Representation varieties and polyhedral products

Alex Suciu

Northeastern University

Special Session

Representation Spaces and Toric Topology

AMS Spring Eastern Sectional Meeting

Hunter College, New York City

May 6, 2017

REFERENCES

[MPPS17] Anca Măcinic, Stefan Papadima, Radu Popescu, and Alex Suciu, *Flat connections and resonance varieties: from rank one to higher ranks*, Trans. Amer. Math. Soc. **369** (2017), no. 2, 1309–1343.

[PS16] Stefan Papadima and Alex Suciu, *Naturality properties and comparison results for topological and infinitesimal embedded jump loci*, arxiv:1609.02768.

OVERVIEW

- The study of analytic germs of G-representation varieties and cohomology jump loci is a basic problem in deformation theory with homological constraints.
- Building on work of Goldman–Millson [1988], it was shown by Dimca–Papadima [2014] that the germs at the origin of those loci are isomorphic to the germs at the origin of infinitesimal jump loci of a CDGA that is a finite model for the space in question.
- Budur and Wang [2015] have extended this result away from the origin, by developing a theory of differential graded Lie algebra modules which control the corresponding deformation problem.

- The universality theorem of Kapovich and Millson [1998] shows that $\mathsf{SL}_2(\mathbb{C})$-representation varieties of Artin groups may have arbitrarily bad singularities away from 1.

- This lead us to focus on germs at the origin of the representation varieties $\mathrm{Hom}(\pi, G)$, and look for explicit descriptions via infinitesimal CDGA methods.

- This approach works very well in the case when $G = \mathrm{SL}(2, \mathbb{C})$ or one of its standard subgroups, and π is a right-angled Artin group, that is, the fundamental group of a polyhedral product of the form $\mathcal{Z}_\Gamma(S^1, *)$, for some finite simplicial graph Γ.

REPRESENTATION VARIETIES

- Let π be a finitely generated group.
- Let G be a complex, linear algebraic group.
- The set $\mathrm{Hom}(\pi, G)$ has a natural structure of an affine variety, called the *G-representation variety of π*.
- Every homomorphism $\varphi\colon \pi \to \pi'$ induces an algebraic morphism, $\varphi^!\colon \mathrm{Hom}(\pi', G) \to \mathrm{Hom}(\pi, G)$.
- Example: $\mathrm{Hom}(F_n, G) = G^n$.
- $\mathrm{Hom}(\mathbb{Z}^2, \mathrm{GL}_k(\mathbb{C}))$ is irreducible, but relatively little is known about the varieties of commuting matrices, $\mathrm{Hom}(\mathbb{Z}^n, \mathrm{GL}_k(\mathbb{C}))$.
- The varieties $\mathrm{Hom}(\pi_1(\Sigma_g), G)$ are connected if $G = \mathrm{SL}_k(\mathbb{C})$, and irreducible if $G = \mathrm{GL}_k(\mathbb{C})$.

COHOMOLOGY JUMP LOCI

- Let (X, x_0) be a pointed, path-connected space, and assume $\pi = \pi_1(X, x_0)$ is finitely generated.
- The *characteristic varieties* of X with respect to a representation $\iota\colon G \to \mathrm{GL}(V)$ are the sets

$$\mathcal{V}^i_r(X, \iota) = \{\rho \in \mathrm{Hom}(\pi, G) \mid \dim_{\mathbb{C}} H^i(X, V_{\iota\circ\rho}) \geqslant r\}.$$

- For all $i \geqslant 0$, these sets form a descending filtration of $\mathrm{Hom}(\pi, G)$.
- The pairs $\big(\mathrm{Hom}(\pi, G), \mathcal{V}^i_r(X, \iota)\big)$ depend only on the homotopy type of X and on the representation ι.
- If X is a finite-type CW-complex, and ι is a rational representation, then the sets $\mathcal{V}^i_r(X, \iota)$ are closed subvarieties of $\mathrm{Hom}(\pi, G)$.

FLAT CONNECTIONS

- The infinitesimal analogue of the G-representation variety is

$$F(A, \mathfrak{g}),$$

  the set of $\mathfrak{g}$-valued flat connections on a commutative, differential graded $\mathbb{C}$-algebra $(A^\bullet, d)$, where $\mathfrak{g}$ is a Lie algebra.

- This set consists of all elements $\omega \in A^1 \otimes \mathfrak{g}$ which satisfy the Maurer–Cartan equation,

$$d\omega + \tfrac{1}{2}[\omega, \omega] = 0.$$

- If A^1 and $\mathfrak{g}$ are finite dimensional, then $F(A, \mathfrak{g})$ is a Zariski-closed subset of the affine space $A^1 \otimes \mathfrak{g}$.

HOLONOMY LIE ALGEBRA

- Let $A = (A^{\bullet}, d)$ be a connected CDGA with $\dim A^1 < \infty$, and set $A_i = (A^i)^*$.
- The *holonomy Lie algebra* $\mathfrak{h}(A)$ is the quotient of the free Lie algebra $\mathbb{L}(A_1)$ by the ideal generated by the image of the map

$$\partial_A := d^* + \mu^* \colon A_2 \to \mathbb{L}^1(A_1) \oplus \mathbb{L}^2(A_1) \subset \mathbb{L}(A_1),$$

  where $d^* \colon A_2 \to A_1 = \mathbb{L}^1(A_1)$ and $\mu^* \colon A_2 \to A_1 \wedge A_1 = \mathbb{L}^2(A_1)$.
- Functoriality: if $\varphi \colon A \to A'$ is a CDGA map, then the linear map $\varphi_1 = (\varphi^1)^* \colon A'_1 \to A_1$ extends to a Lie map $\mathbb{L}(\varphi_1) \colon \mathbb{L}(A'_1) \to \mathbb{L}(A_1)$, which in turn induces a Lie algebra map $\mathfrak{h}(\varphi) \colon \mathfrak{h}(A') \to \mathfrak{h}(A)$.

PROPOSITION (MPPS 2017)

The canonical isomorphism $A^1 \otimes \mathfrak{g} \cong \mathrm{Hom}(A_1, \mathfrak{g})$ restricts to an identification $\mathcal{F}(A, \mathfrak{g}) \cong \mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{h}(A), \mathfrak{g})$.

INFINITESIMAL COHOMOLOGY JUMP LOCI

- For each $\omega \in \mathcal{F}(A, \mathfrak{g})$, we turn $A \otimes V$ into a cochain complex,

$$(A \otimes V, d_\omega)\colon \ A^0 \otimes V \xrightarrow{d_\omega} A^1 \otimes V \xrightarrow{d_\omega} A^2 \otimes V \xrightarrow{d_\omega} \cdots ,$$

  using as differential the covariant derivative $d_\omega = d \otimes \mathrm{id}_V + \mathrm{ad}_\omega$. (The flatness condition on ω insures that $d_\omega^2 = 0$.)

- The *resonance varieties* of the CDGA $(A^\bullet, d)$ with respect to a representation $\theta\colon \mathfrak{g} \to \mathfrak{gl}(V)$ are the sets

$$\mathcal{R}^i_r(A, \theta) = \{\omega \in \mathcal{F}(A, \mathfrak{g}) \mid \dim_{\mathbb{C}} H^i(A \otimes V, d_\omega) \geqslant r\}.$$

- For each $i \geqslant 0$, these sets form a descending filtration of $\mathcal{F}(A, \mathfrak{g})$.
- If A, $\mathfrak{g}$, and V are all finite-dimensional, the sets $\mathcal{R}^i_r(A, \theta)$ are closed subvarieties of $\mathcal{F}(A, \mathfrak{g})$.

- Let $\mathcal{F}^1(A,\mathfrak{g}) = \{\eta \otimes g \in \mathcal{F}(A,\mathfrak{g}) \mid d\eta = 0\}$.
- Let $\Pi(A,\theta) = \{\eta \otimes g \in \mathcal{F}^1(A,\mathfrak{g}) \mid \det(\theta(g)) = 0\}$.
- For $\mathfrak{g} = \mathbb{C}$, we have $\mathcal{F}(A,\mathfrak{g}) \cong H^1(A)$. Also, for $\theta = \mathrm{id}_{\mathbb{C}}$, we get the usual resonance varieties $\mathcal{R}^i_r(A)$.
- In this rank 1 case, $\mathcal{F}^1(A,\mathbb{C}) = \mathcal{F}(A,\mathbb{C})$ and $\Pi(A,\theta) = \{0\}$.

THEOREM (MPPS 2017)

Let $\omega = \eta \otimes g \in \mathcal{F}^1(A,\mathfrak{g})$. Then ω belongs to $\mathcal{R}^k_1(A,\theta)$ if and only if there is an eigenvalue λ of $\theta(g)$ such that $\lambda\eta$ belongs to $\mathcal{R}^k_1(A)$. Moreover,

$$\Pi(A,\theta) \subseteq \bigcap_{k : H^k(A) \neq 0} \mathcal{R}^k_1(A,\theta).$$

ALGEBRAIC MODELS FOR SPACES

- From now on, X will be a connected space having the homotopy type of a finite CW-complex.
- Let $A_{\mathrm{PL}}(X)$ be the Sullivan CDGA of piecewise polynomial $\mathbb{C}$-forms on X. Then $H^\bullet(A_{\mathrm{PL}}(X)) \cong H^\bullet(X,\mathbb{C})$.
- A CDGA (A,d) is a *model* for X if it may be connected by a zig-zag of quasi-isomorphisms to $A_{\mathrm{PL}}(X)$.
- A is a *finite* model if $\dim_{\mathbb{C}} A < \infty$ and A is connected.
- X is *formal* if $(H^\bullet(X,\mathbb{C}), \mathrm{d}=0)$ is a (finite) model for X.
  - E.g.: Compact Kähler manifolds, complements of hyperplane arrangments.
- Thus, if X is formal, then $H^\bullet(X,\mathbb{C})$ is a finite model for X.
  - Converse not true. E.g.: all nilmanifolds, solvmanifolds, Sasakian manifolds, smooth quasi-projective varieties, etc, admit finite models, but many are non-formal.

GERMS OF JUMP LOCI

THEOREM (DIMCA–PAPADIMA 2014)

Suppose X admits a finite CDGA model A. Let $\iota\colon G \to \mathrm{GL}(V)$ be a rational representation, and $\theta\colon \mathfrak{g} \to \mathfrak{gl}(V)$ its tangential representation. There is then an analytic isomorphism of germs,

$$\mathcal{F}(A, \mathfrak{g})_{(0)} \xrightarrow{\simeq} \mathrm{Hom}(\pi_1(X), G)_{(1)},$$

restricting to isomorphisms $\mathcal{R}^i_r(A, \theta)_{(0)} \xrightarrow{\simeq} \mathcal{V}^i_r(X, \iota)_{(1)}$ for all i, r.

Rank 1 case:

- For $G = \mathbb{C}^*$, the representation variety $\mathrm{Hom}(\pi, \mathbb{C}^*) = H^1(X, \mathbb{C}^*)$ is the character group of $\pi = \pi_1(X)$.
- For $\iota\colon \mathbb{C}^* \xrightarrow{\simeq} \mathrm{GL}_1(\mathbb{C})$ and $V = \mathbb{C}$, we get the usual characteristic varieties, $\mathcal{V}^i_r(X)$

- The local analytic isomorphism $H^1(A)_{(0)} \xrightarrow{\simeq} \mathrm{Hom}(\pi_1(X), \mathbb{C}^*)_{(1)}$ is induced by the exponential map $H^1(X, \mathbb{C}) \to H^1(X, \mathbb{C}^*)$.

THEOREM (DIMCA–PAPADIMA 2014, MPPS 2017)

If (A, d) is a finite CDGA such that $A_{\mathrm{PL}}(X) \simeq A$, then

$$\mathrm{TC}_0(\mathcal{R}^i_r(A)) \subseteq \mathcal{R}^i_r(H^\bullet(A)).$$

Moreover, if (A, d) is rationally defined, with positive weights, and $A_{\mathrm{PL}}(X) \simeq A$ over $\mathbb{Q}$, then each $\mathcal{R}^i_r(A)$ is a finite union of rationally defined linear subspaces of $H^1(A)$, and $\mathcal{R}^i_r(A) \subseteq \mathcal{R}^i_r(H^\bullet(A))$.

THEOREM (BUDUR–WANG 2017)

If X admits a finite CDGA model A, then all the components of the characteristic varieties $\mathcal{V}^i_r(X)$ passing through 1 are algebraic subtori.

LINEAR RESONANCE

- Suppose $\mathcal{R}^1_1(A) = \bigcup_{C\in\mathcal{C}} C$, a finite union of linear subspaces.
- Let A_C denote the sub-CDGA of the truncation $A^{\leqslant 2}$ defined by $A^1_C = C$ and $A^2_C = A^2$.

THEOREM (MPPS 2017)

For any Lie algebra $\mathfrak{g}$,

$$\mathcal{F}(A,\mathfrak{g}) \supseteq \mathcal{F}^1(A,\mathfrak{g}) \cup \bigcup_{0\neq C\in\mathcal{C}} \mathcal{F}(A_C,\mathfrak{g}), \qquad (*)$$

where each $\mathcal{F}(A_C,\mathfrak{g})$ is Zariski-closed in $\mathcal{F}(A,\mathfrak{g})$. Moreover, if A has zero differential, and $\mathfrak{g} = \mathfrak{sl}_2$ or $\mathfrak{sol}_2$, then (*) holds as an equality, and

$$\mathcal{R}^1_1(A,\theta) = \Pi(A,\theta) \cup \bigcup_{0\neq C\in\mathcal{C}} \mathcal{F}(A_C,\mathfrak{g}).$$

(For $\mathfrak{g} = \mathfrak{sl}_2$ or $\mathfrak{sol}_2$: if $g, g' \in \mathfrak{g}$, then $[g,g'] = 0$ if and only if $\operatorname{rank}\{g,g'\} \leqslant 1$.)

TORIC COMPLEXES

- Let K be simplicial complex on n vertices.
- Let $T_K = \mathcal{Z}_K(S^1, *)$ be the subcomplex of T^n obtained by deleting the cells corresponding to the missing simplices of K.
- T_K is a minimal CW-complex, of dimension $\dim K + 1$.
- T_K is formal (Notbohm and Ray, 2005).
- The cohomology algebra $A_K = H^*(T_K, \mathbb{C})$ is isomorphic to the *exterior Stanley–Reisner ring* of K.

RIGHT ANGLED ARTIN GROUPS

- The fundamental group $\pi_\Gamma = \pi_1(T_K)$ is the RAAG associated to the graph $\Gamma := K^{(1)} = (\mathsf{V}, \mathsf{E})$,

$$\pi_\Gamma = \langle v \in V \mid [v, w] = 1 \text{ if } \{v, w\} \in E \rangle.$$

- Then $K(\pi_\Gamma, 1) = T_{\Delta_\Gamma}$, where Δ_Γ is the flag complex of Γ.

- The holonomy Lie algebra associated to $(A_\Gamma, d = 0)$ has presentation

$$\mathfrak{h}(\Gamma) = \mathbb{L}(\mathsf{V})/([v, w] = 0 \text{ if } \{v, w\} \in \mathsf{E}).$$

RESONANCE VARIETIES

Identify $H^1(T_K, \mathbb{C}) = \mathbb{C}^{\mathsf{V}}$, the $\mathbb{C}$-vector space with basis $\{v \mid v \in \mathsf{V}\}$.

THEOREM (PAPADIMA–S. 2009)

$$\mathcal{R}^i_r(T_K) = \bigcup_{\substack{\mathsf{W}\subset\mathsf{V} \\ \sum_{\sigma\in K_{\mathsf{V}\setminus\mathsf{W}}} \dim_{\Bbbk} \widetilde{H}_{i-1-|\sigma|}(\mathrm{lk}_{K_{\mathsf{W}}}(\sigma),\Bbbk)\geqslant r}} \mathbb{C}^{\mathsf{W}},$$

where K_{W} is the subcomplex induced by K on W, and $\mathrm{lk}_L(\sigma)$ is the link of a simplex σ in a subcomplex $L \subseteq K$.

In particular (PS 2006): $\mathcal{R}^1_1(\pi_\Gamma) = \bigcup_{\substack{\mathsf{W}\subseteq\mathsf{V} \\ \Gamma_{\mathsf{W}} \text{ disconnected}}} \mathbb{C}^{\mathsf{W}}$.

Similar formulas for the characteristic varieties $\mathcal{V}^i_r(T_K)$.

FLAT CONNECTIONS

- Let $A = (A^{\bullet}_{\Gamma}, d = 0)$, and $\mathfrak{g}$ a finite-dimensional Lie algebra.
- The isomorphism $\mathbb{C}^{\mathsf{V}} \otimes \mathfrak{g} \cong \mathrm{Hom}(\mathbb{C}^{\mathsf{V}}, \mathfrak{g})$ induces an iso $\mathcal{F}(A, \mathfrak{g}) \cong \mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{h}(\Gamma), \mathfrak{g})$.
- View $\omega \in \mathbb{C}^{\mathsf{V}} \otimes \mathfrak{g}$ as a tuple of elements $\omega_v \in \mathfrak{g}$, indexed by $v \in \mathsf{V}$. Then $\omega \in \mathcal{F}(A, \mathfrak{g})$ if and only if $[\omega_u, \omega_v] = 0$ for all $\{u, v\} \in \mathsf{E}$.
- For each subset $\mathsf{W} \subseteq \mathsf{V}$, let $\mathsf{W}_1, \dots, \mathsf{W}_c$ be the connected components of the vertex set of Γ_{W}, let $\overline{\mathsf{W}} = \mathsf{V} \backslash \mathsf{W}$, and put

$$S_{\mathsf{W}} = \left\{ \omega \in \mathbb{C}^{\mathsf{V}} \otimes \mathfrak{g} \;\middle|\; \begin{array}{ll} \omega_v = 0 & \text{for } v \in \overline{\mathsf{W}} \\ \mathrm{rank}\{\omega_v\}_{v \in \mathsf{W}_i} \leqslant 1 & \text{for } 1 \leqslant i \leqslant c \end{array} \right\}.$$

- Then $S_{\mathsf{W}} \cong \prod_{i=1}^{c} \mathrm{cone}\left(\mathbb{P}(\mathbb{C}^{\mathsf{W}_i}) \times \mathbb{P}(\mathfrak{g})\right)$ is a Zariski-closed subset of the affine space $\mathbb{C}^{\mathsf{W}} \otimes \mathfrak{g} \subseteq \mathbb{C}^{\mathsf{V}} \otimes \mathfrak{g}$.

PROPOSITION (MPPS 2017)

Let $\Gamma = (\mathsf{V}, \mathsf{E})$ be a finite simplicial graph, and let $\mathfrak{g}$ be a Lie algebra. Then $\mathcal{F}(A_\Gamma, \mathfrak{g}) \supseteq \bigcup_{\mathsf{W}\subseteq\mathsf{V}} S_{\mathsf{W}}$. Moreover, if $\mathfrak{g} = \mathfrak{sl}_2$ or $\mathfrak{sol}_2$, then

$$\mathcal{F}(A_\Gamma, \mathfrak{g}) = \bigcup_{\mathsf{W}\subseteq\mathsf{V}} S_{\mathsf{W}}.$$

- Let $\theta\colon \mathfrak{g} \to \mathfrak{gl}(V)$ be a finite-dimensional representation.
- Given a subset $\mathsf{W} \subseteq \mathsf{V}$, put

$$P_{\mathsf{W}} = \left\{ \omega \in \mathbb{C}^{\mathsf{V}} \otimes \mathfrak{g} \;\middle|\; \begin{array}{ll} \omega_v = 0 & \text{if } v \in \overline{\mathsf{W}} \\ \omega_v = \lambda_v g_{\mathsf{W}} & \text{if } v \in \mathsf{W} \end{array} \right\}.$$

  where $\lambda_v \in \mathbb{C}$ and $g_{\mathsf{W}} \in V(\det \circ \theta)$.
- Then P_{W} is a Zariski-closed subset of $\mathbb{C}^{\mathsf{V}} \otimes \mathfrak{g}$, and $P_{\mathsf{W}} \subseteq S_{\mathsf{W}}$.

IRREDUCIBLE DECOMPOSITIONS

PROPOSITION (MPPS 2017)

If $\mathfrak{g} = \mathfrak{sl}_2$ *or* $\mathfrak{sol}_2$, *then*

$$\mathcal{R}^1_1(A_\Gamma, \theta) = \bigcup_{\substack{W \subseteq V \\ c(W)=1}} P_W \cup \bigcup_{\substack{W \subseteq \mathcal{V} \\ c(W)>1}} S_W.$$

- For $W \subseteq W' \subseteq V$, let $K_{WW'} \colon \{W_1, \dots, W_c\} \to \{W'_1, \dots, W'_{c'}\}$ be the map from the connected components of Γ_W to those of $\Gamma_{W'}$.
- Define an order relation on the subsets of V by
$$W \preccurlyeq W' \Leftrightarrow W \subseteq W' \text{ and } K_{WW'} \text{ is injective.}$$
- Clearly, if $c(W) > 1$ and $c(W') = 1$, then $W \not\preccurlyeq W'$. Furthermore, if $c(W) = 1$, then $W \preccurlyeq W'$ if and only if $W \subseteq W'$.

THEOREM (MPPS 2017)

If $\Gamma = (\mathsf{V}, \mathsf{E})$ *be a finite, simplicial graph, and let* $\theta\colon \mathfrak{sl}_2 \to \mathfrak{gl}(V)$ *be a finite-dimensional representation. We then have the following decompositions into irreducible components:*

$$\mathcal{F}(A_\Gamma, \mathfrak{sl}_2) = \bigcup_{\mathsf{W}\ \preccurlyeq\text{-maximal}} S_{\mathsf{W}},$$

$$\mathcal{R}^1_1(A_\Gamma, \theta) = \bigcup_{\substack{c(\mathsf{W})=1 \\ \mathsf{W}\ \preccurlyeq\text{-maximal}}} P_{\mathsf{W}} \cup \bigcup_{\substack{c(\mathsf{W})>1 \\ \nexists \mathsf{W} \precnsim \mathsf{W}' \text{ with } c(\mathsf{W}')>1}} S_{\mathsf{W}}.$$

RANK GREATER THAN 1

PROPOSITION (MPPS 2017)

Suppose $\mathfrak{g}$ is a semisimple Lie algebra, $\mathfrak{g} \neq \mathfrak{sl}_2$. There is then a connected, finite simple graph Γ such that $\mathcal{F}(A_\Gamma, \mathfrak{g}) \neq \bigcup_{W \subseteq V} S_W$.

Sketch of proof:

- Let $r = \operatorname{rank} \mathfrak{g}$. By assumption, $r > 1$.
- Let $\{\alpha_1, \ldots, \alpha_r\}$ be a system of simple roots.
- If $r > 2$, let Γ be the graph with vertex set $V = \{\pm\alpha_1, \ldots, \pm\alpha_r\}$ and edges $\{\alpha_i, -\alpha_j\}$ for $i \neq j$. Clearly, Γ is connected.
- Pick $\omega \in \mathcal{F}(A_\Gamma, \mathfrak{g})$ so that ω_α is a generator of the root space $\mathfrak{g}_\alpha \subseteq \mathfrak{g}$, for each $\alpha \in V$. Then $\omega \notin \bigcup_{W \subseteq V} S_W$.
- The case $r = 2$ is similar.

ARTIN GROUPS

- Let $\Gamma = (\mathsf{V}, \mathsf{E}, \ell)$ be a finite simplicial graph with labeling function $\ell \colon \mathsf{E} \to \mathbb{Z}_{\geqslant 2}$. The corresponding *Artin group* is

$$\pi_{\Gamma,\ell} = \langle v \in V \mid \underbrace{vwv\cdots}_{\ell(e)} = \underbrace{wvw\cdots}_{\ell(e)} \text{ if } e = \{v, w\} \in E \rangle.$$

- If $\ell(e) = 2$ for all $e \in \mathsf{E}$, then $\pi_{\Gamma,\ell} = \pi_\Gamma$.
- To each labeled graph (Γ, ℓ) we associate an unlabeled graph, $\tilde{\Gamma}$, called the *odd contraction* of (Γ, ℓ), as follows.
- We first define an unlabeled graph Γ_{odd} by keeping all the vertices of Γ, and retaining only those edges for which the label is odd.
- We then let $\tilde{\Gamma}$ be the graph whose vertices correspond to the connected components of Γ_{odd}, with two distinct components determining an edge $\{c, c'\}$ in $\tilde{\Gamma}$ if and only if there exist vertices $v \in c$ and $v' \in c'$ which are connected by an edge in Γ.

EXAMPLE

Let Γ be the complete graph on $\{1, 2, \dots, n-1\}$, with $\ell(\{i,j\}) = 2$ if $|i-j| > 1$ and $\ell(\{i,j\}) = 3$ if $|i-j| = 1$. Then $\pi_{\Gamma,\ell} = B_n$. Moreover, Γ_{odd} is connected, and so $\tilde{\Gamma} = \bullet$.

- Let $A^{\bullet}_{\Gamma,\ell} = H^{\bullet}(\pi_{\Gamma,\ell}, \mathbb{C})$ and $A^{\bullet}_{\tilde{\Gamma}} = H^{\bullet}(\pi_{\tilde{\Gamma}}, \mathbb{C})$ be the respective cohomology algebras, both endowed with the zero differential.
- Then $\mathfrak{h}(\pi_{\Gamma,\ell}) = \mathfrak{h}(\pi_{\tilde{\Gamma}})$ and
$$(\mathcal{F}(A_{\Gamma,\ell}, \mathfrak{g}), \mathcal{R}^1_1(A_{\Gamma,\ell}, \theta)) \cong (\mathcal{F}(A_{\tilde{\Gamma}}, \mathfrak{g}), \mathcal{R}^1_1(A_{\tilde{\Gamma}}, \theta))$$
- This yields explicit decompositions into irreducible components for the varieties $\mathcal{F}(A_{\Gamma,\ell}, \mathfrak{sl}_2)$ and $\mathcal{R}^1_1(A_{\Gamma,\ell}, \theta)$, for any labeled graph (Γ, ℓ) and any representation $\theta\colon \mathfrak{sl}_2 \to \mathfrak{gl}(V)$.

KAPOVICH–MILLSON UNIVERSALITY

THEOREM (KAPOVICH–MILLSON 1998)

Let $\mathcal{X}$ be an affine variety defined over $\mathbb{Q}$, and let $x \in \mathcal{X}$. There is then a labeled graph (Γ, ℓ) and a non-trivial *representation $\rho\colon \pi_{\Gamma,\ell} \to \mathrm{PSL}_2$ with finite image and trivial centralizer such that*

$$\big(\operatorname{Hom}(\pi_{\Gamma,\ell}, \mathrm{PSL}_2)/\!/ \mathrm{PSL}_2\big)_{([\rho])} \cong \mathcal{X}_{(x)}$$

and $\operatorname{Hom}(\pi_{\Gamma,\ell}, \mathrm{PSL}_2)_{(\rho)} \cong \mathcal{X}_{(x)} \times \mathbb{C}^3_{(0)}$.

At the trivial representation, though, things are completely different.

THEOREM (KAPOVICH–MILLSON 1998)

For any labeled graph (Γ, ℓ), the variety $\operatorname{Hom}(\pi_{\Gamma,\ell}, \mathrm{PSL}_2)$ has at worst a quadratic singularity at $\rho = 1$.

GERMS AT 1 OF REPRESENTATION VARIETIES

- Let $\tilde{\Gamma}$ be the odd contraction of (Γ, ℓ). We then have a local analytic isomorphism

$$\mathrm{Hom}(\pi_{\Gamma,\ell}, \mathrm{PSL}_2)_{(1)} \cong \mathcal{F}(A_{\tilde{\Gamma}}, \mathfrak{sl}_2)_{(0)}$$

  which identifies $\mathcal{V}^1_1(K(\pi_{\Gamma,\ell}, 1), \iota)_{(1)}$ with $\mathcal{R}^1_1(A_{\tilde{\Gamma}}, \theta)_{(0)}$, for every rational representation $\iota\colon \mathrm{PSL}_2 \to \mathrm{GL}(V)$.
- The analytic singularity at 1 of $\mathrm{Hom}(\pi_{\Gamma,\ell}, \mathrm{PSL}_2)$ can then be completely described in terms of the graph $\tilde{\Gamma}$.
- Similarly, $\mathcal{V}^1_1(K(\pi_{\Gamma,\ell}, 1), \iota)$, can be completely described in terms of the graph $\tilde{\Gamma}$ and the tangential representation of ι.