POLYHEDRAL PRODUCTS, DUALITY PROPERTIES, AND COHEN–MACAULAY COMPLEXES

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POLYHEDRAL PRODUCTS

- Let (X, A) be a pair of topological spaces, and let L be a simplicial complex on vertex set $[m]$.
- The corresponding *polyhedral product* (or, *generalized moment-angle complex*) is defined as

$$\mathcal{Z}_L(X, A) = \bigcup_{\sigma \in L} (X, A)^{\sigma} \subset X^{\times m},$$

  where $(X, A)^{\sigma} = \{x \in X^{\times m} \mid x_i \in A \text{ if } i \notin \sigma\}$.
- Homotopy invariance:

$$(X, A) \simeq (X', A') \implies \mathcal{Z}_L(X, A) \simeq \mathcal{Z}_L(X', A').$$

- Converts simplicial joins to direct products:

$$\mathcal{Z}_{K*L}(X, A) \cong \mathcal{Z}_K(X, A) \times \mathcal{Z}_L(X, A).$$

- Takes a cellular pair (X, A) to a cellular subcomplex of $X^{\times m}$.

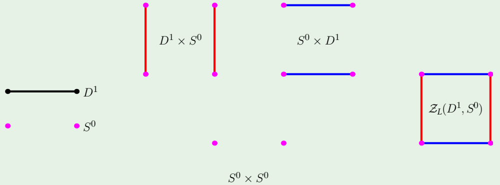
The usual moment-angle complexes (which play an important role in toric topology) are:

- Complex moment-angle complex, $\mathcal{Z}_L(D^2, S^1)$.
    - $\pi_1 = \pi_2 = \{1\}$.
- Real moment-angle complex, $\mathcal{Z}_L(D^1, S^0)$.
    - $\pi_1 = W'_L$, the derived subgroup of W_Γ, the right-angled Coxeter group associated to $\Gamma = L^{(1)}$.

EXAMPLE

Let $L =$ two points. Then:

$$\mathcal{Z}_L(D^2, S^1) = D^2 \times S^1 \cup S^1 \times D^2 = S^3$$
$$\mathcal{Z}_L(D^1, S^0) = D^1 \times S^0 \cup S^0 \times D^1 = S^1$$

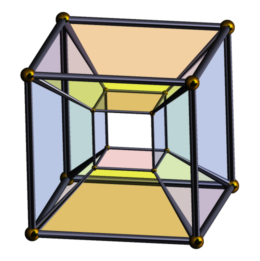


EXAMPLE

Let L be a circuit on 4 vertices. Then:

$$\mathcal{Z}_L(D^2, S^1) = S^3 \times S^3$$

$$\mathcal{Z}_L(D^1, S^0) = S^1 \times S^1$$



EXAMPLE

More generally, let L be an m-gon. Then:

$$\mathcal{Z}_L(D^2, S^1) = \#_{r=1}^{m-3}\, r \cdot \binom{m-2}{r+1} S^{r+2} \times S^{m-r}.$$

(McGavran 1979)

$$\mathcal{Z}_L(D^1, S^0) = \text{an orientable surface of genus } 1 + 2^{m-3}(m-4).$$

(Coxeter 1937)

- If $(M, \partial M)$ is a compact manifold of dimension d, and L is a PL-triangulation of S^m on n vertices, then $\mathcal{Z}_L(M, \partial M)$ is a compact manifold of dimension $(d-1)n + m + 1$.
- (Bosio–Meersseman 2006) If K is a *polytopal* triangulation of S^m, then
  - $\mathcal{Z}_L(D^2, S^1)$ if $n + m + 1$ is even, or
  - $\mathcal{Z}_L(D^2, S^1) \times S^1$ if $n + m + 1$ is odd

  is a complex manifold.
- This construction generalizes the classical constructions of complex structures on $S^{2p-1} \times S^1$ (Hopf) and $S^{2p-1} \times S^{2q-1}$ (Calabi–Eckmann).
- In general, the resulting complex manifolds are *not* symplectic, thus, not Kähler. In fact, they may even be non-formal (Denham–Suciu 2007).

- The GMAC construction enjoys nice functoriality properties in both arguments. E.g:
  - Let $f\colon (X,A) \to (Y,B)$ be a (cellular) map. Then $f^{\times n}\colon X^{\times n} \to Y^{\times n}$ restricts to a (cellular) map $\mathcal{Z}_L(f)\colon \mathcal{Z}_L(X,A) \to \mathcal{Z}_L(Y,B)$.
- Much is known from work of M. Davis about the fundamental group and the asphericity problem for $\mathcal{Z}_L(X) = \mathcal{Z}_L(X,*)$. E.g.:
  - $\pi_1(\mathcal{Z}_L(X,*))$ is the graph product of $G_v = \pi_1(X,*)$ along the graph $\Gamma = L^{(1)} = (\mathsf{V},\mathsf{E})$, where

$$\operatorname{Prod}_\Gamma(G_v) = \mathop{\ast}_{v\in \mathsf{V}} G_v/\{[g_v,g_w] = 1 \text{ if } \{v,w\}\in \mathsf{E},\, g_v\in G_v,\, g_w\in G_w\}.$$

  - Suppose X is aspherical. Then: $\mathcal{Z}_L(X,*)$ is aspherical iff L is a flag complex.

Toric complexes

- Let L be a simplicial complex on vertex set $\mathsf{V} = \{v_1, \dots, v_m\}$.
- Let $T_L = \mathcal{Z}_L(S^1, *)$ be the subcomplex of T^m obtained by deleting the cells corresponding to the missing simplices of L.
- T_L is a connected, minimal CW-complex, of dimension $\dim L + 1$.
- T_L is formal (Notbohm–Ray 2005).
- (Kim–Roush 1980, Charney–Davis 1995) The cohomology algebra $H^*(T_L, \Bbbk)$ is the exterior Stanley–Reisner ring

$$\Bbbk\langle L\rangle = \bigwedge V^*/(v_\sigma^* \mid \sigma \notin L),$$

where $\Bbbk = \mathbb{Z}$ or a field, V is the free $\Bbbk$-module on V, and $V^* = \operatorname{Hom}_{\Bbbk}(V, \Bbbk)$, while $v_\sigma^* = v_{i_1}^* \cdots v_{i_s}^*$ for $\sigma = \{i_1, \dots, i_s\}$.

Right angled Artin groups

- The fundamental group $\pi_\Gamma := \pi_1(T_L, *)$ is the RAAG associated to the graph $\Gamma := L^{(1)} = (\mathsf{V}, \mathsf{E})$,

$$\pi_\Gamma = \langle v \in V \mid [v, w] = 1 \text{ if } \{v, w\} \in E \rangle.$$

- Moreover, $K(\pi_\Gamma, 1) = T_{\Delta_\Gamma}$, where Δ_Γ is the flag complex of Γ.
- (Kim–Makar-Limanov–Neggers–Roush 1980, Droms 1987)

$$\Gamma \cong \Gamma' \Longleftrightarrow \pi_\Gamma \cong \pi_{\Gamma'}.$$

- (Papadima–S. 2006) The associated graded Lie algebra of π_Γ has (quadratic) presentation

$$\operatorname{gr}(\pi_\Gamma) = \mathbb{L}(\mathsf{V})/([v, w] = 0 \text{ if } \{v, w\} \in \mathsf{E}).$$

- (Duchamp–Krob 1992, PS06) The lower central series quotients of π_Γ are torsion-free, with ranks ϕ_k given by

$$\prod_{k=1}^{\infty} (1 - t^k)^{\phi_k} = P_\Gamma(-t),$$

where $P_\Gamma(t) = \sum_{k \geqslant 0} f_k(\Delta_\Gamma) t^k$ is the clique polynomial of Γ.

CHEN RANKS

- The *Chen Lie algebra* of a f.g. group π is the associated graded Lie algebra of its maximal metabelian quotient, $\operatorname{gr}(\pi/\pi'')$.
- Write $\theta_k(\pi) = \operatorname{rank} \operatorname{gr}_k(\pi/\pi'')$ for the Chen ranks.
- (K.-T. Chen 1951) $\operatorname{gr}(F_n/F_n'')$ is torsion-free, with ranks $\theta_1 = n$ and $\theta_k = (k-1)\binom{n+k-2}{k}$ for $k \geqslant 2$.
- (PS 06) $\operatorname{gr}(\pi_\Gamma/\pi_\Gamma'')$ is torsion-free, with ranks given by $\theta_1 = |\mathsf{V}|$ and

$$\sum_{k=2}^{\infty} \theta_k t^k = Q_\Gamma\left(\frac{t}{1-t}\right).$$

- Here $Q_\Gamma(t) = \sum_{j \geqslant 2} c_j(\Gamma) t^j$ is the "cut polynomial" of Γ, with

$$c_j(\Gamma) = \sum_{\mathsf{W} \subset \mathsf{V}\colon |\mathsf{W}|=j} \tilde{b}_0(\Gamma_{\mathsf{W}}).$$

Example

Let Γ be a pentagon, and Γ' a square with an edge attached to a vertex. Then:

- $P_\Gamma = P_{\Gamma'} = 1 + 5t + 5t^2$, and so

$$\phi_k(\pi_\Gamma) = \phi_k(\pi_{\Gamma'}), \quad \text{for all } k \geqslant 1.$$

- $Q_\Gamma = 5t^2 + 5t^3$ but $Q_{\Gamma'} = 5t^2 + 5t^3 + t^4$, and so

$$\theta_k(\pi_\Gamma) \neq \theta_k(\pi_{\Gamma'}), \quad \text{for } k \geqslant 4.$$

COHOMOLOGY JUMP LOCI

- Let X be a connected, finite CW-complex X with $\pi := \pi_1(X)$.
- Fix a field $\Bbbk$ and set $A = H^{\bullet}(X, \Bbbk)$. If $\operatorname{char}(\Bbbk) = 2$, assume $H_1(X, \mathbb{Z})$ is torsion-free. Then, for each $a \in A^1$, we have $a^2 = 0$, and so we get a cochain complex, $(A, \cdot a)\colon\ A^0 \xrightarrow{\cdot a} A^1 \xrightarrow{\cdot a} A^2 \longrightarrow \cdots$.
- The *resonance varieties* of X are defined as

$$\mathcal{R}^i_s(X) = \{a \in A^1 \mid \dim H^i(A, \cdot a) \geqslant s\}.$$

- They are Zariski closed, homogeneous subsets of $A^1 = H^1(X, \Bbbk)$.
- The *characteristic varieties* of X are the jump loci for homology with coefficients in rank-1 local systems,

$$\mathcal{V}^i_s(X, \Bbbk) = \{\rho \in \operatorname{Hom}(\pi, \Bbbk^*) \mid \dim H_i(X, \Bbbk_\rho) \geqslant s\}.$$

- These loci are Zariski closed subsets of the character group. For $i = 1$, they depend only on π/π'' (and $\Bbbk$).

JUMP LOCI OF TORIC COMPLEXES

For a field $\Bbbk$, identify $H^1(T_L, \Bbbk) = \Bbbk^{\mathsf{V}}$, the $\Bbbk$-vector space with basis V.

THEOREM (PAPADIMA–S. 2009)

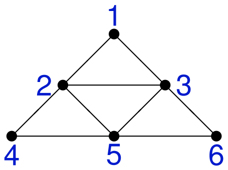
$$\mathcal{R}^i_s(T_L, \Bbbk) = \bigcup_{\substack{\mathsf{W}\subset\mathsf{V} \\ \sum_{\sigma\in L_{\mathsf{V}\setminus\mathsf{W}}} \dim_{\Bbbk} \widetilde{H}_{i-1-|\sigma|}(\operatorname{lk}_{L_{\mathsf{W}}}(\sigma),\Bbbk)\geq s}} \Bbbk^{\mathsf{W}},$$

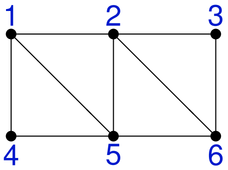
where L_{W} is the subcomplex induced by L on W, and $\operatorname{lk}_K(\sigma)$ is the link of a simplex σ in a subcomplex $K \subseteq L$.

In particular,

$$\mathcal{R}^1_1(\pi_\Gamma) = \bigcup_{\substack{\mathsf{W}\subseteq\mathsf{V} \\ \Gamma_{\mathsf{W}} \text{ disconnected}}} \Bbbk^{\mathsf{W}}.$$

Similar formulas hold for the characteristic varieties $\mathcal{V}^i_s(T_L, \Bbbk)$.





EXAMPLE

Let Γ and Γ' be the two graphs above. Both have

$$P(t) = 1 + 6t + 9t^2 + 4t^3, \quad \text{and} \quad Q(t) = t^2(6 + 8t + 3t^2).$$

Thus, π_Γ and $\pi_{\Gamma'}$ have the same LCS and Chen ranks.
Each resonance variety has 3 components, of codimension 2:

$$\mathcal{R}_1(\pi_\Gamma, \Bbbk) = \Bbbk^{\overline{23}} \cup \Bbbk^{\overline{25}} \cup \Bbbk^{\overline{35}}, \qquad \mathcal{R}_1(\pi_{\Gamma'}, \Bbbk) = \Bbbk^{\overline{15}} \cup \Bbbk^{\overline{25}} \cup \Bbbk^{\overline{26}}.$$

Yet the two varieties are not isomorphic, since

$$\dim(\Bbbk^{\overline{23}} \cap \Bbbk^{\overline{25}} \cap \Bbbk^{\overline{35}}) = 3, \quad \text{but} \quad \dim(\Bbbk^{\overline{15}} \cap \Bbbk^{\overline{25}} \cap \Bbbk^{\overline{26}}) = 2.$$

PROPAGATION OF JUMP LOCI

- We say that the resonance varieties of a graded algebra $A = \bigoplus_{i=0}^{n} A^i$ *propagate* if

$$\mathcal{R}_1^1(A) \subseteq \cdots \subseteq \mathcal{R}_1^n(A).$$

- (Eisenbud–Popescu–Yuzvinsky 2003) If $M(\mathcal{A})$ is the complement of a hyperplane arrangement, then the resonance varieties of the Orlik–Solomon algebra $A = H^*(M(\mathcal{A}), \mathbb{C})$ propagate.

- The resonance varieties of $A = H^*(T_L, \Bbbk)$ may not propagate. E.g., if $L =$  , then $\mathcal{R}_1^1(A) = \Bbbk^4$, yet $\mathcal{R}_1^2(A) = \Bbbk^2 \cup \Bbbk^2$.

THEOREM (DENHAM–S.–YUZVINSKY 2016/17, GENERALIZING EPY)

Suppose the $\Bbbk$-dual of A has a linear free resolution over $E = \bigwedge A^1$. Then the resonance varieties of A propagate.

DUALITY SPACES

In order to study propagation of jump loci in a topological setting, we turn to a notion due to Bieri and Eckmann (1978).

- X is a *duality space* of dimension n if $H^i(X, \mathbb{Z}\pi) = 0$ for $i \neq n$ and $H^n(X, \mathbb{Z}\pi) \neq 0$ and torsion-free.
- Let $D = H^n(X, \mathbb{Z}\pi)$ be the dualizing $\mathbb{Z}\pi$-module. Given any $\mathbb{Z}\pi$-module A, we have $H^i(X, A) \cong H_{n-i}(X, D \otimes A)$.
- If $D = \mathbb{Z}$, with trivial $\mathbb{Z}\pi$-action, then X is a Poincaré duality space.
- If $X = K(\pi, 1)$ is a duality space, then π is a *duality group*.

ABELIAN DUALITY SPACES

We introduce in (DSY17) an analogous notion, by replacing $\pi \rightsquigarrow \pi_{\mathrm{ab}}$.

- X is an *abelian duality space* of dimension n if $H^i(X, \mathbb{Z}\pi_{\mathrm{ab}}) = 0$ for $i \neq n$ and $H^n(X, \mathbb{Z}\pi_{\mathrm{ab}}) \neq 0$ and torsion-free.
- Let $B = H^n(X, \mathbb{Z}\pi_{\mathrm{ab}})$ be the dualizing $\mathbb{Z}\pi_{\mathrm{ab}}$-module. Given any $\mathbb{Z}\pi_{\mathrm{ab}}$-module A, we have $H^i(X, A) \cong H_{n-i}(X, B \otimes A)$.
- The two notions of duality are independent.

THEOREM (DSY)

Let X be an abelian duality space of dimension n. If $\rho\colon \pi_1(X) \to \mathbb{k}^*$ satisfies $H^i(X, \mathbb{k}_\rho) \neq 0$, then $H^j(X, \mathbb{k}_\rho) \neq 0$, for all $i \leqslant j \leqslant n$.

COROLLARY (DSY)

Let X be an abelian duality space of dimension n. Then:

- *The characteristic varieties propagate: $\mathcal{V}^1_1(X,\Bbbk) \subseteq \cdots \subseteq \mathcal{V}^n_1(X,\Bbbk)$.*
- $\dim_{\Bbbk} H^1(X,\Bbbk) \geqslant n-1$.
- *If $n \geqslant 2$, then $H^i(X,\Bbbk) \neq 0$, for all $0 \leqslant i \leqslant n$.*

PROPOSITION (DSY)

Let M be a closed, orientable 3-manifold. If $b_1(M)$ is even and non-zero, then the resonance varieties of M do not propagate.

EXAMPLE

- Let M be the 3-dimensional Heisenberg nilmanifold.
- Characteristic varieties propagate: $\mathcal{V}^i_1(M,\Bbbk) = \{1\}$ for $i \leqslant 3$.
- Resonance does not propagate: $\mathcal{R}^1_1(M,\Bbbk) = \Bbbk^2$, $\mathcal{R}^3_1(M,\Bbbk) = 0$.

ARRANGEMENTS OF SMOOTH HYPERSURFACES

THEOREM (DENHAM–S. 2017)

Let U be a connected, smooth, complex quasi-projective variety of dimension n. Suppose U has a smooth compactification Y for which

1. *Components of $Y \setminus U$ form an arrangement of hypersurfaces $\mathcal{A}$;*
2. *For each submanifold X in the intersection poset $L(\mathcal{A})$, the complement of the restriction of $\mathcal{A}$ to X is a Stein manifold.*

Then:

1. *U is both a duality space and an abelian duality space of dimension n.*
2. *If A is a finite-dimensional representation of $\pi = \pi_1(U)$, and if $A^{\gamma_g} = 0$ for all g in a building set $\mathcal{G}_X$, for some $X \in L(\mathcal{A})$, then $H^i(U, A) = 0$ for all $i \neq n$.*
3. *The ℓ_2-Betti numbers of U vanish for all $i \neq n$.*

LINEAR, ELLIPTIC, AND TORIC ARRANGEMENTS

THEOREM (DS17)

Suppose that $\mathcal{A}$ is one of the following:

1. *An affine-linear arrangement in $\mathbb{C}^n$, or a hyperplane arrangement in $\mathbb{CP}^n$;*
2. *A non-empty elliptic arrangement in E^n;*
3. *A toric arrangement in $(\mathbb{C}^*)^n$.*

Then the complement $M(\mathcal{A})$ is both a duality space and an abelian duality space of dimension $n-r$, $n+r$, and n, respectively, where r is the corank of the arrangement.

This theorem extends several previous results:

1. Davis, Januszkiewicz, Leary, and Okun (2011);
2. Levin and Varchenko (2012);
3. Davis and Settepanella (2013), Esterov and Takeuchi (2014).

THE COHEN–MACAULAY PROPERTY

A simplicial complex L is *Cohen–Macaulay* if for each simplex $\sigma \in L$, the reduced cohomology of $\mathrm{lk}(\sigma)$ is concentrated in degree $\dim L - |\sigma|$ and is torsion-free.

THEOREM (N. BRADY–MEIER 2001, JENSEN–MEIER 2005)

A RAAG π_Γ is a duality group if and only if Δ_Γ is Cohen–Macaulay. Moreover, π_Γ is a Poincaré duality group if and only if Γ is a complete graph.

THEOREM (DSY17)

A toric complex T_L is an abelian duality space (of dimension $\dim L + 1$) if and only if L is Cohen-Macaulay, in which case both the resonance and characteristic varieties of T_L propagate.

BESTVINA–BRADY GROUPS

- The Bestvina–Brady group associated to a graph Γ is defined as $N_\Gamma = \ker(\nu\colon \pi_\Gamma \to \mathbb{Z})$, where $\nu(v) = 1$, for each $v \in V(\Gamma)$.
- A counterexample to either the Eilenberg–Ganea conjecture or the Whitehead conjecture can be constructed from these groups.
- The cohomology ring $H^*(N_\Gamma, \Bbbk)$ was computed by Papadima–S. (2007) and Leary–Saadetoğlu (2011).
- The jump loci $\mathcal{R}^1_1(N_\Gamma, \Bbbk)$ and $\mathcal{V}^1_1(N_\Gamma, \Bbbk)$ were computed in PS07.

THEOREM (DAVIS–OKUN 2012)

Suppose Δ_Γ is acyclic. Then N_Γ is a duality group if and only if Δ_Γ is Cohen–Macaulay.

THEOREM (DSY17)

A Bestvina–Brady group N_Γ is an abelian duality group if and only if Δ_Γ is acyclic and Cohen–Macaulay.

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