Duality, finiteness, and cohomology jump loci

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POINCARÉ DUALITY

- Let M be a compact, connected, orientable, n-dimensional manifold.
- Fix an orientation class $[M] \in H_n(M, \mathbb{Z}) \cong \mathbb{Z}$.
- Let $A = H^{\cdot}(M, \Bbbk)$ be the cohomology ring of M with coefficients in a field $\Bbbk$.
- The Poincaré duality theorem implies that A is a Poincaré duality algebra of dimension n, with orientation $\varepsilon\colon A^n \to \Bbbk$ given by

$$[M] \otimes \Bbbk \in H_n(M, \Bbbk) \cong \operatorname{Hom}_{\Bbbk}(H^n(M, \Bbbk), \Bbbk).$$

Poincaré duality algebras

- Let A be a graded, graded-commutative algebra over a field $\Bbbk$.
  - $A = \bigoplus_{i \geqslant 0} A^i$, where A^i are $\Bbbk$-vector spaces.
  - $\cdot\colon A^i \otimes A^j \to A^{i+j}$.
  - $ab = (-1)^{ij} ba$ for all $a \in A^i$, $b \in A^j$.
- We will assume that A is connected ($A^0 = \Bbbk \cdot 1$), and locally finite (all the Betti numbers $b_i(A) := \dim_{\Bbbk} A^i$ are finite).
- A is a *Poincaré duality* $\Bbbk$*-algebra* of dimension n if there is a $\Bbbk$-linear map $\varepsilon\colon A^n \to \Bbbk$ (called an *orientation*) such that all the bilinear forms $A^i \otimes_{\Bbbk} A^{n-i} \to \Bbbk$, $a \otimes b \mapsto \varepsilon(ab)$ are non-singular.
- Consequently,
  - $b_i(A) = b_{n-i}(A)$, and $A^i = 0$ for $i > n$.
  - ε is an isomorphism.
  - The maps $\mathrm{PD}\colon A^i \to (A^{n-i})^*$, $\mathrm{PD}(a)(b) = \varepsilon(ab)$ are isomorphisms.
  - Each $a \in A^i$ has a *Poincaré dual*, $a^\vee \in A^{n-i}$, such that $\varepsilon(aa^\vee) = 1$.
  - The *orientation class* is defined as $\omega_A = 1^\vee$, so that $\varepsilon(\omega_A) = 1$.

THE ASSOCIATED ALTERNATING FORM

- Associated to a $\Bbbk$-PD_n algebra there is an alternating n-form,

$$\mu_A\colon \bigwedge^n A^1 \to \Bbbk, \quad \mu_A(a_1 \wedge \cdots \wedge a_n) = \varepsilon(a_1 \cdots a_n).$$

- Assume now that $n = 3$, and set $r = b_1(A)$. Fix a basis $\{e_1, \dots, e_r\}$ for A^1, and let $\{e_1^\vee, \dots, e_r^\vee\}$ be the dual basis for A^2.
- The multiplication in A, then, is given on basis elements by

$$e_i e_j = \sum_{k=1}^r \mu_{ijk}\, e_k^\vee, \quad e_i e_j^\vee = \delta_{ij}\omega,$$

where $\mu_{ijk} = \mu(e_i \wedge e_j \wedge e_k)$.
- Alternatively, let $A_i = (A^i)^*$, and let $e^i \in A_1$ be the (Kronecker) dual of e_i. We may then view μ dually as a trivector,

$$\mu = \sum \mu_{ijk}\, e^i \wedge e^j \wedge e^k \in \bigwedge^3 A_1,$$

which encodes the algebra structure of A.

POINCARÉ DUALITY IN 3-MANIFOLDS

- Sullivan (1975): for every finite-dimensional $\mathbb{Q}$-vector space V and every alternating 3-form $\mu \in \bigwedge^3 V^*$, there is a closed 3-manifold M with $H^1(M, \mathbb{Q}) = V$ and cup-product form $\mu_M = \mu$.
- Such a 3-manifold can be constructed via "Borromean surgery."

- For instance, 0-surgery on the Borromean rings yields the 3-torus, whose intersection form is $\mu = e^1 e^2 e^3$.
- If M bounds an oriented 4-manifold W such that the cup-product pairing on $H^2(W, M)$ is non-degenerate (e.g., if M is the link of an isolated surface singularity), then $\mu_M = 0$.

DUALITY SPACES

A more general notion of duality is due to Bieri and Eckmann (1978).

Let X be a connected, finite-type CW-complex, and set $\pi = \pi_1(X, x_0)$.

- X is a *duality space* of dimension n if $H^i(X, \mathbb{Z}\pi) = 0$ for $i \neq n$ and $H^n(X, \mathbb{Z}\pi) \neq 0$ and torsion-free.
- Let $D = H^n(X, \mathbb{Z}\pi)$ be the dualizing $\mathbb{Z}\pi$-module. Given any $\mathbb{Z}\pi$-module A, we have $H^i(X, A) \cong H_{n-i}(X, D \otimes A)$.
- If $D = \mathbb{Z}$, with trivial $\mathbb{Z}\pi$-action, then X is a Poincaré duality space.
- If $X = K(\pi, 1)$ is a duality space, then π is a *duality group*.

ABELIAN DUALITY SPACES

We introduce in [Denham–S.–Yuzvinsky 2016/17] an analogous notion, by replacing $\pi \rightsquigarrow \pi_{ab}$.

- X is an *abelian duality space* of dimension n if $H^i(X, \mathbb{Z}\pi_{ab}) = 0$ for $i \neq n$ and $H^n(X, \mathbb{Z}\pi_{ab}) \neq 0$ and torsion-free.
- Let $B = H^n(X, \mathbb{Z}\pi_{ab})$ be the dualizing $\mathbb{Z}\pi_{ab}$-module. Given any $\mathbb{Z}\pi_{ab}$-module A, we have $H^i(X, A) \cong H_{n-i}(X, B \otimes A)$.
- The two notions of duality are independent:

EXAMPLE

- Surface groups of genus at least 2 are not abelian duality groups, though they are (Poincaré) duality groups.
- Let $\pi = \mathbb{Z}^2 * G$, where
$$G = \langle x_1, \dots, x_4 \mid x_1^{-2} x_2 x_1 x_2^{-1}, \dots, x_4^{-2} x_1 x_4 x_1^{-1} \rangle$$
is Higman's acyclic group. Then π is an abelian duality group (of dimension 2), but not a duality group.

THEOREM (DSY)

Let X be an abelian duality space of dimension n. Then:

- $b_1(X) \geqslant n-1$.
- $b_i(X) \neq 0$, *for* $0 \leqslant i \leqslant n$ *and* $b_i(X) = 0$ *for* $i > n$.
- $(-1)^n \chi(X) \geqslant 0$.
- *Let* $\rho\colon \pi_1(X) \to \mathbb{C}^*$ *be a character such that* $H^i(X, \mathbb{C}_\rho) \neq 0$, *for some* $i > 0$. *Then* $H^j(X, \mathbb{C}_\rho) \neq 0$, *for all* $i \leqslant j \leqslant n$.

THEOREM (DENHAM–S. 2018)

Let U be a connected, smooth, complex quasi-projective variety of dimension n. Suppose U has a smooth compactification Y for which

1. *Components of $Y \backslash U$ form an arrangement of hypersurfaces $\mathcal{A}$;*
2. *For each submanifold X in the intersection poset $L(\mathcal{A})$, the complement of the restriction of $\mathcal{A}$ to X is a Stein manifold.*

Then U is both a duality space and an abelian duality space of dimension n.

LINEAR, ELLIPTIC, AND TORIC ARRANGEMENTS

THEOREM (DENHAM–S. 2018)

Suppose that $\mathcal{A}$ is one of the following:

1. *An affine-linear arrangement in $\mathbb{C}^n$, or a hyperplane arrangement in $\mathbb{CP}^n$;*
2. *A non-empty elliptic arrangement in E^n;*
3. *A toric arrangement in $(\mathbb{C}^*)^n$.*

Then the complement $M(\mathcal{A})$ is both a duality space and an abelian duality space of dimension $n-r$, $n+r$, and n, respectively, where r is the corank of the arrangement.

This theorem extends several previous results:

1. Davis, Januszkiewicz, Leary, and Okun (2011);
2. Levin and Varchenko (2012);
3. Davis and Settepanella (2013), Esterov and Takeuchi (2018).

FINITENESS PROPERTIES FOR SPACES AND GROUPS

- A recurring theme in topology is to determine the geometric and homological finiteness properties of spaces and groups.
- For instance, to decide whether a path-connected space X is homotopy equivalent to a CW-complex with finite k-skeleton.
- A group G has property F_k if it admits a classifying space $K(G,1)$ with finite k-skeleton.
  - F_1: G is finitely generated;
  - F_2: G is finitely presentable.
- G has property FP_k if the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ admits a projective $\mathbb{Z}G$-resolution which is finitely generated in all dimensions up to k.
- The following implications (none of which can be reversed) hold:

$$\begin{aligned} G \text{ is of type } \mathrm{F}_k &\Rightarrow G \text{ is of type } \mathrm{FP}_k \\ &\Rightarrow H_i(G,\mathbb{Z}) \text{ is finitely generated, for all } i \leqslant k \\ &\Rightarrow b_i(G) < \infty, \text{ for all } i \leqslant k. \end{aligned}$$

- Moreover, FP_k & $\mathrm{F}_2 \Rightarrow \mathrm{F}_k$.

BIERI–NEUMANN–STREBEL–RENZ INVARIANTS

- (Bieri–Neumann–Strebel 1987) For a f.g. group G, let
$$\Sigma^1(G) = \{\chi \in S(G) \mid \mathcal{C}_\chi(G) \text{ is connected}\},$$
where $S(G) = (\operatorname{Hom}(G, \mathbb{R}) \backslash \{0\}) / \mathbb{R}^+$ and $\mathcal{C}_\chi(G)$ is the induced subgraph of $\operatorname{Cay}(G)$ on vertex set $G_\chi = \{g \in G \mid \chi(g) \geqslant 0\}$.
- $\Sigma^1(G)$ is an open set, independent of generating set for G.
- (Bieri, Renz 1988)
$$\Sigma^k(G, \mathbb{Z}) = \{\chi \in S(G) \mid \text{the monoid } G_\chi \text{ is of type } \mathrm{FP}_k\}.$$
In particular, $\Sigma^1(G, \mathbb{Z}) = \Sigma^1(G)$.
- The Σ-invariants control the finiteness properties of normal subgroups $N \lhd G$ for which G/N is free abelian:
$$N \text{ is of type } \mathrm{FP}_k \iff S(G, N) \subseteq \Sigma^k(G, \mathbb{Z})$$
where $S(G, N) = \{\chi \in S(G) \mid \chi(N) = 0\}$. In particular:
$$\ker(\chi\colon G \twoheadrightarrow \mathbb{Z}) \text{ is f.g.} \iff \{\pm\chi\} \subseteq \Sigma^1(G).$$

- Fix a connected CW-complex X with finite k-skeleton, for some $k \geqslant 1$. Let $G = \pi_1(X, x_0)$.
- For each $\chi \in S(X) := S(G)$, set

$$\widehat{\mathbb{Z}G}_{\chi} = \Big\{\lambda \in \mathbb{Z}^G \mid \{g \in \operatorname{supp}\lambda \mid \chi(g) < c\} \text{ is finite, } \forall c \in \mathbb{R}\Big\}.$$

This is a ring, contains $\mathbb{Z}G$ as a subring; hence, a $\mathbb{Z}G$-module.

- (Farber, Geoghegan, Schütz 2010)

$$\Sigma^q(X, \mathbb{Z}) := \{\chi \in S(X) \mid H_i(X, \widehat{\mathbb{Z}G}_{-\chi}) = 0,\ \forall\, i \leqslant q\}.$$

- (Bieri) G is of type $\mathrm{FP}_k \implies \Sigma^q(G, \mathbb{Z}) = \Sigma^q(K(G,1), \mathbb{Z})$, $\forall q \leqslant k$.

DWYER–FRIED SETS

- For a fixed $r \in \mathbb{N}$, the connected, regular covers $Y \to X$ with group of deck-transformations $\mathbb{Z}^r$ are parametrized by the Grassmannian of r-planes in $H^1(X, \mathbb{Q})$.
- Moving about this variety, and recording when $b_1(Y), \dots, b_i(Y)$ are finite defines subsets $\Omega^i_r(X) \subseteq \mathrm{Gr}_r(H^1(X, \mathbb{Q}))$, which we call the *Dwyer–Fried invariants* of X.
- These sets depend only on the homotopy type of X. Hence, if G is a f.g. group, we may define $\Omega^i_r(G) := \Omega^i_r(K(G, 1))$.

THEOREM

Let G be a f.g. group, and $\nu \colon G \twoheadrightarrow \mathbb{Z}^r$ an epimorphism, with kernel Γ. Suppose $\Omega^k_r(G) = \emptyset$, and Γ is of type F_{k-1}. Then $b_k(\Gamma) = \infty$.

Proof: Set $X = K(G, 1)$; then $X^\nu = K(\Gamma, 1)$. Since Γ is of type F_{k-1}, $b_i(X^\nu) < \infty$ for $i \leqslant k - 1$. But now $\Omega^k_r(X) = \emptyset$ implies $b_k(X^\nu) = \infty$.

COROLLARY

Let G be a f.g. group, and suppose $\Omega^3_1(G) = \emptyset$. Let $\nu\colon G \twoheadrightarrow \mathbb{Z}$ be an epimorphism. If the group $\Gamma = \ker(\nu)$ is f.p., then $b_3(\Gamma) = \infty$.

EXAMPLE (THE STALLINGS GROUP)

- Let $Y = S^1 \vee S^1$ and $X = Y \times Y \times Y$. Clearly, X is a classifying space for $G = F_2 \times F_2 \times F_2$.
- Let $\nu\colon G \to \mathbb{Z}$ be the homomorphism taking each standard generator to 1. Set $\Gamma = \ker(\nu)$.
- Stallings (1963) showed that Γ is finitely presented.
- Using a Mayer-Vietoris argument, he also showed that $H_3(\Gamma, \mathbb{Z})$ is not finitely generated.
- Alternate explanation: $\Omega^3_1(X) = \emptyset$. Thus, by the previous Corollary, a stronger statement holds: $b_3(\Gamma)$ is not finite.

KOLLÁR'S QUESTION

QUESTION (J. KOLLÁR 1995)

Given a smooth, projective variety M, is the fundamental group $G = \pi_1(M)$ commensurable, up to finite kernels, with another group, π, admitting a $K(\pi, 1)$ which is a quasi-projective variety?

(Two groups, G_1 and G_2, are said to be *commensurable up to finite kernels* if there is a zig-zag of groups and homomorphisms connecting them, with all arrows of finite kernel and cofinite image.)

THEOREM (DIMCA–PAPADIMA–S. 2009)

For each $k \geqslant 3$, there is a smooth, irreducible, complex projective variety M of complex dimension $k-1$, such that $\pi_1(M)$ is of type F_{k-1}, but not of type FP_k.

Further examples given by Llosa Isenrich and Bridson (2016–2019).

Support loci

- Let $\Bbbk$ be an (algebraically closed) field.
- Let S be a commutative, finitely generated $\Bbbk$-algebra.
- Let $\operatorname{Spec}(S) = \operatorname{Hom}_{\Bbbk\text{-alg}}(S, \Bbbk)$ be the maximal spectrum of S.
- Let $E\colon \ \cdots \to E_i \xrightarrow{d_i} E_{i-1} \to \cdots \to E_0 \to 0$ be an S-chain complex.
- The *support varieties* of E are the subsets of $\operatorname{Spec}(S)$ given by

$$\mathcal{W}^i_d(E) = \operatorname{supp}\Big(\bigwedge^d H_i(E)\Big).$$

- They depend only on the chain-homotopy equivalence class of E.
- For each $i \geqslant 0$, $\operatorname{Spec}(S) = \mathcal{W}^i_0(E) \supseteq \mathcal{W}^i_1(E) \supseteq \mathcal{W}^i_2(E) \supseteq \cdots$.
- If all E_i are finitely generated S-modules, then the sets $\mathcal{W}^i_d(E)$ are Zariski closed subsets of $\operatorname{Spec}(S)$.

HOMOLOGY JUMP LOCI

- The *homology jump loci* of the S-chain complex E are defined as

$$\mathcal{V}^i_d(E) = \{\mathfrak{m} \in \operatorname{Spec}(S) \mid \dim_{S/\mathfrak{m}} H_i(E \otimes_S S/\mathfrak{m}) \geqslant d\}.$$

- They depend only on the chain-homotopy equivalence class of E.
- Get stratifications $\operatorname{Spec}(S) = \mathcal{V}^i_0(E) \supseteq \mathcal{V}^i_1(E) \supseteq \mathcal{V}^i_2(E) \supseteq \cdots$.

THEOREM (PAPADIMA–S. 2014)

Suppose E is a chain complex of free, *finitely generated S-modules. Then:*

- *Each $\mathcal{V}^i_d(E)$ is a Zariski closed subset of $\operatorname{Spec}(S)$.*
- *For each q,*

$$\bigcup_{i\leqslant q} \mathcal{V}^i_1(E) = \bigcup_{i\leqslant q} \mathcal{W}^i_1(E).$$

RESONANCE VARIETIES OF A CDGA

- Let $A = (A^{\bullet}, \mathrm{d})$ be a commutative, differential graded algebra over a field $\Bbbk$ of characteristic 0. That is:
  - $A = \bigoplus_{i \geqslant 0} A^i$, where A^i are $\Bbbk$-vector spaces.
  - The multiplication $\cdot\colon A^i \otimes A^j \to A^{i+j}$ is graded-commutative, i.e., $ab = (-1)^{|a||b|} ba$ for all homogeneous a and b.
  - The differential $\mathrm{d}\colon A^i \to A^{i+1}$ satisfies the graded Leibniz rule, i.e., $\mathrm{d}(ab) = \mathrm{d}(a)b + (-1)^{|a|} a\,\mathrm{d}(b)$.
- We assume A is connected (i.e., $A^0 = \Bbbk \cdot 1$) and of finite-type (i.e., $\dim A^i < \infty$ for all i).
- For each $a \in Z^1(A) \cong H^1(A)$, we have a cochain complex,

$$(A^{\bullet}, \delta_a)\colon \; A^0 \xrightarrow{\delta_a^0} A^1 \xrightarrow{\delta_a^1} A^2 \xrightarrow{\delta_a^2} \cdots,$$

  with differentials $\delta_a^i(u) = a \cdot u + \mathrm{d}(u)$, for all $u \in A^i$.
- The *resonance varieties* of A are the affine varieties

$$\mathcal{R}_s^i(A) = \{a \in H^1(A) \mid \dim_{\Bbbk} H^i(A^{\bullet}, \delta_a) \geqslant s\}.$$

- Fix a $\Bbbk$-basis $\{e_1, \dots, e_r\}$ for A^1, and let $\{x_1, \dots, x_r\}$ be the dual basis for $A_1 = (A^1)^*$.
- Identify $\mathrm{Sym}(A_1)$ with $S = \Bbbk[x_1, \dots, x_r]$, the coordinate ring of the affine space A^1.
- Build a cochain complex of free S-modules, $\mathbf{L}(A) := (A^{\bullet} \otimes S, \delta)$:

$$\cdots \longrightarrow A^i \otimes S \xrightarrow{\delta^i} A^{i+1} \otimes S \xrightarrow{\delta^{i+1}} A^{i+2} \otimes S \longrightarrow \cdots,$$

where $\delta^i(u \otimes f) = \sum_{j=1}^r e_j u \otimes f x_j + \mathrm{d}\, u \otimes f$.

- The specialization of $(A \otimes S, \delta)$ at $a \in Z^1(A)$ is (A, δ_a).
- Hence, $\mathcal{R}^i_s(A)$ is the zero-set of the ideal generated by all minors of size $b_i(A) - s + 1$ of the block-matrix $\delta^{i+1} \oplus \delta^i$.

CHARACTERISTIC VARIETIES

- Let X be a connected, finite-type CW-complex. Then $\pi = \pi_1(X, x_0)$ is a finitely presented group, with $\pi_{ab} \cong H_1(X, \mathbb{Z})$.
- The ring $R = \mathbb{C}[\pi_{ab}]$ is the coordinate ring of the character group, $\mathrm{Char}(X) = \mathrm{Hom}(\pi, \mathbb{C}^*) \cong (\mathbb{C}^*)^r \times \mathrm{Tors}(\pi_{ab})$, where $r = b_1(X)$.
- The *characteristic varieties* of X are the homology jump loci

$$\mathcal{V}^i_s(X) = \{\rho \in \mathrm{Char}(X) \mid \dim_{\mathbb{C}} H_i(X, \mathbb{C}_\rho) \geqslant s\}.$$

- These varieties are homotopy-type invariants of X, with $\mathcal{V}^1_s(X)$ depending only on $\pi = \pi_1(X)$.
- Set $\mathcal{V}^1_1(\pi) := \mathcal{V}^1_1(K(\pi, 1))$; then $\mathcal{V}^1_1(\pi) = \mathcal{V}_1(\pi/\pi'')$.

EXAMPLE

Let $f \in \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ be a Laurent polynomial, $f(1) = 0$. There is then a finitely presented group π with $\pi_{ab} = \mathbb{Z}^n$ such that $\mathcal{V}^1_1(\pi) = V(f)$.

TANGENT CONES

- Let $\exp\colon H^1(X,\mathbb{C}) \to H^1(X,\mathbb{C}^*)$ be the coefficient homomorphism induced by $\mathbb{C} \to \mathbb{C}^*$, $z \mapsto e^z$.
- Let $W = V(I)$, a Zariski closed subset of $\operatorname{Char}(G) = H^1(X,\mathbb{C}^*)$.
- The *tangent cone* at 1 to W is $\operatorname{TC}_1(W) = V(\operatorname{in}(I))$.
- The *exponential tangent cone* at 1 to W:

$$\tau_1(W) = \{z \in H^1(X,\mathbb{C}) \mid \exp(\lambda z) \in W,\ \forall \lambda \in \mathbb{C}\}.$$

- Both tangent cones are homogeneous subvarieties of $H^1(X,\mathbb{C})$; are non-empty iff $1 \in W$; depend only on the analytic germ of W at 1; commute with finite unions and arbitrary intersections.
- $\tau_1(W) \subseteq \operatorname{TC}_1(W)$, with $=$ if all irred components of W are subtori, but $\neq$ in general.
- (Dimca–Papadima–S. 2009) $\tau_1(W)$ is a finite union of rationally defined subspaces.

ALGEBRAIC MODELS FOR SPACES

- A CDGA map $\varphi\colon A \to B$ is a *quasi-isomorphism* if $\varphi^*\colon H^{\boldsymbol{\cdot}}(A) \to H^{\boldsymbol{\cdot}}(B)$ is an isomorphism.
- φ is a q-quasi-isomorphism (for some $q \geqslant 1$) if φ^* is an isomorphism in degrees $\leqslant q$ and is injective in degree $q+1$.
- Two CDGAs, A and B, are *(q-) equivalent* if there is a zig-zag of (q-) quasi-isomorphisms connecting A to B.
- A is *formal* (or just *q-formal*) if it is (q-) equivalent to $(H^{\bullet}(A), d = 0)$.
- A CDGA is *q-minimal* if it is of the form $(\bigwedge V, d)$, where the differential structure is the inductive limit of a sequence of Hirsch extensions of increasing degrees, and $V^i = 0$ for $i > q$.
- Every CDGA A with $H^0(A) = \Bbbk$ admits a *q-minimal model*, $\mathcal{M}_q(A)$ (i.e., a q-equivalence $\mathcal{M}_q(A) \to A$ with $\mathcal{M}_q(A) = (\bigwedge V, d)$ a q-minimal cdga), unique up to iso.

- Given any (path-connected) space X, there is an associated Sullivan $\mathbb{Q}$-cdga, $A_{\mathrm{PL}}(X)$, such that $H^\bullet(A_{\mathrm{PL}}(X)) = H^\bullet(X, \mathbb{Q})$.
- An *algebraic (q-)model* (over $\Bbbk$) for X is a $\Bbbk$-cgda (A, d) which is (q-) equivalent to $A_{\mathrm{PL}}(X) \otimes_{\mathbb{Q}} \Bbbk$.
- If M is a smooth manifold, then $\Omega_{\mathrm{dR}}(M)$ is a model for M (over $\mathbb{R}$).
- Examples of spaces having finite-type models include:
  - Formal spaces (such as compact Kähler manifolds, hyperplane arrangement complements, toric spaces, etc).
  - Smooth quasi-projective varieties, compact solvmanifolds, Sasakian manifolds, etc.

THE TANGENT CONE THEOREM

Let X be a connected CW-complex with finite q-skeleton. Suppose X admits a q-finite q-model A.

THEOREM

For all $i \leqslant q$ and all s:

- *(DPS 2009, Dimca–Papadima 2014)* $\mathcal{V}^i_s(X)_{(1)} \cong \mathcal{R}^i_s(A)_{(0)}$.
- *(Budur–Wang 2017) All the irreducible components of $\mathcal{V}^i_s(X)$ passing through the origin of $\operatorname{Char}(X)$ are algebraic subtori.*

Consequently,

$$\tau_1(\mathcal{V}^i_s(X)) = \mathrm{TC}_1(\mathcal{V}^i_s(X)) = \mathcal{R}^i_s(A).$$

THEOREM (PAPADIMA–S. 2017)

A f.g. group G admits a 1-finite 1-model if and only if the Malcev Lie algebra $\mathfrak{m}(G)$ is the LCS completion of a finitely presented Lie algebra.

INFINITESIMAL FINITENESS OBSTRUCTIONS

THEOREM

Let X be a connected CW-complex with finite q-skeleton. Suppose X admits a q-finite q-model A. Then, for all $i \leqslant q$ and all s,

- *(Dimca–Papadima 2014)* $\mathcal{V}^i_s(X)_{(1)} \cong \mathcal{R}^i_s(A)_{(0)}$.
  In particular, if X is q-formal, then $\mathcal{V}^i_s(X)_{(1)} \cong \mathcal{R}^i_s(X)_{(0)}$.
- *(Macinic, Papadima, Popescu, S. 2017)* $\mathrm{TC}_0(\mathcal{R}^i_s(A)) \subseteq \mathcal{R}^i_s(X)$.
- *(Budur–Wang 2017) All the irreducible components of $\mathcal{V}^i_s(X)$ passing through the origin of $H^1(X, \mathbb{C}^*)$ are* algebraic subtori.

EXAMPLE

Let G be a f.p. group with $G_{\mathrm{ab}} = \mathbb{Z}^n$ and $\mathcal{V}^1_1(G) = \{t \in (\mathbb{C}^*)^n \mid \sum_{i=1}^n t_i = n\}$. Then G admits no 1-finite 1-model.

THEOREM (PAPADIMA–S. 2017)

Suppose X is $(q+1)$ finite, or X admits a q-finite q-model. Then $b_i(\mathcal{M}_q(X)) < \infty$, for all $i \leqslant q+1$.

COROLLARY

Let G be a f.g. group. Assume that either G is finitely presented, or G has a 1-finite 1-model. Then $b_2(\mathcal{M}_1(G)) < \infty$.

EXAMPLE

- Consider the free metabelian group $G = \mathrm{F}_n / \mathrm{F}_n''$ with $n \geqslant 2$.
- We have $\mathcal{V}^1(G) = \mathcal{V}^1(\mathrm{F}_n) = (\mathbb{C}^*)^n$, and so G passes the Budur–Wang test.
- But $b_2(\mathcal{M}_1(G)) = \infty$, and so G admits no 1-finite 1-model (and is not finitely presented).

BOUNDING THE Σ AND Ω-INVARIANTS

Let $\mathcal{V}^i(X) = \bigcup_{j\leqslant i} \mathcal{V}^i_1(X)$.

THEOREM (PAPADIMA–S. 2010)

$$\Sigma^i(X,\mathbb{Z}) \subseteq S(X)\backslash S(\tau_1^{\mathbb{R}}(\mathcal{V}^i(X))).$$

EXAMPLE (KOBAN–MCCAMMOND–MEIER 2015)

$$\Sigma^1(P_n) = \mathcal{R}^1(P_n,\mathbb{R})^{\complement}.$$

Given a homogeneous variety $V \subset \mathbb{k}^n$, the set $\sigma_r(V) = \{P \in \mathrm{Gr}_r(\mathbb{k}^n) \mid P \cap V \neq \{0\}\}$ is Zariski closed.

THEOREM (S. 2012/2014)

$$\Omega^i_r(X) \subseteq \mathrm{Gr}_r(H^1(X,\mathbb{Q}))\backslash\sigma_r(\tau_1^{\mathbb{Q}}(\mathcal{V}^i(X))).$$

If the upper bound for the Σ-invariants is attained, then the upper bound for the Ω-invariants is also attained.

RESONANCE VARIETIES OF PD-ALGEBRAS

- Let A be a PD_n algebra.
- For all $0 \leqslant i \leqslant n$ and all $a \in A^1$, the square

$$\begin{array}{ccc} (A^{n-i})^* & \xrightarrow{(\delta_a^{n-i-1})^*} & (A^{n-i-1})^* \\ \mathrm{PD}\uparrow\cong & & \mathrm{PD}\uparrow\cong \\ A^i & \xrightarrow{\delta_a^i} & A^{i+1} \end{array}$$

commutes up to a sign of $(-1)^i$.

- Consequently,

$$\left(H^i(A,\delta_a)\right)^* \cong H^{n-i}(A,\delta_{-a}).$$

- Hence, for all i and s,

$$\mathcal{R}^i_s(A) = \mathcal{R}^{n-i}_s(A).$$

- In particular, $\mathcal{R}^n_1(A) = \{0\}$.

3-DIMENSIONAL POINCARÉ DUALITY ALGEBRAS

- Let A be a PD_3-algebra with $b_1(A) = n > 0$. Then
  - $\mathcal{R}^3_1(A) = \mathcal{R}^0_1(A) = \{0\}$.
  - $\mathcal{R}^2_s(A) = \mathcal{R}^1_s(A)$ for $1 \leqslant s \leqslant n$.
  - $\mathcal{R}^i_s(A) = \emptyset$, otherwise.
- Write $\mathcal{R}_s(A) = \mathcal{R}^1_s(A)$. Then
  - $\mathcal{R}_{2k}(A) = \mathcal{R}_{2k+1}(A)$ if n is even.
  - $\mathcal{R}_{2k-1}(A) = \mathcal{R}_{2k}(A)$ if n is odd.
- If μ_A has rank $n \geqslant 3$, then $\mathcal{R}_{n-2}(A) = \mathcal{R}_{n-1}(A) = \mathcal{R}_n(A) = \{0\}$.
- If $n \geqslant 4$, and $\Bbbk = \bar{\Bbbk}$, then $\dim \mathcal{R}_1(A) \geqslant \mathrm{null}(\mu_A) \geqslant 2$.
- If n is even, then $\mathcal{R}_1(A) = \mathcal{R}_0(A) = A^1$.
- If $n = 2g + 1 > 1$, then $\mathcal{R}_1(A) \neq A^1$ if and only if μ_A is "generic", i.e., $\exists c \in A^1$ such that the 2-form $\gamma_c \in \bigwedge^2 A_1$, $\gamma_c(a \wedge b) = \mu_A(a \wedge b \wedge c)$, has maximal rank, i.e., $\gamma_c^g \neq 0$ in $\bigwedge^{2g} A_1$.

THEOREM (S. 2018)

Suppose $\operatorname{rank}\gamma_c > 2$, *for all non-zero* $c \in A^1$. *Then:*

- *If* n *is odd, then* $\mathcal{R}^1_1(A)$ *is a hypersurface of degree* $(n-3)/2$ *which is smooth if* $n \leqslant 7$, *and singular in codimension* 5 *if* $n \geqslant 9$.
- *If* n *is even, then* $\mathcal{R}^1_2(A)$ *is a subvariety of codimension* 3 *and degree* $\frac{1}{4}\binom{n-1}{3} + 1$, *which is smooth if* $n \leqslant 10$, *and is singular in codimension* 7 *if* $n \geqslant 12$.

THEOREM (S. 2019)

Let M *be a closed, orientable,* 3*-dimensional manifold.*

- *If* n *is odd and* μ_M *is generic, then* $\operatorname{TC}_1(\mathcal{V}^1_1(M)) = \mathcal{R}^1_1(M)$.
- *If* n *is even, then* $\operatorname{TC}_1(\mathcal{V}^1_1(M)) = \mathcal{R}^1_1(M)$ *if and only if* $\Delta_M = 0$.

RESONANCE VARIETIES OF 3-FORMS OF LOW RANK

n	μ	$\mathcal{R}_1$
3	123	0

n	μ	$\mathcal{R}_1 = \mathcal{R}_2$	$\mathcal{R}_3$
5	125+345	$\{x_5 = 0\}$	0

n	μ	$\mathcal{R}_1$	$\mathcal{R}_2 = \mathcal{R}_3$	$\mathcal{R}_4$
6	123+456	$\mathbb{C}^6$	$\{x_1 = x_2 = x_3 = 0\} \cup \{x_4 = x_5 = x_6 = 0\}$	0
	123+236+456	$\mathbb{C}^6$	$\{x_3 = x_5 = x_6 = 0\}$	0

n	μ	$\mathcal{R}_1 = \mathcal{R}_2$	$\mathcal{R}_3 = \mathcal{R}_4$	$\mathcal{R}_5$
7	147+257+367	$\{x_7 = 0\}$	$\{x_7 = 0\}$	0
	456+147+257+367	$\{x_7 = 0\}$	$\{x_4 = x_5 = x_6 = x_7 = 0\}$	0
	123+456+147	$\{x_1 = 0\} \cup \{x_4 = 0\}$	$\{x_1 = x_2 = x_3 = x_4 = 0\} \cup \{x_1 = x_4 = x_5 = x_6 = 0\}$	0
	123+456+147+257	$\{x_1 x_4 + x_2 x_5 = 0\}$	$\{x_1 = x_2 = x_4 = x_5 = x_7^2 - x_3 x_6 = 0\}$	0
	123+456+147+257+367	$\{x_1 x_4 + x_2 x_5 + x_3 x_6 = x_7^2\}$	0	0

n	μ	$\mathcal{R}_1$	$\mathcal{R}_2 = \mathcal{R}_3$	$\mathcal{R}_4 = \mathcal{R}_5$
8	147+257+367+358	$\mathbb{C}^8$	$\{x_7 = 0\}$	$\{x_3 = x_5 = x_7 = x_8 = 0\} \cup \{x_1 = x_3 = x_4 = x_5 = x_7 = 0\}$
	456+147+257+367+358	$\mathbb{C}^8$	$\{x_5 = x_7 = 0\}$	$\{x_3 = x_4 = x_5 = x_7 = x_1 x_8 + x_6^2 = 0\}$
	123+456+147+358	$\mathbb{C}^8$	$\{x_1 = x_5 = 0\} \cup \{x_3 = x_4 = 0\}$	$\{x_1 = x_3 = x_4 = x_5 = x_2 x_6 + x_7 x_8 = 0\}$
	123+456+147+257+358	$\mathbb{C}^8$	$\{x_1 = x_5 = 0\} \cup \{x_3 = x_4 = x_5 = 0\}$	$\{x_1 = x_2 = x_3 = x_4 = x_5 = x_7 = 0\}$
	123+456+147+257+367+358	$\mathbb{C}^8$	$\{x_3 = x_5 = x_1 x_4 - x_7^2 = 0\}$	$\{x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = x_7 = 0\}$
	147+268+358	$\mathbb{C}^8$	$\{x_1 = x_4 = x_7 = 0\} \cup \{x_8 = 0\}$	$\{x_1 = x_4 = x_7 = x_8 = 0\} \cup \{x_2 = x_3 = x_5 = x_6 = x_8 = 0\}$
	147+257+268+358	$\mathbb{C}^8$	$L_1 \cup L_2 \cup L_3$	$L_1 \cup L_2$
	456+147+257+268+358	$\mathbb{C}^8$	$C_1 \cup C_2$	$L_1 \cup L_2$
	147+257+367+268+358	$\mathbb{C}^8$	$L_1 \cup L_2 \cup L_3 \cup L_4$	$L'_1 \cup L'_2 \cup L'_3$
	456+147+257+367+268+358	$\mathbb{C}^8$	$C_1 \cup C_2 \cup C_3$	$L_1 \cup L_2 \cup L_3$
	123+456+147+268+358	$\mathbb{C}^8$	$C_1 \cup C_2$	L
	123+456+147+257+268+358	$\mathbb{C}^8$	$\{f_1 = \cdots = f_{20} = 0\}$	0
	123+456+147+257+367+268+358	$\mathbb{C}^8$	$\{g_1 = \cdots = g_{20} = 0\}$	0

References

- G. Denham, A.I. Suciu, and S. Yuzvinsky, *Abelian duality and propagation of resonance*, Selecta Math. **23** (2017), no. 4, 2331–2367.
- G. Denham and A.I. Suciu, *Local systems on arrangements of smooth, complex algebraic hypersurfaces*, Forum of Mathematics, Sigma **6** (2018), e6, 20 pages.
- S. Papadima and A.I. Suciu, *Infinitesimal finiteness obstructions*, J. London Math. Soc. **99** (2019), no. 1, 173–193.
- A.I. Suciu, *Poincaré duality and resonance varieties*, arXiv:1809.01801.
- A.I. Suciu, *Cohomology jump loci of* 3*-manifolds*, arXiv:1901.01419.