Mathematisches Forschungsinstitut Oberwolfach

Report No. 11/2002

Miniworkshop: Cohomology Jumping Loci

March 3rd – March 9th, 2002

1. Overview

This Mini-Workshop was organized by A. Suciu (Boston) and S. Yuzvinsky (Eugene). While continuing a sequence of conferences and workshops on arrangements in recent years, it was the first meeting, ever, devoted entirely to cohomology jumping loci.

The main focus was on the varieties of jumping loci for cohomology with coefficients in rank one local systems, and the related resonance varieties, arising from complex hyperplane arrangements. These varieties have emerged as central objects of study, providing deep and varied information about the topology of the complement of an arrangement.

Given the multifaceted nature of the topic, the meeting brought together people with a variety of backgrounds, including algebra, topology, discrete geometry, and singularity theory. Almost half the participants were recent Ph.D.'s, most of them on their first visit to Oberwolfach. In addition to experts and young people in the field, there were several participants with complementary expertise. In all, there were 13 people attending the workshop (including the organizers), coming from the United States, Canada, France, Japan, Spain, Sweden, and Switzerland. The lively atmosphere and the free-flow of ideas led to significant progress in solving long-standing open problems, and to fruitful insights on how to attack new problems.

2. Characteristic and Resonance Varieties

If M is the complement of an arrangement of n hyperplanes, then the fundamental group $G = \pi_1(M)$ has abelianization $H_1(G) \cong \mathbb{Z}^n$, and the representation variety of G is the algebraic torus $\mathrm{Hom}(G, \mathbb{C}^*) \cong (\mathbb{C}^*)^n$. The jumping loci for k-dimensional cohomology—also known as the *characteristic varieties*—are the subvarieties

$$V_d^k = \{\mathbf{t} \in (\mathbb{C}^*)^n \mid \dim_{\mathbb{C}} H^k(M, \mathbb{C}_{\mathbf{t}}) \geq d\}, \quad d \geq 1.$$

The irreducible components of V_d^k are subtori of the character torus, possibly translated by roots of unity. Counting certain torsion points on the character torus, according to their depth with respect to the stratification by the characteristic varieties, yields information about the homology of finite abelian covers of the complement, and the homology of the Milnor fiber.

Closely related to the characteristic varieties are the *resonance varieties*, which arise from the theory of graded modules over an exterior algebra E (over the field $\mathbb{C}$). There the action of an element $a \in E_1$ converts an E-module A into a cochain complex. The cohomology of this complex serves as a measure of the non-regularity of a. The resonance varieties of the module A are the jumping loci for this cohomology:

$$R_d^k = \{a \in E_1 \mid \dim_{\mathbb{C}} H^k(A, \cdot a) \geq d\}, \quad d \geq 1.$$

If $A = H^*(M, \mathbb{C})$ is the cohomology ring of the complement of an arrangement (the so-called Orlik-Solomon algebra), the resonance varieties are linear approximations of the characteristic varieties, and a purely combinatorial description of R_d^1 is known. This description is useful in studying the free resolution of A over E, as well as the lower central series quotients of G and G/G''. Understanding the cohomology jumping loci V_d^k and R_d^k in dimensions $k > 1$, as well as the cohomology jumping loci over fields of positive characteristic, are some of the most challenging problems in this area.

Another challenging problem is that of distinguishing the homotopy types of complements of hyperplane arrangements that are combinatorially equivalent. Since the complements are formal, rational homotopy theory cannot pick up such a difference. On the other hand, there is an example (not yet well-understood), that suggests Massey products over $\mathbb{Z}_p$ may detect such homotopy types. Higher-order resonance varieties, defined by means of certain Massey products, should prove useful in solving such problems.

3. Format of the Mini-Workshop

The Mini-Workshop at Oberwolfach provided a lively forum for discussing a host of questions related to the structure and applications of cohomology jumping loci. The day-by-day schedule was kept very flexible, and was agreed upon on short notice, making it possible to shape the program on-site, and in response to the interests expressed by the participants. The borderlines between problem sessions and formal lectures were often blurred, and almost no time constraints were imposed on the speakers. Some lectures ran up to two hours or came in several parts, while others were informal twenty-minute discussions of an open problem—a format which proved to be very popular with both the speakers and the audience.

Spending a concentrated and highly intense week in a relatively small group allowed for in-depth and continuing conversations, in particular with new acquaintances. These opportunities (difficult to find at larger meetings) were enhanced by the diversity of backgrounds of the participants. This speaks to the fact that the usual, more rigid conference climate was superseded by an open and creative workshop atmosphere.

There was general agreement that the Mini-Workshop created an effective and stimulating research atmosphere. Many people expressed the sentiment that this was the most useful and pleasant meeting they ever attended.

During the week of the workshop, marked progress was made in solving old and new problems, both directly and indirectly related to cohomology jumping loci. The work initiated at Oberwolfach is continuing now in several research groups. The intense interactions at the meeting gave rise to new projects, which should start bearing fruit soon.

A record of the activities of the Oberwolfach Mini-Workshop on Cohomology Jumping Loci is included below. This record consists of abstracts of some of the more formal talks, as well as summaries of several of the less formal problem sessions. A web-based version of the Problem Sessions is being made available at `http://www.matematik.su.se/events/cjl2002`.

4. Abstracts of Talks

Essential Coordinate Components

José Ignacio Cogolludo-Agustín

Consider $\mathcal{C}$ a plane affine curve with r irreducible components. Denote by X its complement in $\mathbb{C}^2$. Consider its characteristic variety, $V_1(\mathcal{C})$, as a union of translated subtori of $\operatorname{Spec}(\mathbb{C}[H_1(X,\mathbb{Z})])$. For a distinguished basis of $H_1(X,\mathbb{Z})$ given by meridians of each component, it makes sense to distinguish coordinate and non-coordinate components of $V_1(\mathcal{C})$.

The existence of coordinate components might be due to the existence of components of characteristic varieties of subarrangements of $\mathcal{C}$. Such components are called non-essential. In addition, there might be certain coordinate components that don't appear in any subarrangement of $\mathcal{C}$. Such components are called essential coordinate components. Their existence is unknown for line arrangements.

We present an example of two rational arrangements with the same number and type of singularities but non-homeomorphic complements. The obstruction is given by the existence of essential coordinate components in one arrangement that is lacking in the other. Such an example seems to correlate essential coordinate components with the position of singularities.

Triples of arrangements and local systems
Characteristic varieties and links at infinity

Daniel C. Cohen

We discuss several topics relating to the question of determining the extent to which the characteristic varieties of the complement of a complex hyperplane arrangement are determined by combinatorial data.

First, we extend a well known result relating the constant coefficient cohomology of the complements of arrangements in a deletion-restriction triple to certain nontrivial local coefficient systems. This extension is used to produce infinite families $\{\mathcal{A}_r\}$ of arrangements whose first characteristic varieties $V_1^1(\mathcal{A}_r)$ have components translated by characters of order r, and a combinatorial explanation of the existence of these translated components.

Next, we relate the characteristic varieties of a central arrangement $\mathcal{A}$ to those of the links at infinity of various decones of $\mathcal{A}$. The resulting "virtual characteristic varieties" $W_d^1(\mathcal{A})$ provide combinatorial "upper bounds" for the characteristic varieties: There is a containment $V_d^1(\mathcal{A}) \subseteq W_d^1(\mathcal{A})$. This puts combinatorial constraints on which translated subtori of the character torus can arise as components of $V_d^1(\mathcal{A})$.

Alexander Invariants and Monodromy of Polynomial Functions

Alexandru Dimca

Let X be an affine complex hypersurface given by a polynomial equation $f = 0$ in the affine space $\mathbb{C}^n$. The Alexander invariants of X describe the topology of the complement $\mathbb{C}^n \setminus X$. I will discuss relations between these invariants and the monodromy associated to the function $f\colon \mathbb{C}^n \to \mathbb{C}$, based on joint work with A. Némethi.

Resonance varieties in positive characteristic

Michael Falk

Using a combinatorial characterization of resonant pairs of weights, we are able to give a precise description of the resonance variety $\mathcal{R}_d(\mathcal{A}, \mathbb{K})$, for any sufficiently large, e.g., algebraically closed field $\mathbb{K}$. In particular, we confirm the conjecture that irreducible components of $\mathcal{R}_d(\mathcal{A}, \mathbb{K})$ are linear.

Components of $\mathcal{R}_d(\mathcal{A}, \mathbb{K})$ are determined by neighbourly partitions of $\mathcal{A}$. Such a partition π determines a subspace $\mathbb{P}^k$ of $\mathbb{P}^n$, and a rational mapping $\mathbb{P}^k \to \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, given by the collection of projections corresponding to the blocks of π. The variety of poles of this mapping is a union of linear spaces. Each component C of the pole variety consists entirely of weights of fixed depth, determined by the codimension of the component, in the resonance variety, and a component of $\mathcal{R}_d(\mathcal{A}, \mathbb{K})$ is a trivial bundle over C with fiber equal to the annihilator of a generic weight on C.

Free resolutions and resonance varieties

Henry K. Schenck

Let X be the complement of a hyperplane arrangement. Components of the cohomology jump loci of the cohomology ring E/I of X give rise to linear first syzygies in the resolution of E/I over E. The number of linear first syzygies,

$$\dim_k \operatorname{Tor}_2^E(E/I, k)_3,$$

is also the rank of the third lower central series quotient ϕ_3. Passing to the initial ideal of I, results of Aramova, Herzog, and Hibi on the exterior Stanley-Reisner ring give very good bounds on ϕ_3.

Torsion in the homology of the Milnor fiber of a multi-arrangement

Alexander I. Suciu

In a recent paper with Daniel Matei (IMRN 2002:9), we give a formula that computes the mod q first Betti number of a finite abelian cover, $\hat{X}$, of a finite CW-complex X, directly from the stratification of the character variety $\operatorname{Hom}(\pi_1(X), \mathbb{K}^*)$ by the cohomology jumping loci $V_d^1(X, \mathbb{K})$, provided the field $\mathbb{K}$ is sufficiently large, and $q = \operatorname{char} \mathbb{K}$ does not divide the order of the cover. If the variety $V_1^1(X, \mathbb{C})$ contains a subtorus, translated by a q-th root of unity, then this formula can be used to detect q-torsion in $H_1(\hat{X}; \mathbb{Z})$.

Using this method, we give an example of a multi-arrangement $\mathcal{A}$ (obtained by deleting a plane from the reflection B_3 arrangement, and taking the remaining planes with suitable multiplicities), for which the Milnor fiber $F(\mathcal{A})$ has 2-torsion in first homology.

It remains an open question whether a *bona-fide* hyperplane arrangement $\mathcal{A}$ can have torsion in the homology of its Milnor fiber, or whether the Betti numbers $b_i(F(\mathcal{A}))$—let alone the integral homology groups $H_i(F(\mathcal{A}), \mathbb{Z})$—are combinatorially determined. For more on this last question, see the next talk.

Virtual jumping loci

Alexander I. Suciu

The characteristic varieties of a complex hyperplane arrangement, $\mathcal{A}$, consist of torsion-translated subtori of the character torus, $(\mathbb{C}^*)^{|\mathcal{A}|}$. The components of $V_d^1(\mathcal{A})$ passing through the origin admit a combinatorial interpretation, in terms of the intersection lattice. No such interpretation is known for the other components, except in some special cases.

As a step towards a better understanding of the translated tori in the jumping loci, we introduce the *virtual characteristic varieties* of an arrangement, $W_d^1(\mathcal{A})$. These varieties are closely related (though in general not equal) to $V_d^1(\mathcal{A})$. They are unions of affine subgroups of the character torus, of the form $W_d^1(\mathcal{A}) = \bigcup \Gamma_i \cdot T_i$, where T_i are subtori of $(\mathbb{C}^*)^{|\mathcal{A}|}$ and Γ_i are finite (abelian) subgroups.

From the intersection lattice, we may easily compute the invariant factors of the translation groups Γ_i, thereby obtaining certain *a priori* information about the possible orders of translation of the components of $V_d^1(\mathcal{A})$. This information has practical uses in the detection of translated tori in $V_d^1(\mathcal{A})$, and may provide clues as to whether such components are combinatorially determined.

The Poincaré series of the coordinate ring of the complement of a hyperplane arrangement

Hiroaki Terao

(joint work with Hiroki Horiuchi)

Let Δ be a finite set of nonzero linear forms in several variables with coefficients in a field $\mathbf{K}$ of characteristic zero. Consider the $\mathbf{K}$-algebra $R(\Delta)$ of rational functions on V which are regular outside $\bigcup_{\alpha\in\Delta} \ker\alpha$. It is the coordinate ring of the complement of the hyperplane arrangement. Then the ring $R(\Delta)$ is naturally doubly filtered by the degrees of denominators and of numerators.

An explicit combinatorial formula for the Poincaré series in two variables of the associated bigraded vector space $\overline{R}(\Delta)$ is given. The details are posted at `math.CO/0202296`.

5. Problem Sessions

Does Resonance Propagate?

Daniel C. Cohen

Let $\mathcal{A}$ be an arrangement of hyperplanes in $\mathbb{C}^\ell$, with complement M, and let $\mathcal{L}$ be a (complex rank one) local system on M. Assume that $\mathcal{A}$ contains ℓ linearly independent hyperplanes.

For a generic or "nonresonant" local system $\mathcal{L}$, the local coefficient Betti numbers $b_p(M;\mathcal{L}) = \dim_{\mathbb{C}} H^p(M;\mathcal{L})$ are minimal. Explicitly, $b_p(M;\mathcal{L}) = 0$ for $p \neq \ell$ and $b_\ell(M;\mathcal{L}) = |\chi(M)|$, where $\chi(M)$ is the Euler characteristic.

Suppose $\mathcal{L}$ is a resonant local system, and that $p \neq \ell$ is minimal so that $b_p(M;\mathcal{L}) \neq 0$.

Question. Does resonance propagate? If $p < \ell$ is minimal so that $b_p(M;\mathcal{L}) \neq 0$, is it the case that $b_q(M;\mathcal{L}) \neq 0$ for $p \leq q \leq \ell - 1$ and $b_\ell(M;\mathcal{L}) > |\chi(M)|$?

Discussion. If $b_0(M;\mathcal{L}) \neq 0$, then the rank one local system $\mathcal{L}$ is trivial, and the answer to the above question is **yes**.

If $p = \ell - 1$ is minimal so that $b_p(M;\mathcal{L}) \neq 0$, then an Euler characteristic argument shows that $b_\ell(M;\mathcal{L}) = |\chi(M)| + b_{\ell-1}(M;\mathcal{L}) > |\chi(M)|$. So the answer to the above question is **yes** in this instance as well.

If $p = \ell - 2$ is minimal so that $b_p(M;\mathcal{L}) \neq 0$, let S be a $(\ell-1)$-dimensional affine subspace in $\mathbb{C}^\ell$ that is transverse to $\mathcal{A}$. Then one can show (for instance, using stratified Morse theory) that the inclusion $S \cap M \to M$ induces an isomorphism $H^i(M;\mathcal{L}) \to H^i(S \cap M;\mathcal{L})$ for $i \leq \ell - 2$, and that there is a short exact sequence

$$0 \to H^{\ell-1}(M;\mathcal{L}) \to H^{\ell-1}(S \cap M;\mathcal{L}) \to H^\ell(M, S \cap M;\mathcal{L}) \to H^\ell(M;\mathcal{L}) \to 0.$$

By an Euler characteristic argument as above, we have $b_{\ell-1}(S \cap M;\mathcal{L}) > |\chi(S \cap M)|$. Using this, together with the above exact sequence, one can show that $b_{\ell-1}(M;\mathcal{L}) \neq 0$.

More generally, one can similarly show that if $b_p(M;\mathcal{L}) \neq 0$ for some $p \leq \ell - 2$, then $b_{p+1}(M;\mathcal{L}) \neq 0$. Thus, resonance does propagate, at least to some extent.

A 30-line Hessian-like fibered $K(\pi, 1)$ arrangement

Michael Falk

Suppose $\mathcal{A}$ is a projective line arrangement that supports resonant weights. Then, according to Libgober and Yuzvinsky, following Arapura, there is a pencil of degree d plane curves that includes $\bigcup \mathcal{A}$ in the union of its singular fibers. If $\bigcup \mathcal{A}$ exhausts the union of singular fibers,, then the pencil defines a fibration of the complement of $\mathcal{A}$ by punctured surfaces, hence $\mathcal{A}$ is a $K(\pi, 1)$ arrangement. The converse is also true: if $\bigcup \mathcal{A}$ consists of singular degree d curves which lie in a pencil, then $\mathcal{A}$ supports resonant weights. The correspondence between resonant weights and pencils is very explicit; each can be computed from the other.

Such fibered $K(\pi, 1)$ arrangements are few and far between: at this point there are three infinite families and one sporadic example known. The sporadic example is the Hessian arrangement, which consists of four singular fibers in a pencil of cubic curves, each of which consists of three lines in general position. This is the only example with more than three singular fibers.

The structure of such a pencil is governed by the Hurwitz formula, relating the degree d to the Euler characteristics of the singular curves. The formula allows for the existence of a pencil of degree six curves, with precisely five singular fibers, each of which is a union of six lines in general position. The union of the singular fibers thus forms a $K(\pi, 1)$ arrangement of 30 lines, partitioned into five blocks of six lines each. There are 36 base points each of multiplicity five in $\mathcal{A}$, and $5 \cdot 15 = 75$ points of multiplicity two in $\mathcal{A}$. The problem is to find such an arrangement.

Zeta functions of Orlik-Solomon algebras

Alexander I. Suciu

Let A be an algebra over $\mathbb{Z}$, with underlying group free abelian, of finite rank. Grunewald, Segal, and Smith defined the zeta function of A as

$$\zeta_A(s) = \sum_{k=1}^{\infty} a_k(A) k^{-s},$$

where $a_k(A)$ is the number of index k subalgebras of A. Similarly, $\zeta_A^{\triangleleft}(s) = \sum_{k=1}^{\infty} a_k^{\triangleleft}(A) k^{-s}$, where $a_k^{\triangleleft}(A)$ is the number of index k ideals of A.

Now assume $A = A^*$ is a graded, graded-commutative, associative algebra, generated in degree 1. Set $\zeta_A^*(s) = \sum_{k=1}^{\infty} a_k^*(A) k^{-s}$, where $a_k^*(A)$ is the number of index k subalgebras of A, generated in degree 1.

Problem. Let $\mathcal{A}$ be a complex hyperplane arrangement, with complement M, and let $A = H^*(M; \mathbb{Z})$ be its Orlik-Solomon algebra. Compute the zeta functions $\zeta_A(s)$, $\zeta_A^{\triangleleft}(s)$, and $\zeta_A^*(s)$ in terms of the intersection lattice of $\mathcal{A}$.

An analogous, but easier problem can be stated for the truncated Orlik-Solomon algebra $\overline{A} = A/A^{\geq 3}$.

Conjecture. Let $R_d^1(A, \mathbb{F}_p)$ be the resonance varieties of A over the field $\mathbb{F}_p$. Then:

$$\sharp\{B^* < \overline{A}^* \mid [A^1 : B^1] = p,\ [A^2 : B^2] = p^d\} = \frac{\sharp(R_d^1(A, \mathbb{F}_p) \setminus R_{d+1}^1(A, \mathbb{F}_p))}{p-1}.$$

This formula would compute the local zeta functions $\zeta_{\overline{A},p}^*(s)$—and thus, by Euler factorization, the zeta function $\zeta_{\overline{A}}^*(s)$—in terms of the resonance varieties of A (or $\overline{A}$) in positive characteristic. In turn, these varieties can be determined explicitly in terms of the intersection lattice, by work of M. Falk, announced at the Oberwolfach Mini-Workshop. Thus, the Conjecture would solve the above Problem for the graded zeta function of $\overline{A}$.

Three problems on jumping loci and spectral sequences

Sergey Yuzvinsky

In a paper from 1995, the author introduced a spectral sequence converging to the cohomology of the Orlik-Solomon algebra and used it to show that this cohomology vanishes in middle dimensions for a generic differential. This spectral sequence seems to be analogous to the Leray spectral sequence for the embedding $M \hookrightarrow \mathbb{C}^n$ of the arrangement complement and a rank 1 local system $\mathcal{L}$ on M. The latter spectral sequence converges to the cohomology of $\mathcal{L}$ and, as A. Dimca remarked, it can be used to prove the vanishing of this cohomology for a generic $\mathcal{L}$ (a result due to T. Kohno).

This discussion justifies the following three problems.

Problem 1: Relate the author's spectral sequence to the resonance varieties.

Problem 2: Relate the Leray spectral sequence to the characteristic varieties.

Problem 3: Relate the two types of varieties to each other using the above spectral sequences.

Edited by Alexander I. Suciu

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