(X;\mathbf{F})$ has $\mathrm{swgt}(u) \geq k$, where $k(r+1) \leq \ell < (k+1)(r+1)$ and $\mathbf{F}$ is a field. Then there exists an element $\phi(u) \in H^\ell(X \times X;\mathbf{F})$, of the form*

$$\phi(u) = 1 \times u + \theta(u), \quad \theta(u) \in \bigoplus_{i+j=\ell,\, i>0} H^i(X;\mathbf{F}) \otimes H^j(X;\mathbf{F}),$$

which has $\mathrm{wgt}_\pi(\phi(u)) \geq k$. If u is indecomposable, then $\phi(u) = \overline{u} = 1 \times u - u \times 1$.

Question. Is there an analogue of Theorem 3 in the non-simply-connected case?

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Global versus local algebraic fundamental groups

Stefan Papadima

1. Questions

1.1. **Global groups.** An old theme, initiated by Serre 50 years ago, is to characterize *global* (or *quasi-projective*) groups, i.e., finitely presented groups G realizable as $\pi_1(M)$, where M is a smooth, irreducible, quasi-projective complex variety. A basic obstruction was found by Morgan [7]: the *Malcev Lie algebra* E_G of such a group has generators of weight 1 or 2, and relations of weight 2, 3 or 4. Another important obstruction is due to Arapura [1]: the *cohomology jumping locus in degree* 1 of such a group must be a finite union of (translated) subtori of the character torus of the group. A recent new restriction involves the *Alexander polynomial* $\Delta^G \in \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$, where $n := b_1(G)$.

Theorem 1 (Dimca-Papadima-Suciu [3])**.** *If G is a global group with $n := b_1(G) \geq 3$, then Δ^G* has a single essential variable*, that is, $\Delta^G = P(u)$, with $P \in \mathbb{Z}[u^{\pm 1}]$ and $u = t_1^{e_1} \cdots t_n^{e_n}$.*

1.2. **Local groups.** A *local* group is by definition realizable as $\pi_1(\mathcal{M})$, where $\mathcal{M}$ is the analytic germ at o of $X \setminus Y$, with X and Y complex algebraic varieties, $Y \subset X$ a divisor, and $o \in Y$ an isolated singularity on both X and Y. I will focus here mainly on *plane curve singularity* groups, for which Y is a curve and $X = \mathbb{C}^2$, a case going back to Zariski's work from the early 1930s. As follows from a result of Durfee and Hain [4], all plane curve singularity groups verify Morgan's test from §1.1, in stronger form. More precisely, they all are 1-*formal* groups, i.e., their Malcev Lie algebras have only quadratic relations. Moreover, all plane curve singularity groups also pass the Arapura test from §1.1, as a consequence of recent results by Libgober [6]. Two natural questions thus arise:

Question 1. How many plane curve singularity groups (besides those coming from quasi-homogeneous curves) are global?

Question 2. Classify the Malcev Lie algebras of global plane curve singularity groups.

As noticed by Eisenbud and Neumann in [5], plane curve singularity groups are particular cases of fundamental groups of *graph links.* A graph link, (Σ, L), is a non-empty, oriented, n-component link L in a compact, oriented, $\mathbb{Z}$-homology sphere Σ^3, having the property that all geometric pieces of the link exterior, M_L, are Seifert fibered. Fundamental groups of *Seifert* graph links are both global and local, see [5].

Question 3. Which graph links (besides Seifert links) have global fundamental groups?

2. Answers

2.1. **Graph links.** The approach to Questions 1 and 3 uses Theorem 1, together with the Eisenbud-Neumann calculus of Alexander polynomials from [5].

Graph links are classified by (minimal) *splice diagrams*. It will be assumed from now on that splice diagrams are connected, with non-zero edge weights (since this always happens for the diagram of a plane curve singularity link). Consequently, all geometric pieces of the corresponding link exterior are non-empty Seifert links.

Definition 1. A node of a splice diagram is *essential* if it has an adjacent arrowhead vertex in the diagram.

Theorem 2. *The Alexander polynomial of a minimal splice diagram has a single essential variable if and only if the diagram has at most one essential node.*

In conjunction with Theorem 1, this gives partial answers to Questions 1 and 3, as follows.

Theorem 3. *For any $n \geq 3$, there exist infinitely many non-isomorphic plane curve singularity groups, with $b_1 = n$, that are not global.*

Theorem 4. *Let (Σ, L) be a graph link such that all geometric pieces of M_L are Seifert links with at least 2 components, and $b_1(M_L) \geq 3$. Then the group $\pi_1(\Sigma \setminus L)$ is global if and only if the given link is Seifert.*

2.2. **Positive deficiency.** Since plane curve singularity groups have positive deficiency (i.e., they admit presentations with more generators than relators), the result below fully answers Question 2. The *resonance obstructions* to quasi-projectivity from [2] provide a key ingredient for the proof.

Theorem 5. *Let G be a 1-formal group with positive deficiency. If G is global, then $E_G \cong E_{\pi_1(M)}$, where M is either a smooth complex algebraic curve, or the complement of a central line arrangement in $\mathbb{C}^2$.*

Complete proofs of the results from Section 2 will appear elsewhere.

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Completely reducible fibers of a pencil of curves

Sergey Yuzvinsky

The main new result in this talk is the following theorem.

Theorem 1. *In a pencil of plane curves with irreducible generic fiber of degree $d > 1$ there are at most four completely reducible fibers. Moreover if the number k of completely reducible fibers is four then the pencil is linearly isomorphic to the Hesse pencil generated by the smooth cubic and its Hessian.*

Equivalently this result can be formulated in the following form.

Theorem 1'. *The maximal dimension of the first non-local resonance component of any hyperplane arrangement is three, i.e., the maximal dimension of the first cohomology of an Orlik-Solomon algebra on any non-local component is two. Moreover the only known example (of the Hesse arrangement) with the two-dimensional cohomology is unique up-to a linear isomorphism.*

The completely reducible fibers of a pencil of curves were considered in at least two classical papers: by G.Halphen (e.g., see Oeuvres de G.-H.Halphen, tome III, Gauthier-Villars, 1921, 1-260) and J.Hadamard ("Sur les conditions de decompositions des formes", Bull. SMF 27 (1899), 34-47). However the amazingly simple answer given by Theorem 1 seemed to stay unnoticed. Let us mention for comparison that for the number of reducible fibers the upper bounds are much more complicated and depend on d. The best known result was obtained by A.Vistoli (Invent. Math. 112 (1993)) and the newest one was obtained by Arnaud Bodin in "Reducibility of rational functions in several variables", math.NT/0510434.

For the speaker the inspiration came from the question answered by Theorem 1'. The cohomology algebras A of hyperplane arrangement complements in $\mathbb{C}^n$ are well known and the next important question is to study the 'secondary' cohomology, i.e., the cohomology of A under multiplication by an element of degree 1. The elements of degree one for which the first cohomology of A does not vanish form a subvariety (so called first resonance variety) in the affine space A^1 and the existence of an irreducible resonance component is equivalent to the arrangement containing the union of some completely reducible fibers of a pencil of hypersurfaces in $\mathbb{C}P^{n-1}$. Moreover the dimension of the component is $k-1$ where k is the number of the fibers and $\dim H^1(A,a) = k-2$ for every a from the component. The local components, i.e., the ones lying in a pencil of degree 1 are easy to study and their dimensions are not bounded uniformly. For non-local components only examples of dimensions two and three had been known.

The history of the results mentioned above is as follows. The equivalence of the two theorems was contained implicitly in a paper by Libgober and Yuzvinsky (2000) who also proved the inequality $k < 6$ for pencils with the maximal size of the base ($= d^2$), equivalently for the arrangements that support so called **nets**. This equivalence was proved explicitly by Falk and Y. (2006) who also proved that $k < 6$ for the pencils with reduced completely reducible fibers. In 2007, the inequality $k < 6$ was proved in full generality by Pereira and Y. They proved also that the pencils with $k > 2$ in any projective space are linear pullbacks from $\mathbb{C}P^4$ (similarly, for $k > 3$ from $\mathbb{C}P^2$). Then in 2007 the Dissertation of J.Stipins appeared where he proved for nets that $k < 5$ and that any arrangement with $k = 4$ is isomorphic to the Hesse one. Finally Yuzvinsky has generalized Stipins' proof to all pencils.

The answer to the problem 1 suggests many natural questions. For instance,

Question 1. What are the similar upper bounds for linear systems of higher dimensions?

Question 2. What are the similar bounds for curves of degree larger than 1 in the space of hypersurfaces (or homogeneous polynomials)?

Question 3. Do there exist pencils in $\mathbb{C}P^4$ with $k > 2$ that are not linear pullbacks of pencils from $\mathbb{C}P^3$?

In the mentioned above paper by Pereira and Y., there is an infinite family of pencils in $\mathbb{C}P^3$ with $k = 3$ that are not pullbacks from smaller projective spaces.

Resonance and zeros of logarithmic one-forms with hyperplane poles

Michael J. Falk

(joint work with Daniel C. Cohen, Graham Denham and Alexander Varchenko)

Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be an essential central arrangement of complex hyperplanes. Each H_i is the kernel of a linear functional $\alpha_i \colon \mathbb{C}^\ell \to \mathbb{C}$ well-defined up to multiplicative constant. Let $M = \mathbb{C}^\ell \setminus \bigcup_{i=1}^n H_i$, and $\omega_i = \frac{d\alpha_i}{\alpha_i}$, a 1-form on M with logarithmic poles along H_i. Let $A = A(\mathcal{A})$ be the $\mathbb{C}$-subalgebra of the (holomorphic) de Rham complex of M generated by $\{\omega_1, \ldots, \omega_n\}$. By a theorem of Orlik and Solomon, A is isomorphic to the cohomology ring $H^*(M, \mathbb{C})$. Note $A^1 \cong \mathbb{C}^n$.

For $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$, set $\omega_\lambda = \sum_{i=1}^n \lambda_i \omega_i \in A^1$. We are concerned with the following question: how does the zero locus $Z(\omega_\lambda) := \{x \in M \mid \omega_\lambda(x) = 0\}$ depend on λ? Note that $\omega_\lambda = d\log(\Phi_\lambda) = \frac{d\Phi_\lambda}{\Phi_\lambda}$ where $\Phi_\lambda = \prod_{i=1}^n \alpha_i^{\lambda_i}$ is the corresponding (generally multi-valued) *master function.* Thus $Z(\omega_\lambda)$ coincides with the critical locus $\operatorname{crit}(\Phi_\lambda)$ of Φ_λ. This is our original motiviation: for $\mathcal{A}$ a discriminantal arrangement (in the sense of [6]) and parameter vectors λ constructed from certain representations of $\mathfrak{sl}_n$, $\operatorname{crit}(\Phi_\lambda)$ gives rise to solutions of the $\mathfrak{sl}_n$ Knizhnik-Zamolodchikov equation via the Bethe Ansatz.

We always assume $\sum_{i=1}^n \lambda_i = 0$, so that ω_λ descends to a form on the projective image $\overline{M}$ of M. If λ is generic, then $\operatorname{crit}(\Phi_\lambda)$ consists of $\chi(\overline{M})$ isolated points [8, 5]. On the other hand, if $\mathcal{A}$ is a discriminantal arrangement, and certain integral weight vectors λ, $\operatorname{crit}(\Phi_\lambda)$ is a 1-dimensional (complex) subvariety of M [7]. Moreover, for those special weights ω_λ is a *resonant* 1-form [1], in a sense we now describe.

Left-multiplication by ω_λ makes the graded algebra A into a cochain complex. The cohomology of this complex is denoted $H^*(A, \omega_\lambda)$. This cohomology is approximates the cohomology of the local system with connection one-form ω_λ, and is isomorphic to the local system cohomology for most λ. If λ is generic, then $H^i(A, \omega_\lambda) = 0$ for $i < \ell - 1$, and $\dim H^{\ell-1}(A, \omega_\lambda) = \chi(\overline{M})$, which coincidentally matches the number of points in the discrete set $\mathrm{crit}(\Phi)$.

We say ω_λ is *resonant in degree p* if $H^i(A, \omega_\lambda)$ vanishes for $i < p$ and is nonzero for $i = p$. One might conjecture that $\mathrm{crit}(\Phi_\lambda)$ has codimension equal to p when ω_λ is resonant in degree p, and that the number of components equals $\dim H^p(A, \omega_\lambda)$. Both statements are false without some further hypotheses on ω_λ. For instacne we have the following result.

Theorem 1. *Suppose ω_λ is resonant in degree p. Suppose ω_λ possesses a nontrivial decomposable p-cocycle, i.e., there exist $\lambda_1, \ldots, \lambda_p$ such that $\omega_\lambda \wedge (\omega_{\lambda_1} \wedge \ldots \wedge \omega_{\lambda_p}) = 0$, and $\omega_{\lambda_1} \wedge \ldots \wedge \omega_{\lambda_p}$ is not a multiple of ω_λ. Then, for generic λ' in the $\mathbb{C}$-span of $\{\lambda, \lambda_1, \ldots, \lambda_p\}$, $\mathrm{crit}(\Phi_{\lambda'})$ has codimension p. Moreover, components of $\mathrm{crit}(\Phi_{\lambda'})$ are intersections of level sets of $\Phi_\lambda, \Phi_{\eta_1}, \ldots, \Phi_{\lambda_p}$.*

This theorem is proved by considering the map $\Phi = (\Phi_\lambda, \Phi_{\lambda_1}, \ldots, \Phi_{\lambda_p})$. The hypothesis implies the image Σ of Φ has codimension one in $\mathbb{C}^{p+1}$. In other words, the rational functions $\Phi_\lambda, \Phi_{\lambda_1}, \ldots, \Phi_{\lambda_p}$ satisfy a nontrivial algebraic dependence relation. A generic form $\sigma = \sum_{i=0}^p a_i \frac{dx_i}{x_i}$, restricted to Σ, will have isolated zeros that miss the singularities of Σ. The preimage of $Z(\sigma|_\Sigma)$ is the zero set of $\omega_{\lambda'}$, where $\lambda' = a_0\lambda + \sum_{i=1}^p a_i\lambda_i$, and it has the claimed properties. The genericity conditions can be written explicitly in terms of a defining equation for Σ. In particular, if Σ is known, one can check whether λ itself satsifies the conditions.

In related work we have shown a similar result, for arbitrary ω_λ resonant in degree p, if $\mathcal{A}$ is a free arrangement, though the conclusion is weaker: the codimension of $\hat{Z}(\omega_\lambda) := \{x \in \mathbb{C}^\ell \mid \omega_\lambda(x) = 0\}$ is at most p. But it may be the case that $Z(\omega_\lambda) = \hat{Z}(\omega_\lambda) \cap M = \emptyset$. See the talk by G. Denham in this workshop.

In case $p = 1$, one can use the results of [3] to draw more precise conclusions.

Theorem 2. *If ω_λ is resonant in degree 1, then, with finitely many exceptions, $\mathrm{crit}(\Phi_\lambda)$ has codimension one and has $\dim H^1(A, \omega_\lambda)$ components.*

The finitely many exceptions in the above statement can be explicitly identified.

The proof of the theorem above indicates the importance of algebraic dependence relations among master functions. One can approach this question using the methods of tropical algebraic geometry [2]. The function Φ above can be factored into the linear map $\alpha = (\alpha_1, \ldots, \alpha_n)\colon \mathbb{C}^\ell \to \mathbb{C}^n$ followed by a monomial map $m\colon \mathbb{C}^n \to \mathbb{C}^{p+1}; x \mapsto (x^\lambda, x^{\lambda_1}, \ldots, x^{\lambda_p})$, in the usual vector notation for monomials. The image Σ of Φ has codimension one precisely when its tropicalization $\tau(\Sigma)$, a polyhedral fan, has codimension one in $\mathbb{R}^{p+1}$. But the tropicalizations of the factors of Φ are well-understood. The image of α tropicalizes to the Bergman fan of $\mathcal{A}$, and the tropicalization of m is the linear map given by the matrix of weights $\lambda, \lambda_1 \ldots \lambda_p$. By [4], the Bergman fan of $\mathcal{A}$ is subdivided by the nested set fan of $\mathcal{A}$, whose cones are generated by incidence vectors of certain flags in the intersection lattice of $\mathcal{A}$. Using this one can write rank conditions that guarantee that the tropical variety $\tau(\Sigma)$ has positive codimension. Whether this tropical approach can yield more definitive results on $\mathrm{crit}(\Phi_\lambda)$ remains a subject of current research.

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Zeroes of 1-forms and Resonance of Free Arrangements

Graham Denham

(joint work with D. Cohen, M. Falk, A. Varchenko)

Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be an arrangement of hyperplanes in $\mathbb{C}^\ell$, with complement $M = M(\mathcal{A}) = \mathbb{C}^\ell \setminus \bigcup_{j=1}^n H_j$. Fix coordinates $\mathbf{x} = (x_1, \ldots, x_\ell)$ on $\mathbb{C}^\ell$, and for each hyperplane H_j of $\mathcal{A}$, let f_j be a linear polynomial for which $H_j = \{\mathbf{x} \colon f_j(\mathbf{x}) = 0\}$. A collection $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ of complex weights with $\sum_{i=1}^n \lambda_i = 0$ determines a *master function*

$$\Phi_\lambda = \prod_{j=1}^n f_j^{\lambda_j}, \tag{1}$$

a multi-valued holomorphic function with zeros and poles on the variety $\bigcup_{j=1}^n H_j$ defined by $\mathcal{A}$. The master function Φ_λ determines a one-form

$$\omega_\lambda = \mathrm{d}\log\Phi_\lambda = \sum_{j=1}^n \lambda_j \frac{\mathrm{d}f_j}{f_j} \tag{2}$$

in the Orlik-Solomon algebra $A = A(\mathcal{A}) \cong H^\cdot(M; \mathbb{C})$.

Let $\Sigma_\lambda = V(\omega_\lambda) \subset M$, the variety defined by the vanishing of the one-form ω_λ or, equivalently, by the critical equations of the master function Φ_λ on M. For certain arrangements, this variety is of interest in mathematical physics: the

Bethe ansatz equations in the Gaudin model associated with a complex simple Lie algebra $\mathfrak{g}$ are critical equations of a suitable master function: see [5, 8].

In [7], Varchenko showed that, for generic weights λ and real equations $\{f_i\}$, the master function is a cone over $|\chi(\mathbb{P}(M))|$ isolated critical points. The generalization to arbitrary complex arrangements was proven by Orlik and Terao [3].

On the other hand, a one-form $\omega_\lambda \in A^1$ is said to be resonant (in dimension p) if the pth cohomology group of the complex

$$(A, \omega_\lambda) = A^0 \stackrel{\omega_\lambda}{\to} A^1 \stackrel{\omega_\lambda}{\to} A^2 \cdots \stackrel{\omega_\lambda}{\to} A^\ell$$

is nonzero, where the differential is given by multiplication by ω_λ. Yuzvinsky showed in [9] that, for generic λ, the one-form ω_λ is resonant only in dimensions $\ell-1, \ell$. This and further work in [4, 1, 6] leads to the (elegant, but easily falsified) conjecture that the codimension of the variety Σ_λ equals the least dimension p for which ω_λ is resonant.

Instead, let $R = \mathbb{C}[\mathbf{x}]$, and let $\mathrm{Der}(\mathcal{A})$ denote the R-module of logarithmic derivations. The arrangement $\mathcal{A}$ is said to be free if $\mathrm{Der}(\mathcal{A})$ is free as an R-module. Such arrangements have an extensive literature. Let $I_\lambda = \langle \mathrm{Der}(\mathcal{A}), \omega_\lambda \rangle$, the ideal of R given by evaluating logarithmic derivations on ω_λ via the canonical (duality) pairing. Let $\overline{\Sigma}_\lambda = V(I_\lambda)$, an affine variety. Clearly $\overline{\Sigma}_\lambda \cap M = \Sigma_\lambda$. Then our main result is the following:

Theorem 1. *If $\mathcal{A}$ is a free arrangement, and p is the least integer for which a one-form $\omega_\lambda \in A^1$ is resonant, then the variety $\overline{\Sigma}_\lambda$ has codimension p.*

On the other hand, we see the converse is false: for this, consider the master function

$$\Phi_\lambda = \frac{x^2(y^2 - z^2)}{y^2(x^2 - z^2)}$$

on the "deleted B_3-arrangement" with hyperplanes

$$\{x, y, x-y, x+y, x-z, x+z, y-z, y+z\}.$$

The arrangement is free, and λ is generic (that is, ω_λ is resonant only for dimensions $2, 3$); however, Σ_λ and $\overline{\Sigma}_\lambda$ have codimension 1: the latter is the union of hyperplanes $z = 0, x = y, x = -y$, the first of which is not in the arrangement. This example has very special properties (a positive-dimensional component in a characteristic variety, discovered by Alex Suciu: see [2]), although the conceptual relationship with the problem considered here remains to be worked out.

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Arrangements of hypersurfaces and Bestvina-Brady groups

Daniel Matei

(joint work with Enrique Artal Bartolo and Jose Ignacio Cogolludo)

We show that quasi-projective Bestvina-Brady groups are fundamental groups of complements to hyperplane arrangements. We thus obtain examples of hypersurface complements whose fundamental groups satisfy various finiteness properties.

In a series of papers [5] from 1960's C.T.C. Wall studied general finiteness properties of groups and CW-complexes. A group G is said to be of type F_n if it has an Eilenberg-MacLane complex $K(G,1)$ with finite n-skeleton. Clearly G is finitely generated if and only if it is F_1 and finitely presented iff it is F_2. An interesting example of a finitely presented group which is not finitely presented was given by Stallings in [4].

A group G is said to be of type FP_n if the ZG-module Z admits a projective resolution which is finitely generated in dimensions $\leq n$. If X is a $K(G,1)$ complex, then the action of G on the universal cover $\widetilde{X}^{(n)}$ induces such a resolution, hence the property F_n implies the FP_n property. Note that G is of type FP_1 if and only if it is finitely generated, and that G is of type FP_2 if it is finitely presented. But, as shown by Bestvina and Brady [1] FP_2 does not imply finite presentation. The first example of a group which is finitely presented but not of type FP_3 was given by Stallings in [4]. Afterwards Bieri [2] generalized Stallings' examples to the following family: Let $G_n = \mathbb{F}_2 \times \cdots \times \mathbb{F}_2$ be the direct product of n free groups, each of rank 2. Then the kernel of the map taking each generator to $1 \in \mathbb{Z}$ is F_{n-1} but not F_n. Stallings' examples mentioned above are the cases $n = 2$ and $n = 3$.

Let Γ be a finite simplicial graph and Δ the flag simplicial complex it generates. If $1, \dots, s$ are the vertices of Γ, the *right-angled Artin group* A_Γ associated to Γ is the group with generators $\sigma_1, \dots, \sigma_s$ and relations $\sigma_i\sigma_j = \sigma_j\sigma_i$, one for each edge ij of Γ. For example if $\Gamma = K_{n_1,\dots,n_r}$ the complete multipartite graph, then the right-angled Artin group A_Γ is the product of free groups $\mathbb{F}_{n_1} \times \cdots \times \mathbb{F}_{n_r}$. Thus the Stallings-Bieri examples are Bestvina-Brady kernels.

The *Bestvina-Brady group* associated to Γ is the kernel N_Γ of the homomorphism $A_\Gamma \to \mathbb{Z}$ that sends each σ_i to 1. The group N_Γ is finitely presented if and only if Δ is simply-connected. Furthermore, Bestvina and Brady show in [1] that N_Γ is FP_n if and only Δ is $(n-1)$-acyclic.

In [3] all the quasi-projective Bestvina-Brady groups N_Γ are determined. The graph Γ is either a tree, or a complete multipartite graph $K_{n_1,\dots,n_r}$ with either some $n_i = 1$ or all $n_i \geq 2$ and $r \geq 3$. The class of groups corresponding to these graphs consists of the following two distinct types:

(1) a product of free groups $\mathbb{Z}^r \times \mathbb{F}_{n_1} \times \cdots \times \mathbb{F}_{n_s}$, with $r \geq 0, s \geq 0$ and all $n_i \geq 2$;
(2) $N_{K_{n_1,\dots,n_r}}$, with all $n_i \geq 2$ and $r \geq 3$.

We will show that all these groups can be realized by fundamental groups of complements to line arrangements. It is not hard to realize $\mathbb{Z}^r \times \mathbb{F}_{n_1} \times \cdots \times \mathbb{F}_{n_s}$ as the fundamental group of a complement to a line arrangement: Consider s distinct directions in $\mathbb{C}^2$ and take n_i lines parallel to the i^{th} direction, and r other lines in general position. So, it remains to deal with $N_{K_{n_1,\dots,n_r}}$.

Theorem 1. *Any quasi-projective Bestvina-Brady group N_Γ is an arrangement group. More precisely, if $\mathcal{A}_{n_1,\dots,n_r}$ is the arrangement of $n = n_1 + \cdots + n_r$ lines in $\mathbb{P}^2$ consisting of the r sides of an r-gon together with $n_i - 1$ lines in a pencil at vertex i of the polygon, for all i, such that these $n - r$ lines intersect generically away from the r-gon, then N_Γ is isomorphic to $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_{n_1,\dots,n_r})$.*

It follows from Bestvina and Brady [1] that $N_{K_{n_1,\dots,n_r}}$ is F_{r-1} but not FP_r, and so the arrangement groups $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_{n_1,\dots,n_r})$ enjoy the same finiteness properties.

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The equivariant spectral sequence and cohomology with local coefficients

Alexander I. Suciu
(joint work with Stefan Papadima)

In his pioneering work from the late 1940s, J.H.C. Whitehead established the category of CW-complexes as the natural framework for much of homotopy theory. A key role in this theory is played by the cellular chain complex of the universal cover of a connected CW-complex, which in turn is tightly connected to (co-)homology with local coefficients. In [8], we revisit these classical topics, drawing much of the motivation from recent work on the topology of complements of complex hyperplane arrangements, and the study of cohomology jumping loci.

A spectral sequence. Let X be a connected CW-complex, π its fundamental group, and $\Bbbk\pi$ the group ring over a coefficient ring $\Bbbk$. The cellular chain complex of the universal cover, $C_\bullet(\widetilde{X}, \Bbbk)$, is a chain complex of left $\Bbbk\pi$-modules, and so it is filtered by the powers of the augmentation ideal. We investigate the spectral sequence associated to this filtration, with coefficients in an arbitrary right $\Bbbk\pi$-module M. To start with, we identify the d^1 differential.

Theorem 2. *There is a second-quadrant spectral sequence, $\{E^r(X, M), d^r\}_{r\geq 1}$, with $E^1_{-p,p+q}(X, M) = H_q(X, \operatorname{gr}^p(M))$. If $\Bbbk$ is a field, or $\Bbbk = \mathbb{Z}$ and $H_*(X, \mathbb{Z})$ is torsion-free, then $E^1_{-p,p+q}(X, M) = \operatorname{gr}^p(M) \otimes_\Bbbk H_q(X, \Bbbk)$, and the d^1 differential decomposes as*

$$\begin{array}{ccc} \operatorname{gr}^p(M)\otimes_\Bbbk H_q \xrightarrow{\ \mathrm{id}\otimes\nabla_X\ } & \operatorname{gr}^p(M)\otimes_\Bbbk (H_1\otimes_\Bbbk H_{q-1}) & \\ & \big\downarrow\cong & \\ & (\operatorname{gr}^p(M)\otimes_\Bbbk \operatorname{gr}^1(\Bbbk\pi))\otimes_\Bbbk H_{q-1} & \xrightarrow{\ \operatorname{gr}(\mu_M)\otimes\mathrm{id}\ } \operatorname{gr}^{p+1}(M)\otimes_\Bbbk H_{q-1}\,, \end{array}$$

where ∇_X is the comultiplication map on $H_* = H_*(X, \Bbbk)$, and $\mu_M \colon M \otimes_\Bbbk \Bbbk\pi \to M$ is the multiplication map of the module M.

Under fairly general assumptions, $E^\bullet(X, M)$ has an E^∞ term. In general, though, $E^\bullet(X, \Bbbk\pi)$ does not converge, even if X has only finitely many cells.

Base change. To obtain more structure in the spectral sequence, we restrict to a special situation. Suppose $\nu\colon \pi \twoheadrightarrow G$ is an epimorphism onto a group G; then the group ring $\Bbbk G$ becomes a right $\Bbbk\pi$-module, via extension of scalars. The resulting spectral sequence, $E^\bullet(X, \Bbbk G_\nu)$, is a spectral sequence in the category of left $\operatorname{gr}_J(\Bbbk G)$-modules, where J is the augmentation ideal of $\Bbbk G$.

Now let G be an abelian group. Assuming X is of finite type and $\Bbbk$ is a field, the spectral sequence $E^\bullet(X, \Bbbk G_\nu)$ does converge, and computes the J-adic completion of $H_*(X, \Bbbk G_\nu) = H_*(Y, \Bbbk)$, where $Y \to X$ is the Galois G-cover defined by ν. As a particular case, we recover in dual form a result of A. Reznikov [9] on the mod p cohomology of cyclic p-covers of aspherical complexes.

Monodromy action. Let X be a connected, finite-type CW-complex. Suppose $\nu\colon \pi_1(X) \twoheadrightarrow \mathbb{Z}$ is an epimorphism, and $\Bbbk$ is a field. Let $(H^*(X, \Bbbk), \cdot\nu_\Bbbk)$ be the cochain complex defined by left-multiplication by $\nu_\Bbbk \in H^1(X, \Bbbk)$, the cohomology class corresponding to ν.

Theorem 3. *For each $q \geq 0$, the $\operatorname{gr}_J(\Bbbk\mathbb{Z})$-module structure on $E^\infty(X, \Bbbk\mathbb{Z}_\nu)$ determines P^q_0 and P^q_{t-1}, the free and $(t-1)$-primary parts of $H_q(X, \Bbbk\mathbb{Z}_\nu)$, viewed as a module over $\Bbbk\mathbb{Z} = \Bbbk[t^{\pm 1}]$. Moreover, the monodromy action of $\mathbb{Z}$ on $P^j_0 \oplus P^j_{t-1}$ is trivial for all $j \leq q$ if and only if $H^j(H^*(X, \Bbbk), \cdot\nu_\Bbbk) = 0$, for all $j \leq q$.*

Particularly interesting is the case of a smooth manifold X fibering over the circle, with $\nu = p_*\colon \pi \twoheadrightarrow \mathbb{Z}$ the homomorphism induced by the projection map, $p\colon X \to S^1$. The homology of the resulting infinite cyclic cover was studied by J. Milnor in [7]. This led to another spectral sequence, introduced by M. Farber, and further developed by S.P. Novikov, see [6]. The Farber-Novikov spectral sequence has (E_1, d_1)-page dual to our $(E^1(X, \Bbbk\mathbb{Z}_\nu), d^1_\nu)$-page, and higher differentials given by certain Massey products. Their spectral sequence, though, converges to the free part of $H_*(X, \Bbbk\mathbb{Z}_\nu)$, and thus misses the information on the $(t-1)$-primary part captured by the equivariant spectral sequence.

Formality and Jordan blocks. As an application of our machinery, we develop a new 1-formality obstruction for groups, based on the interplay of two ingredients: the connection between the formality property (in the sense of D. Sullivan) and the cohomology jumping loci, established in [4], and the connection between the monodromy action and the Aomoto complex, established in Theorem 3.

Theorem 4. *Let N be the kernel of an epimorphism $\nu\colon \pi \twoheadrightarrow \mathbb{Z}$. Suppose π is 1-formal, and $b_1(N, \mathbb{C}) < \infty$. Then the eigenvalue 1 of the monodromy action of $\mathbb{Z}$ on $H_1(N, \mathbb{C})$ has only 1×1 Jordan blocks.*

Given a reduced polynomial function $f\colon (\mathbb{C}^2, \mathbf{0}) \to (\mathbb{C}, 0)$, there are two standard fibrations associated with it. The above result helps explain the radically different properties of these two fibrations.

- The Milnor fibration, $S^3_\epsilon \setminus K \to S^1$, has total space the complement of the link at the origin. As shown in [5], this space is formal. Theorem 4 allows us then to recover the well-known fact that the algebraic monodromy has no Jordan blocks of size greater than 1 for the eigenvalue $\lambda = 1$.
- The fibration $f^{-1}(D^*_\epsilon) \to D^*_\epsilon$ is obtained by restricting f to the preimage of a small punctured disk around 0. As pointed out by Alex Dimca at the Oberwolfach Mini-Workshop, the algebraic monodromy of this fibration can have larger Jordan blocks for $\lambda = 1$, see [1]. In such a situation, the total space, $f^{-1}(D^*_\epsilon)$, is non-formal, by Theorem 4.

Bounds on twisted cohomology ranks. Our approach yields readily computable upper bounds on the ranks of the cohomology groups of a space, with coefficients in a prime-power order, rank one local system.

Theorem 5. *Let X be a connected, finite-type CW-complex, and let $\rho\colon \pi_1(X) \to \mathbb{C}^\times$ be a character given by $\rho(g) = \zeta^{\nu(g)}$, where $\nu\colon \pi \to \mathbb{Z}$ is a homomorphism, and ζ is a root of unity of order a power of a prime p. Then, for all $q \geq 0$,*

$$\dim_{\mathbb{C}} H^q(X, {}_\rho\mathbb{C}) \leq \dim_{\mathbb{F}_p} H^q(X, \mathbb{F}_p).$$

If, moreover, $H_*(X, \mathbb{Z})$ is torsion-free,

$$\dim_{\mathbb{C}} H^q(X, {}_\rho\mathbb{C}) \leq \dim_{\mathbb{F}_p} H^q(H^*(X, \mathbb{F}_p), \nu_{\mathbb{F}_p}).$$

Neither of these inequalities can be sharpened further. Indeed, we give examples showing that both the prime-power hypothesis on the order of ζ, and the torsion-free hypothesis on $H_*(X, \mathbb{Z})$ are really necessary. The second inequality above generalizes a result of D. Cohen and P. Orlik [2], valid only for complements of complex hyperplane arrangements.

Minimality and linearization. Suppose now X has a *minimal* cell structure, i.e., the number of q-cells of X coincides with the (rational) Betti number $b_q(X)$, for every $q \geq 0$; in particular, $H_*(X, \mathbb{Z})$ is torsion-free. Let $\Bbbk = \mathbb{Z}$, or a field. Pick a basis $\{e_1, \dots, e_n\}$ for $H_1 = H_1(X, \Bbbk)$, and identify the symmetric algebra on H_1 with the polynomial ring $S = \Bbbk[e_1, \dots, e_n]$.

Theorem 6. *Under the above assumptions, the linearization of the equivariant cochain complex of the universal abelian cover of X coincides with the universal Aomoto complex, $(H^*(X, \Bbbk) \otimes_{\Bbbk} S, D)$, with differentials $D(\alpha \otimes 1) = \sum_{i=1}^{n} e_i^* \cdot \alpha \otimes e_i$.*

This theorem generalizes results from [2] and [3], and answers a question posed by M. Yoshinaga in [10].

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Floer-Novikov homology in the cotangent bundle

Mihai Damian

Let M^n be a closed connected manifold and T^*M its cotangent bundle endowed with the standard symplectic structure $\omega_M = d\lambda_M$, where λ_M is the Liouville form $\lambda_M = \sum_i p_i \, dq_i$. Let $L^n \hookrightarrow T^*M$ be an exact Lagrangian submanifold, i.e. a submanifold of maximal dimension such that $\lambda_M | L$ is an exact 1-form. The only known examples of exact Lagrangian submanifolds are the graphs of functions $f : M \to \mathbb{R}$

$$L_f := \{(q, df_q) \,|\, q \in M\}$$

and their images by Hamiltonian vector flows. The question of the existence of other examples was first evoked by V.I. Arnold in his survey "First steps in symplectic topology" [1]. It is far from being solved. A positive answer was given by R. Hind in the case $L = M = \mathbf{S}^2$. The other related results which were proved up to now are topological obstructions to the existence of exact Lagrangian embeddings $L \hookrightarrow T^*M$. We summarize them in the statement below :

Theorem 0. *Let M be a closed manifold and $L \hookrightarrow T^*M$ an exact Lagrangian embedding of a closed manifold L. Denote by p the projection of L on the base space M. Then we have :*
a) If L and M are orientable, then $\chi(L) = deg^2(p)\chi(M)$. If L and M are not orientable the same equality is valid modulo 4.
b) The index $[\pi_1(M) : p_*(\pi_1(L))]$ is finite.
c) If M is simply connected then L can not be aspherical (i.e. Eilenberg-Mac Lane).
d) If M is simply connected and L is spin with vanishing Maslov class, then $H^*(L, K) \approx H^*(M, K)$, where K is an arbitrary field of non-zero characteristic.

The statement 0.a was proved by M. Audin in [2], 0.b was proved by F. Lalonde and J-C Sikorav in [5] and 0.c is a result of C. Viterbo [9] (see also [8]) . More recently, 0.d was proved independently by K. Fukaya, P. Seidel and I. Smith [4] and D. Nadler [6]. For $M = \mathbf{S}^n$ and L simply connected this was proved previously by P. Seidel [7] and by L. Buhovsky [3].

The aim of this talk is to present other obstructions which are obtained in the case where M is a total space of a fibration over the circle, by means of a non-Hamiltonian version of the Floer homology theory. Our main results are :

Theorem 1. *Let $M^{n\geq 3}$ be a closed manifold which is the total space of a fibration over $\mathbf{S}^1$ and let $L \hookrightarrow T^*M$ be an exact Lagrangian embedding of a closed manifold L. Then we have :*
a) Let $< g_1, g_2, \ldots, g_p \,|\, r_1, r_2, \ldots, r_q >$ be an arbitrary presentation of the fundamental group $\pi_1(L)$. Then $p - q \leq 1$.
b) The fundamental group $\pi_1(L)$ is not isomorphic to the free product $G_1 * G_2$ of two non-trivial (finitely presented) groups.

Here are some examples of non-embedding statements which can be inferred from this theorem :

Corollary 1. *Let P, Q, L be closed manifolds and suppose that P is simply connected (or more generally that $\pi_1(P)$ is finite).*
a) Suppose that $\chi(L) \neq 0$. Then there is no exact Lagrangian embedding $L \times P \hookrightarrow T^*(Q \times \mathbf{S}^1)$.

In particular, let Σ_g be a (non necessary orientable) surface of genus $g \geq 2$. Then there is no exact Lagrangian embedding of $\Sigma_g \times P$ into $T^*(Q \times \mathbf{S}^1)$. More *generally, for surfaces* Σ_{g_i} *as above there is no exact Lagrangian embedding*

$$\Sigma_{g_1} \times \Sigma_{g_2} \times \cdots \times \Sigma_{g_k} \times P \hookrightarrow T^*(Q \times \mathbf{S}^1).$$

b) Let $L^{n \geq 4}$ *be the connected sum* $L_1 \# L_2$ *of two manifolds which are not homeomorphic to the* n*-sphere. Then there is no exact Lagrangian embedding* $L \times P \hookrightarrow T^*(Q \times \mathbf{S}^1)$.
c) Suppose that there is an exact Lagrangian embedding

$$L \times T^l \hookrightarrow T^*(T^m \times Q),$$

where T^k *is the* k*-dimensional torus and* $m > l$*. Then* L *satisfies the conditions a, b of Theorem 1.*

Idea of the proof
Let $f : M \to \mathbf{S}^1$ be a fibration. The closed 1-form $\alpha = f^* d\theta$ has no zeroes. Let L be an exact embedding into T^*M. Consider the Lagrangian isotopy

$$L_t \,=\, L + t\alpha.$$

It follows that $L_t \cap L = \emptyset$ for t large enough. The Lagrangian manifolds L_t are not exact but they satisfy

$$\omega_M|_{\pi_2(T^*M, L_t)} = 0$$

just like an exact Lagrangian manifold. Under this hypothesis one can define a Floer-type complex $C_\bullet(L, L_t)$, which is spanned by the intersection points $L \cap L_t$. Therefore, this complex vanishes for $t >> 0$.

We then compute the homology of this complex and we show that it is isomorphic to the Novikov homology $H_*(L, p^*u)$, where $u \in H^1(M, \mathbb{Z})$ is the cohomology class of α and $p : L \to M$ is the projection. In particular it is independent of t. It follows :

Theorem 2. $H_*(L, p^*u) = 0$

In order to prove Theorem 1 one has to argue in the following way : suppose that 1.a is false. Then one can show that the Novikov homology $H_*(L, v)$ does not vanish for any $v \in H^1(L, \mathbb{R})$, contradicting thus Theorem 2. A similar argument works for the proof of 1.b.

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Refinement of the Ray-Singer Torsion

Maxim Braverman
(joint work with Thomas Kappeler)

We construct a canonical element, called the *refined analytic torsion*, of the determinant line of the cohomology of a closed oriented odd-dimensional manifold M with coefficients in a flat complex vector bundle E, which depends holomorphically on the flat connection. It encodes the information about both, the Ray-Singer η-invariant of the Atiyah-Patodi-Singer odd signature operator. In particular, when the bundle E is acyclic, the refined analytic torsion is a non-zero complex number, whose absolute value is equal (up to an explicit correction term) to the Ray-Singer torsion and whose phase is expressed in terms of the η-invariant. The fact that the Ray-Singer torsion and the η-invariant can be combined into one holomorphic function allows to use the methods of complex analysis to study both invariants. We present several applications of these methods. In particular, we compute the ratio of the refined analytic torsion and the Farber-Turaev refinement of the combinatorial torsion.

Definition of the refined analytic torsion. For $\alpha \in \mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$ we denote by E_α the flat vector bundle over M whose monodromy is equal to α. Let ∇_α be the flat connection on E_α. We defined a canonical non-zero element

$$\rho_{\mathrm{an}}(\alpha) \;=\; \rho_{\mathrm{an}}(\nabla_\alpha) \;\in\; \mathrm{Det}\big(H^\bullet(M, E_\alpha)\big),$$

called the *refined analytic torsion*, of the determinant line $\mathrm{Det}\big(H^\bullet(M, E_\alpha)\big)$ of the cohomology $H^\bullet(M, E_\alpha)$ of M with coefficients in E_α. The construction is based on the study of the graded determinant of the Atiyah-Patodi-Singer odd signature operator. If the representation α is not unitary, this operator is not self-adjoint. To carry out the construction of the refined analytic torsion we proved several new results about determinants of non-self-adjoint operators, which have an independent interest.

Analyticity of the refined analytic torsion. The disjoint union of the lines $\mathrm{Det}\big(H^\bullet(M, E_\alpha)\big)$, $(\alpha \in \mathrm{Rep}(\pi_1(M), \mathbb{C}^n))$, forms a line bundle

$$\mathcal{D}et \to \mathrm{Rep}(\pi_1(M), \mathbb{C}^n),$$

called the *determinant line bundle*. It admits a nowhere vanishing section, given by the Farber-Turaev torsion, and, hence, has a natural structure of a trivializable holomorphic bundle.

We prove that $\rho_{\mathrm{an}}(\alpha)$ is a nowhere vanishing *holomorphic* section of the bundle $\mathcal{D}et$. It means that the ratio of the refined analytic and the Farber-Turaev torsions is a holomorphic function on $\mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$. For an acyclic representation α, the determinant line $\mathrm{Det}\,\big(H^\bullet(M, E_\alpha)\big)$ is canonically isomorphic to $\mathbb{C}$ and $\rho_{\mathrm{an}}(\alpha)$ can be viewed as a non-zero complex number. We show that $\rho_{\mathrm{an}}(\alpha)$ is a holomorphic function on the open set $\mathrm{Rep}_0(\pi_1(M), \mathbb{C}^n) \subset \mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$ of acyclic representations.

Recently, Burghelea and Haller [5, 6] constructed another holomorphic function on the space of acyclic representations, whose absolute value is related to the Ray-Singer torsion. Their function is different from ours and is not related to the η-invariant. In [4], we show that the Burghelea-Haller torsion can be computed in terms of the refined analytic torsion.

Comparison with the Farber-Turaev torsion. In [9, 10], Turaev constructed a refined version of the combinatorial torsion associated to a representation α, which depends on additional combinatorial data, denoted by ϵ and called the *Euler structure*, as well as on the *cohomological orientation* of M, i.e., on the orientation $\mathfrak{o}$ of the determinant line of the cohomology $H^\bullet(M, \mathbb{R})$ of M. In [8], the Turaev torsion was redefined as a non-zero element $\rho_{\epsilon,\mathfrak{o}}(\alpha)$ of the determinant line $\mathrm{Det}\,\big(H^\bullet(M, E_\alpha)\big)$.

One of our main results states that, for each connected component $\mathcal{C}$ of the space $\mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$, there exists a constant $\theta \in \mathbb{R}$, such that

$$\frac{\rho_{\mathrm{an}}(\alpha)}{\rho_{\epsilon,\mathfrak{o}}(\alpha)} = e^{i\theta} \cdot f_{\epsilon,\mathfrak{o}}(\alpha), \tag{1}$$

where $f_{\epsilon,\mathfrak{o}}(\alpha)$ is a holomorphic function of $\alpha \in \mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$, given by an explicit local expression.

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Problem Session

ALL PARTICIPANTS

During the workshop, we had problem sessions to discuss various questions in the fields. Many of these can be found in the abstracts above, but the remaining ones are collected here.

BNS invariants and resonance

Problem 1 (A. Suciu)**.** What is the relationship between the Bieri-Neumann-Strebel invariants and the resonance varieties of a finitely generated group G?

Here is a more precise formulation. Pick a finite generating set for G, and let $\mathcal{C}(G)$ be the corresponding Cayley graph. Given an additive real character $\chi\colon G \to \mathbb{R}$, let $\mathcal{C}_\chi(G)$ be the full subgraph on vertex set $\{g \in G \mid \chi(g) \geq 0\}$. In [2], Bieri, Neumann, and Strebel define the (first) BNS invariant of G to be:

$$\Sigma^1(G) = \{\chi \in \operatorname{Hom}(G, \mathbb{R}) \setminus \{0\} \mid \mathcal{C}_\chi(G) \text{ is connected}\}.$$

Clearly, $\Sigma^1(G)$ is a conical subset of the vector space $\operatorname{Hom}(G, \mathbb{R}) = H^1(G; \mathbb{R})$. It turns out that $\Sigma^1(G)$ does not depend on the choice of generating set for G.

Problem 2 (A. Suciu)**.** Give conditions on G such that

$$\Sigma^1(G) \cup -\Sigma^1(G) = H^1(G; \mathbb{R}) \setminus R^1_1(G; \mathbb{R}). \tag{1}$$

In the case when G is a right-angled Artin group equality (1) holds: the left hand side was computed by Meier-VanWyk [10], the right hand side by Papadima-Suciu [12], and both agree. Still one should ask for a conceptual reason of this equality in greater generality.

One has to keep in mind that (1) cannot hold in full generality.

Topological complexity

The numerical invariant $\mathbf{TC}(X)$ measures navigational complexity of a topological space X viewed as the configuration space of a mechanical system, see [6].

Problem 3 (M. Farber)**.** If $F \to E \to B$ is a locally trivial fibration, is it true that

$$\mathbf{TC}(E) \leq \mathbf{TC}(F) \cdot \mathbf{TC}(B)?$$

It is shown by M. Farber and M. Grant in [8] that

$$\mathbf{TC}(E) \leq \mathbf{TC}(F) \cdot \operatorname{cat}(B \times B).$$

but there may well be counterexamples to Question 3.

Problem 4 (M. Farber)**.** Let N_g be the non-orientable surface of genus g. What is $\mathbf{TC}(N_g)$?

It is known that $\mathbf{TC}(N_1) = 4$ (Farber, Tabachnikov and Yuzvinsky [7]) and $\mathbf{TC}(N_2) = 4$ (A. Costa) but for $g > 2$ the answer is still unknown.

Consider polygon spaces: Let $\ell = (l_1, \dots, l_n)$, where each $l_i > 0$. Look at all closed planar polygons with given side lengths up to rotation and translation, and call this space M_ℓ. Let N_ℓ be the spatial ($\mathbb{R}^3$) n-gons with the same condition. For generic ℓ, M_ℓ is a compact manifold of dimension $n-3$ and N_ℓ a compact manifold of dimension $2(n-3)$.

Problem 5 (M. Farber)**.** What is $\mathbf{TC}(M_\ell)$?

It is known that $\mathbf{TC}(N_\ell) = 2(n-3)+1$ which uses a symplectic structure and simple-conectedness, but for M_ℓ it depends on the length vector ℓ.

By normalizing one can assume that $l_1 + \cdots + l_n = 1$, so that $\ell \in \Delta^{n-1}$.

Problem 6. Compute $\int_{\Delta^{n-1}} \mathbf{TC}(M_\ell)\, d\mu(\ell)$.

Problem 7 (A. Suciu)**.** Let M be a closed 3-manifold. Does $\mathbf{TC}(M)$ depend only on $\pi_1(M)$? In particular, is $\mathbf{TC}(L(p,q)) = \mathbf{TC}(L(p,1))$?

As shown in [9], the LS category of M depends only on $\pi_1(M)$: it is 2, 3, or 4, according to whether $\pi_1(M)$ is trivial, a non-trivial free group, or not a free group.

Almost direct products of groups

A split extension of groups, $G = B \rtimes A$, is called an almost direct product if A acts on B by automorphisms inducing the identity on $H_*(B)$.

Problem 8 (D. Cohen)**.** For $G = F_m \rtimes F_n$ an almost direct product of free groups, when is G 1-formal?

Let $P\Sigma_n$ be the group of basis-conjugating automorphisms of the free group F_n.

Problem 9 (D. Cohen)**.** Let $G = P\Sigma_n$.

(1) Prove (or disprove) that G is not an almost direct product of free groups.
(2) Determine whether the cohomology ring $H^*(G;\mathbb{Q})$ is a Koszul algebra.

Novikov-Sikorav homology

Problem 10 (J.-C. Sikorav)**.** Let M be a closed 3-dimensional aspherical manifold and $N \subset H^1(M;\mathbb{R})$ be the set of classes represented by nonsingular one-forms. Does there exist an element $\widetilde{\Delta}(M) \in \Lambda$ with the following properties:

(i) $\widetilde{\Delta}(M) = \widetilde{\Delta}(M)^*$
(ii) its image in $\overline{\Lambda}$ is the Alexander polynomial $\Delta(M) = \gcd\det(D_2)_{i,j}$.
(iii) the Thurston norm is
$$||\xi||_T = \max_{g,h\in \mathrm{supp}(\widetilde{\Delta}(M))} |\xi(g)-\xi(h)| - 2 = 2 \max_{g\in \mathrm{supp}(\widetilde{\Delta}(M))} |\xi(g)| - 2$$
(iv) $\xi \in N \Leftrightarrow \widetilde{\Delta}(M)$ is invertible in Λ_ξ (or in $\Lambda_{-\xi}$) $\Leftrightarrow$ is there is only one element in $\mathrm{supp}(\widetilde{\Delta})$ with minimal (or maximal) ξ-value, and its coefficient is $\pm 1 \Leftrightarrow H_1(M;\xi) = 0$.

Problem 11 (J.-C. Sikorav)**.** Let G be a finitely generated group. Let A be a matrix in $\mathrm{Mat}_{p,p}(\mathbb{Z}[G])$, and let $N \subset \mathrm{Hom}(G, \mathbb{R}) \setminus \{0\}$ be the set of ξ such that A is invertible over $\mathbb{Z}[G]_\xi$ (or $\mathbb{Q}[G]_\xi$). Is N defined by a finite number of integral inequalities?

Line arrangements

Let $\mathcal{A}$ be a line arrangement in $\mathbb{P}^2$ and $G = \pi_1(\mathbb{P}^2 - \bigcup_{\ell \in \mathcal{A}} \ell)$. According to a fundamental result of Arapura [1], every positive-dimensional component of $\mathcal{V}^1(G)$ is a coset of a subtorus by a torsion point of $(\mathbb{C}^*)^n$. The component is *translated* if it is not the identity coset.

Problem 12 (M. Falk)**.** Prove or provide a counter-example to the following assertion: all positive-dimensional translated components of $\mathcal{V}^1(G)$ have dimension one.

Alex Dimca showed that any translated component of $\mathcal{V}^1(G)$ of dimension two or greater is a coset of a subtorus component of $\mathcal{V}^1(G)$, see [4]. Combining results of [5] with those of [3] may enable one to prove the assertion above.

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