# Algebraic models for Sasakian manifolds and weighted-homogeneous surface singularities

Alex Suciu

Northeastern University

Joint work with Stefan Papadima (IMAR)

arxiv:1511.08948

Workshop on Singularities and Topology

Université de Nice

March 9, 2016

# CDGAs

- Let $A = (A^{\bullet}, \mathrm{d})$ be a commutative, differential graded $\mathbb{C}$-algebra.
  - Multiplication $\cdot : A^i \otimes A^j \to A^{i+j}$ is graded-commutative.
  - Differential $\mathrm{d} : A^i \to A^{i+1}$ satisfies the graded Leibniz rule.
- Fix integer $q \geqslant 1$ (or $q = \infty$). We say that $A$ is *$q$-finite* if
  - $A$ is connected, i.e., $A^0 = \mathbb{C}$.
  - $A^i$ is finite-dimensional, for each $i \leqslant q$ (or $i < \infty$).
- Two CDGAs have the same $q$-type if there is a zig-zag of connecting morphisms, each one inducing isomorphisms in homology up to degree $q$ and a monomorphism in degree $q + 1$.
- A CDGA $(A, d)$ is *$q$-formal* if it has the same $q$-type as $(H^{\bullet}(A), d = 0)$.

# REGULAR SEQUENCES

- Let $H^\bullet$ be a connected, commutative graded algebra.
- A sequence $\{e_\alpha\}$ of homogeneous elements in $H^{>0}$ is said to be *q-regular* if for each $\alpha$, the class of $e_\alpha$ in
$$\overline{H}_\alpha = H/\sum_{\beta<\alpha} e_\beta H$$
is a non-zero divisor up to degree $q$. That is, the multiplication map $\bar{e}_\alpha\cdot\colon \overline{H}^i_\alpha \to \overline{H}^{i+n_\alpha}_\alpha$ is injective, for all $i \leqslant q$, where $n_\alpha = \deg(e_\alpha)$.

**THEOREM**

*Suppose $e_1,\dots,e_r$ is an even-degree, $q$-regular sequence in $H$. Then the Hirsch extension $A = H \otimes_\tau \bigwedge(t_1,\dots,t_r)$ with $d = 0$ on $H$ and $dt_\alpha = \tau(t_\alpha) = e_\alpha$ has the same $q$-type as*
$$\Big(H/\textstyle\sum_\alpha e_\alpha H, d = 0\Big).$$
*In particular, $A$ is $q$-formal.*

# THE SULLIVAN MODEL

- To a large extent, the rational homotopy type of a space can be reconstructed from algebraic models associated to it.
- If the space is a smooth manifold $M$, the standard model is the de Rham algebra $\Omega_{\mathrm{dR}}(M)$.
- More generally, any space $X$ has an associated Sullivan CDGA, $A_{\mathrm{PL}}(X)$, which serves as the reference algebraic model. In particular, $H^*(A_{\mathrm{PL}}(X)) = H^*(X, \mathbb{C})$.
- A CDGA $(A, d)$ is a $q$-*model* for $X$ if $A$ has the same $q$-type as $A_{\mathrm{PL}}(X)$.
- We say $X$ is $q$-*formal* if $A_{\mathrm{PL}}(X)$ has this property.

# THE MALCEV LIE ALGEBRA

- The $1$-formality property of a connected CW-complex $X$ with finite $1$-skeleton depends only on its fundamental group, $\pi = \pi_1(X)$.
- Let $\mathfrak{m}(\pi) = \text{Prim}(\widehat{\mathbb{C}[\pi]})$ be the Malcev Lie algebra of $\pi$, where $\widehat{\ }$ is completion with respect to powers of the augmentation ideal.
- The $1$-formality of the group $\pi$ is equivalent to

$$\mathfrak{m}(\pi) \cong \widehat{L},$$

for some quadratic, finitely generated Lie algebra $L$, where $\widehat{\ }$ is the degree completion.

- If $\mathfrak{m}(\pi) \cong \widehat{L}$, where $L$ is merely assumed to have homogeneous relations, then $\pi$ is said to be *filtered formal* (see [SW] for details).

# COHOMOLOGY JUMP LOCI

- Let $X$ be a connected, finite-type CW-complex, and let $G$ be a complex linear algebraic group.
- The *characteristic varieties* of $X$ with respect to a rational, finite-dimensional representation $\varphi\colon G \to \mathrm{GL}(V)$ are the sets

$$\mathcal{V}^i_s(X,\varphi) = \Big\{\rho \in \mathrm{Hom}(\pi, G) \mid \dim H^i(X, V_{\varphi\circ\rho}) \geqslant s\Big\}.$$

- In degree $i=1$, these varieties depend only on the group $\pi = \pi_1(X)$, and so we may denote them as $\mathcal{V}^1_s(\pi,\varphi)$.
- When $G = \mathbb{C}^*$ and $\varphi = \mathrm{id}_{\mathbb{C}^*}$, we simply write these sets as $\mathcal{V}^i_s(X)$.

# COMPACT LIE GROUP ACTIONS

- Let $M$ be a compact, connected, smooth manifold (with $\partial M = \varnothing$).
- Suppose a compact, connected Lie group $K$ acts smoothly and *almost freely* on $M$ (i.e., all the isotropy groups are finite).
- Let $K \to EK \times M \to M_K$ be the Borel construction on $M$.
- Let $\tau\colon H^{\bullet}(K,\mathbb{C}) \to H^{\bullet+1}(M_K,\mathbb{C})$ be the transgression in the Serre spectral sequence of this bundle.
- Let $N = M/K$ be the orbit space (a smooth orbifold).
- The projection map $\mathrm{pr}\colon M_K \to N$ induces an isomorphism $\mathrm{pr}^*\colon H^{\bullet}(N,\mathbb{C}) \to H^{\bullet}(M_K,\mathbb{C})$.
- By a theorem of H. Hopf, we may dentify $H^{\bullet}(K,\mathbb{C}) = \bigwedge P^{\bullet}$, where $P = \mathrm{span}\{t_1,\dots,t_r\}$ where $m_\alpha = \deg(t_\alpha)$ is odd.

### THEOREM

*There is a map* $\sigma\colon P^\bullet \to Z^{\bullet+1}(A_{\mathrm{PL}}(N))$ *such that* $\mathrm{pr}^* \circ [\sigma] = \tau$ *and*

$$A_{\mathrm{PL}}(M) \simeq A_{\mathrm{PL}}(N) \otimes_\sigma \bigwedge P.$$

### THEOREM

*Suppose that*

- *The orbit space* $N = M/K$ *is* $k$*-formal, for some* $k > \max\{m_\alpha\}$.
- *The characteristic classes* $e_\alpha = (\mathrm{pr}^*)^{-1}(\tau(t_\alpha)) \in H^{m_\alpha+1}(N, \mathbb{C})$ *form a* $q$*-regular sequence in* $H^\bullet = H^\bullet(N, \mathbb{C})$, *for some* $q \leqslant k$.

*Then the* CDGA

$$\Big(H^\bullet / \sum_\alpha e_\alpha H^\bullet, d = 0\Big)$$

*is a finite* $q$*-model for* $M$*. In particular,* $M$ *is* $q$*-formal.*

### THEOREM

*Suppose the orbit space $N = M/K$ is $2$-formal. Then:*

1. *The group $\pi = \pi_1(M)$ is filtered-formal. In fact, $\mathfrak{m}(\pi)$ is the degree completion of $\mathbb{L}/\mathfrak{r}$, where $\mathbb{L} = \mathrm{Lie}(H_1(\pi, \mathbb{C}))$ and $\mathfrak{r}$ is a homogeneous ideal generated in degrees $2$ and $3$.*
2. *For every complex linear algebraic group $G$, the germ at $1$ of the representation variety $\mathrm{Hom}(\pi, G)$ is defined by quadrics and cubics only.*

The projection map $p\colon M \to M/K$ induces an epimorphism $p_\sharp\colon \pi_1(M) \twoheadrightarrow \pi_1^{\text{orb}}(M/K)$ between orbifold fundamental groups.

### THEOREM

*Suppose that the transgression $P^\bullet \to H^{\bullet+1}(M/K,\mathbb{C})$ is injective in degree $1$. Then:*

1. *If the orbit space $N = M/K$ has a $2$-finite $2$-model, then $p_\sharp$ induces analytic isomorphisms*

$$\mathcal{V}_s^1(\pi_1^{\text{orb}}(N))_{(1)} \cong \mathcal{V}_s^1(\pi_1(M))_{(1)}.$$

2. *If $N$ is $2$-formal, then $p_\sharp$ induces an analytic isomorphism*

$$\text{Hom}(\pi_1^{\text{orb}}(N),\text{SL}_2(\mathbb{C}))_{(1)} \cong \text{Hom}(\pi_1(M),\text{SL}_2(\mathbb{C}))_{(1)}.$$

# SASAKIAN MANIFOLDS AND $q$-FORMALITY

- Sasakian geometry is an odd-dimensional analogue of Kähler geometry.
- Every compact Sasakian manifold $M$ admits an almost-free circle action with orbit space $N = M/S^1$ a Kähler orbifold.
- The Euler class of the action coincides with the Kähler class of the base, $h \in H^2(N, \mathbb{Q})$.
- The class $h$ satisfies the Hard Lefschetz property, i.e., $\cdot h^k \colon H^{n-k}(N, \mathbb{C}) \to H^{n+k}(N, \mathbb{C})$ is an isomorphism, for each $1 \leqslant k \leqslant n$. Thus, $\{h\}$ is an $(n-1)$-regular sequence in $H^\bullet(N, \mathbb{C})$

**EXAMPLE**

Let $N$ be a compact Kähler manifold such that the Kähler class is integral, i.e., $h \in H^2(N, \mathbb{Z})$, and let $M$ be the total space of the principal $S^1$-bundle classified by $h$. Then $M$ is a (regular) Sasakian manifold.

- As shown by Deligne, Griffiths, Morgan, and Sullivan, compact Kähler manifolds are formal.
- As shown by A. Tievsky, every compact Sasakian manifold $M$ has a finite model of the form $(H^\bullet(N,\mathbb{C}) \otimes \bigwedge(t), d)$, where $d$ vanishes on $H^\bullet(N,\mathbb{C})$ and sends $t$ to $h$.

**THEOREM**

*Let $M$ be a compact Sasakian manifold of dimension $2n+1$. Then $M$ is $(n-1)$-formal.*

- This result is optimal: for each $n \geqslant 1$, the $(2n+1)$-dimensional Heisenberg compact nilmanifold (with orbit space $T^{2n}$) is a Sasakian manifold, yet it is not $n$-formal.
- This theorem strengthens a statement of H. Kasuya, who claimed that, for $n \geqslant 2$, a Sasakian manifold $M^{2n+1}$ is $1$-formal. The proof of that claim, though, has a gap.

# SASAKIAN GROUPS

- A group $\pi$ is said to be a *Sasakian group* if it can be realized as the fundamental group of a compact, Sasakian manifold.
- Open problem: Which finitely presented groups are Sasakian?
- A first, well-known obstruction is that $b_1(\pi)$ must be even.

**THEOREM**

*Let $\pi = \pi_1(M^{2n+1})$ be the fundamental group of a compact Sasakian manifold of dimension $2n+1$. Then:*

1. *The group $\pi$ is filtered-formal, and in fact $1$-formal if $n > 1$.*
2. *All irreducible components of the characteristic varieties $\mathcal{V}^1_s(\pi)$ passing through $1$ are algebraic subtori of $\mathrm{Hom}(\pi, \mathbb{C}^*)$.*
3. *If $G$ is a complex linear algebraic group, then the germ at $1$ of $\mathrm{Hom}(\pi, G)$ is defined by quadrics and cubics only, and in fact by quadrics only if $n > 1$.*

# RESONANCE VARIETIES

- Once again, let $(A, d)$ be a CDGA model for a connected, finite-type CW-complex $X$. Let $\pi = \pi_1(X)$, let $G$ be a complex algebraic group, and let $\mathfrak{g}$ be its Lie algebra.
- The infinitesimal analogue (around the origin) of the $G$-representation variety $\mathrm{Hom}(\pi, G)$ is the set $\mathcal{F}(A, \mathfrak{g})$ of $\mathfrak{g}$-valued flat connections on a CDGA $A$,

$$\mathcal{F}(A, \mathfrak{g}) = \Big\{\omega \in A^1 \otimes \mathfrak{g} \mid d\omega + \tfrac{1}{2}[\omega, \omega] = 0\Big\}.$$

- If $\dim A^1 < \infty$, then $\mathcal{F}(A, \mathfrak{g})$ is a Zariski-closed subset, which contains the closed subvariety

$$\mathcal{F}^1(A, \mathfrak{g}) = \{\eta \otimes g \in A^1 \otimes \mathfrak{g} \mid d\eta = 0\}.$$

- Next, we define the infinitesimal counterpart of the characteristic varieties.

- Let $\theta\colon \mathfrak{g} \to \mathfrak{gl}(V)$ be a finite-dimensional representation.
- To each $\omega \in \mathcal{F}(A, \mathfrak{g})$ there is an associated covariant derivative,

$$d_\omega\colon A^\bullet \otimes V \to A^{\bullet+1} \otimes V,$$

given by $d_\omega = d \otimes \mathrm{id}_V + \mathrm{ad}_\omega$. By flatness, $d_\omega^2 = 0$.

- The *resonance varieties* of $A$ with respect to $\theta$ are the sets

$$\mathcal{R}^i_s(A, \theta) = \left\{\omega \in \mathcal{F}(A, \mathfrak{g}) \mid \dim H^i(A \otimes V, d_\omega) \geqslant s\right\}.$$

If $A$ is $q$-finite, these sets are Zariski-closed in $\mathcal{F}(A, \mathfrak{g})$, $\forall i \leqslant q$.

- If $H^i(A) \neq 0$, then $\mathcal{R}^i_1(A, \theta)$ contains the closed subvariety

$$\Pi(A, \theta) = \{\eta \otimes g \in \mathcal{F}^1(A, \mathfrak{g}) \mid \det \theta(g) = 0\}.$$

- When $\mathfrak{g} = \mathbb{C}$, $\theta = \mathrm{id}_{\mathbb{C}}$ we have that $\mathcal{F}(A, \mathfrak{g}) = H^1(A)$ and $\mathcal{R}^i_s(A)$ are the usual resonance varieties of $(A, d)$.

# SMOOTH QUASI-PROJECTIVE VARIETIES

- Let $X$ be a smooth quasi-projective variety.
- Let $\mathcal{E}(X)$ be the (finite!) set of regular, surjective maps $f\colon X \to S$ for which the generic fiber is connected and is a smooth curve $S$ with $\chi(S) < 0$, up to reparametrization at the target.
- All such maps extend to regular maps $\bar{f}\colon \overline{X} \to \overline{S}$, for some 'convenient' compactification $\overline{X} = X \cup D$.
- The variety $X$ admits a finite CDGA model with positive weights, $A(X) = A(\overline{X}, D)$. Such a 'Gysin' model was constructed by Morgan, and was recently improved upon by C. Dupont.

### THEOREM (ARAPURA)

*The correspondence $f \rightsquigarrow f^*(H^1(S,\mathbb{C}^*))$ gives a bijection between the set $\mathcal{E}(X)$ and the set of positive-dimensional irreducible components of $\mathcal{V}^1_1(X)$ passing through the identity of the character group $H^1(X,\mathbb{C}^*)$.*

### THEOREM (DIMCA–PAPADIMA)

*Let $X$ be smooth, quasi-projective variety $X$, and let $A$ be a finite CDGA model with positive weights. The set $\mathcal{E}(X)$ is then in bijection with the set of positive-dimensional, irreducible components of $\mathcal{R}^1_1(A) \subseteq H^1(A) = H^1(X,\mathbb{C})$ via the correspondence $f \rightsquigarrow f^!(H^1(S,\mathbb{C}))$.*

# WEIGHTED HOMOGENEOUS SINGULARITIES

- Let $X$ be a complex affine surface endowed with a 'good' $\mathbb{C}^*$-action and having a normal, isolated singularity at $0$.
- The punctured surface $X^* = X \setminus \{0\}$ is a smooth quasi-projective variety which deform-retracts onto the singularity link, $M$.
- The almost free $\mathbb{C}^*$-action on $X^*$ restricts to an $S^1$-action on $M$ with finite isotropy subgroups. In particular, $M$ is an orientable Seifert fibered $3$-manifold.
- The orbit space, $M/S^1 = X^*/\mathbb{C}^*$, is a smooth projective curve $\Sigma_g$, of genus $g = \frac{1}{2} b_1(M)$. The canonical projection, $f\colon X^* \to X^*/\mathbb{C}^*$, induces an isomorphism on first homology.
- It turns out that $\mathcal{E}(X^*) = \emptyset$ if $g = 1$ and $\mathcal{E}(X^*) = \{f\}$ if $g > 1$.

### THEOREM

*Let* $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, *and let* $\theta\colon \mathfrak{g} \to \mathfrak{gl}(V)$ *be a finite-dimensional rep. There is then a convenient compactification of* $X^*$ *such that*

$$\mathcal{F}(A(X^*), \mathfrak{g}) = \mathcal{F}^1(A(X^*), \mathfrak{g}) \cup \bigcup_{f \in \mathcal{E}(X^*)} f^!(\mathcal{F}(A(S), \mathfrak{g})),$$

$$\mathcal{R}^1_1(A(X^*), \theta) = \Pi(A(X^*), \theta) \cup \bigcup_{f \in \mathcal{E}(X^*)} f^!(\mathcal{F}(A(S), \mathfrak{g})).$$

- For the proof, we replace $X^*$ (up to homotopy) by the singularity link $M$, and $A(X^*)$ by a finite model $A$ for this almost free $S^1$-manifold.
- As shown in [MPPS], the inclusions $\supseteq$ hold for arbitrary smooth, quasi-projective varieties $X$, with equality if $X$ is $1$-formal.
- We do not know (yet) whether the above equalities hold for arbitrary smooth, quasi-projective varieties.

# REFERENCES


- Stefan Papadima and Alex Suciu, *The topology of compact Lie group actions through the lens of finite models*, arxiv:1511.08948.

- Anca Măcinic, Stefan Papadima, Radu Popescu, and Alex Suciu, *Flat connections and resonance varieties: from rank one to higher ranks*, arxiv:1312.1439, to appear in Trans. Amer. Math. Soc.

- Alex Suciu and He Wang, *Formality properties of finitely generated groups and Lie algebras*, arxiv:1504.08294.