ARRANGEMENT COMPLEMENTS AND MILNOR FIBRATIONS

Alex Suciu

Northeastern University

Special Session

Advances in Arrangement Theory

Mathematical Congress of the Americas

Montréal, Canada

July 25, 2017

INTERSECTION LATTICE AND COMPLEMENT

- An *arrangement of hyperplanes* is a finite set $\mathcal{A}$ of codimension 1 linear subspaces in a finite-dimensional $\mathbb{C}$-vector space V.
- The *intersection lattice*, $L(\mathcal{A})$, is the poset of all intersections of $\mathcal{A}$, ordered by reverse inclusion, and ranked by codimension.
- The *complement*, $M(\mathcal{A}) = V \setminus \bigcup_{H \in \mathcal{A}} H$, is a connected, smooth quasi-projective variety, and also a Stein manifold.
- It has the homotopy type of a minimal CW-complex of dimension equal to $\dim V$. In particular, $H_\bullet(M(\mathcal{A}), \mathbb{Z})$ is torsion-free.
- The fundamental group $\pi = \pi_1(M(\mathcal{A}))$ admits a finite presentation, with generators x_H for each $H \in \mathcal{A}$.

COHOMOLOGY RING

- For each $H \in \mathcal{A}$, let α_H be a linear form s.t. $H = \ker(\alpha_H)$. The logarithmic 1-form $\omega_H = \frac{1}{2\pi i}\, d\log \alpha_H \in \Omega_{dR}(M)$ is a closed form, representing a class $e_H \in H^1(M, \mathbb{Z})$.
- Let E be the $\mathbb{Z}$-exterior algebra on $\{e_H \mid H \in \mathcal{A}\}$, and let $\partial\colon E^{\bullet} \to E^{\bullet-1}$ be the differential given by $\partial(e_H) = 1$.
- (Orlik–Solomon 1980). The cohomology ring $A = H^{\bullet}(M(\mathcal{A}), \mathbb{Z})$ is determined by the intersection lattice: $A = E/I$, where

$$I = \text{ideal}\left\{ \partial\Big(\prod_{H\in\mathcal{B}} e_H \Big) \,\Big|\, \mathcal{B} \subseteq \mathcal{A} \text{ and } \operatorname{codim} \bigcap_{H\in\mathcal{B}} H < |\mathcal{B}| \right\}.$$

- The map $e_H \mapsto \omega_H$ extends to a cdga quasi-isomorphism, $(H^{\bullet}(M, \mathbb{R}), d = 0) \xrightarrow{\simeq} \Omega^{\bullet}_{dR}(M)$. Therefore, $M(\mathcal{A})$ is formal.
- Also, $M(\mathcal{A})$ is minimally pure (i.e., $H^k(M(\mathcal{A}), \mathbb{Q})$ is pure of weight $2k$, for all k), which again implies formality (Dupont 2016).

RESONANCE VARIETIES

- For a connected, finite CW-complex X, set $A = H^{\bullet}(X, \mathbb{C})$. For each $a \in A^1$, we have a cochain complex

$$(A, \cdot a)\colon\; A^0 \xrightarrow{\cdot a} A^1 \xrightarrow{\cdot a} A^2 \longrightarrow \cdots .$$

- The *resonance varieties* of X are defined as

$$\mathcal{R}^q_s(X) = \{a \in A^1 \mid \dim H^q(A, \cdot a) \geqslant s\}.$$

- They are Zariski closed, homogeneous subsets of affine space A^1.
- Now let $M = M(\mathcal{A})$. Since M is formal, its resonance varieties are unions of linear subspaces of $H^1(M, \mathbb{C}) \cong \mathbb{C}^{|\mathcal{A}|}$.
- (Falk–Yuzvinsky 2007) The irreducible components of $\mathcal{R}^1_1(M)$ arise from multinets on sub-arrangements of $\mathcal{A}$: each such k-multinet yields a (linear) component of dimension $k - 1 \geqslant 2$.

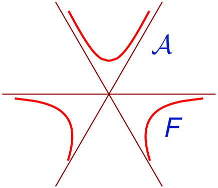
CHARACTERISTIC VARIETIES

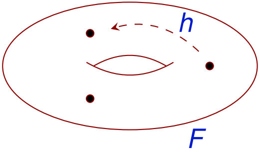
- Let X be a connected, finite cell complex, and set $\pi = \pi_1(X, x_0)$.
- The *characteristic varieties* of X are the jump loci for homology with coefficients in rank-1 local systems,

$$\mathcal{V}^q_s(X) = \{\rho \in \operatorname{Hom}(\pi, \mathbb{C}^*) \mid \dim H_q(X, \mathbb{C}_\rho) \geqslant s\}.$$

- These loci are Zariski closed subsets of the character group. For $q = 1$, they depend only on π/π''.
- They determine the characteristic polynomial of the algebraic monodromy of every regular $\mathbb{Z}_n$-cover $Y \to X$.
- Now let $M = M(\mathcal{A})$ be an arrangement complement. Since M is a smooth, quasi-projective variety, the characteristic varieties of M are unions of torsion-translated algebraic subtori of the character torus, $\operatorname{Hom}(\pi, \mathbb{C}^*) \cong (\mathbb{C}^*)^{|\mathcal{A}|}$.

MILNOR FIBER AND MONODROMY

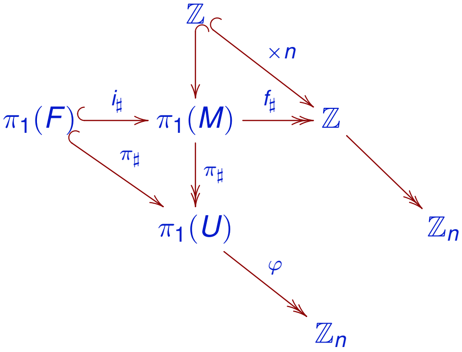




- Let $\mathcal{A}$ be an arrangement of n hyperplanes in $\mathbb{C}^{d+1}$, $d \geqslant 1$.
- (Milnor 1968). The polynomial map $f := \prod_{H \in \mathcal{A}} \alpha_H \colon \mathbb{C}^{d+1} \to \mathbb{C}$ restricts to a smooth fibration, $f \colon M(\mathcal{A}) \to \mathbb{C}^*$.
- Define the *Milnor fiber* of $\mathcal{A}$ as $F(\mathcal{A}) := f^{-1}(1)$.
- The monodromy diffeo, $h \colon F \to F$, is given by $h(z) = e^{2\pi i/n} z$.
- F is a Stein manifold. It has the homotopy type of a connected, finite cell complex of dimension d.
- In general, F is neither formal, nor minimal.

A REGULAR $\mathbb{Z}_n$-COVER

- The Hopf fibration $\mathbb{C}^* \to \mathbb{C}^{d+1}\backslash\{0\} \xrightarrow{\pi} \mathbb{CP}^d$ restricts to a trivial fibration $\mathbb{C}^* \to M(\mathcal{A}) \xrightarrow{\pi} U(\mathcal{A}) := \mathbb{P}(M(\mathcal{A}))$.
- In turn, this fibration restricts to a regular $\mathbb{Z}_n$-cover $\pi\colon F \to U$, classified by the homomorphism $\varphi\colon \pi_1(U) \twoheadrightarrow \mathbb{Z}_n$ taking each meridional loop x_H to 1.



THE ALGEBRAIC MONODROMY

- Let $\Delta_{\mathcal{A}}(t)$ be the characteristic polynomial of $h_*\colon H_1(F,\mathbb{C})\circlearrowleft$.
- WLOG, we may assume $\bar{A}=\mathbb{P}(\mathcal{A})$ is a line arrangement in $\mathbb{CP}^2$.
- Let $\beta_p(\mathcal{A})=\dim_{\mathbb{F}_p} H^1(H^{\bullet}(M(\mathcal{A}),\mathbb{F}_p),\cdot\sigma)$, where $\sigma=\sum_{H\in\mathcal{A}} e_H$. (An integer depending only on $L(\mathcal{A})$ and on the prime p.)

THEOREM (PAPADIMA–S. 2017)

If $\bar{A}$ has only points of multiplicity 2 and 3, then

$$\Delta_{\mathcal{A}}(t)=(t-1)^{n-1}(t^2+t+1)^{\beta_3(\mathcal{A})}.$$

CONJECTURE

If $\operatorname{rank}(\mathcal{A})\geqslant 3$, then

$$\Delta_{\mathcal{A}}(t)=(t-1)^{|\mathcal{A}|-1}((t+1)(t^2+1))^{\beta_2(\mathcal{A})}(t^2+t+1)^{\beta_3(\mathcal{A})}.$$

THE BOUNDARY MANIFOLD

- As before, let $\mathcal{A}$ be a central arrangement of hyperplanes in $V = \mathbb{C}^{d+1}$ ($d \geqslant 1$).
- Let $\overline{U}(\mathcal{A}) = \mathbb{CP}^d \setminus \operatorname{int}(N)$, where N is a (closed) regular neighborhood of the hypersurface $\bigcup_{H\in\mathcal{A}} \mathbb{P}(H) \subset \mathbb{CP}^d$.
- The *boundary manifold* of the arrangement, $\partial\overline{U} = \partial N$, is a compact, orientable, smooth manifold of dimension $2d-1$.

EXAMPLE

- Let $\mathcal{A}$ be a pencil of n hyperplanes in $\mathbb{C}^{d+1}$. If $n = 1$, then $\partial\overline{U} = S^{2d-1}$. If $n > 1$, then $\partial\overline{U} = \sharp^{n-1} S^1 \times S^{2(d-1)}$.
- Let $\mathcal{A}$ be a near-pencil of n planes in $\mathbb{C}^3$. Then $\partial\overline{U} = S^1 \times \Sigma_{n-2}$, where $\Sigma_g = \sharp^g S^1 \times S^1$.

- When $d = 2$, the boundary manifold $\partial\overline{U}$ is a 3-dimensional graph-manifold M_Γ, where
  - Γ is the incidence graph of $\mathcal{A}$, with $V(\Gamma) = L_1(\mathcal{A}) \cup L_2(\mathcal{A})$ and $E(\Gamma) = \{(L, P) \mid P \in L\}$.
  - Vertex manifolds $M_v = S^1 \times \left(S^2 \setminus \bigcup_{\{v,w\} \in E(\Gamma)} D^2_{v,w}\right)$ are glued along edge manifolds $M_e = S^1 \times S^1$ via flip maps.

THEOREM (JIANG–YAU 1993)

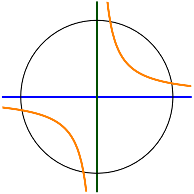
$U(\mathcal{A}) \cong U(\mathcal{A}') \;\Rightarrow\; M_\Gamma \cong M_{\Gamma'} \;\Rightarrow\; \Gamma \cong \Gamma' \;\Rightarrow\; L(\mathcal{A}) \cong L(\mathcal{A}')$.

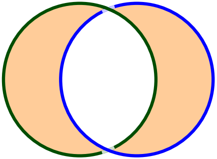
THEOREM (COHEN–S. 2008)

$\mathcal{V}^1_1(M_\Gamma) = \bigcup_{v \in V(\Gamma)\,:\,\deg(v) \geqslant 3} \{\prod_{i \in v} t_i = 1\}$. *Moreover, TFAE:*

- M_Γ *is formal.*
- $\mathrm{TC}_1(\mathcal{V}^1_1(M_\Gamma)) = \mathcal{R}^1_1(M_\Gamma)$.
- $\mathcal{A}$ *is a pencil or a near-pencil.*

THE BOUNDARY OF THE MILNOR FIBER





- Let $\overline{F}(\mathcal{A}) = F(\mathcal{A}) \cap D^{2(d+1)}$ be the *closed Milnor fiber* of $\mathcal{A}$.
- The *boundary of the Milnor fiber* of $\mathcal{A}$ is the compact, smooth, orientable, $(2d-1)$-manifold $\partial\overline{F} = F \cap S^{2d+1}$.
- The pair $(\overline{F}, \partial\overline{F})$ is $(d-1)$-connected. In particular, if $d \geqslant 2$, then $\partial\overline{F}$ is connected, and $\pi_1(\partial\overline{F}) \to \pi_1(\overline{F})$ is surjective.
- If $\mathcal{A}$ is the Boolean arrangement in $\mathbb{C}^n$, then $F = (\mathbb{C}^*)^{n-1}$. Hence, $\overline{F} = T^{n-1} \times D^{n-1}$, and so $\partial\overline{F} = T^{n-1} \times S^{n-2}$.
- If $\mathcal{A}$ is a near-pencil of n planes in $\mathbb{C}^3$, then $\partial\overline{F} = S^1 \times \Sigma_{n-2}$.

- The Hopf fibration $\pi\colon \mathbb{C}^{d+1}\backslash\{0\} \to \mathbb{CP}^d$ restricts to regular, cyclic n-fold covers, $\pi\colon \overline{F} \to \overline{U}$ and $\pi\colon \partial\overline{F} \to \partial\overline{U}$.
- Assume now that $d = 2$. The fundamental group of $\partial\overline{U} = M_\Gamma$ has generators $\overline{x}_H$ for $H \in \mathcal{A}$ and generators y_c for the cycles of Γ.

PROPOSITION (S. 2014)

The $\mathbb{Z}_n$-cover $\pi\colon \partial\overline{F} \to \partial\overline{U}$ is classified by the homomorphism $\pi_1(\partial\overline{U}) \twoheadrightarrow \mathbb{Z}_n$ given by $x_H \mapsto 1$ and $y_c \mapsto 0$.

THEOREM (NÉMETHI–SZILARD 2012)

The characteristic polynomial of $h_*\colon H_1(\partial\overline{F}, \mathbb{C})\circlearrowleft$ is given by

$$\delta_{\mathcal{A}}(t) = \prod_{v \in L_2(\mathcal{A})} (t-1)(t^{\gcd(m_v,n)} - 1)^{m_v-2}.$$

- Note: $H_1(\partial\overline{F}, \mathbb{Z})$ may have torsion. E.g., if $\mathcal{A}$ is generic, then $H_1(\partial\overline{F}, \mathbb{Z}) = \mathbb{Z}^{n(n-1)/2} \oplus \mathbb{Z}_n^{(n-2)(n-3)/2}$.

TRIVIAL ALGEBRAIC MONODROMY

- Let $\mathcal{A}$ be an arrangement of n planes in $\mathbb{C}^3$.
- Let $F \xrightarrow{i} M \xrightarrow{f} \mathbb{C}^*$ be the Milnor fibration, with monodromy $h\colon F \to F$, and let $\pi\colon F \to U$ be the corresponding $\mathbb{Z}_n$-cover.

THEOREM (DIMCA, PAPADIMA 2011)

If $h_*\colon H_1(F,\mathbb{C}) \circlearrowleft$ is the identity, then:

- *F is 1-formal.*
- *The map $\pi^*\colon H^1(U,\mathbb{C}) \to H^1(F,\mathbb{C})$ is an isomorphism which identifies $\mathcal{R}^1_s(M)$ with $\mathcal{R}^1_s(F)$, for all $s \geqslant 1$.*
- *The map $\pi^*\colon H^1(U,\mathbb{C}^*) \to H^1(F,\mathbb{C}^*)$ is a surjection with finite kernel, which establishes a bijection between the sets of irreducible components of $\mathcal{V}^1_s(U)$ and $\mathcal{V}^1_s(F)$ passing through 1.*

If $p\colon Y \to X$ is a regular $\mathbb{Z}_n$-cover, and the monodromy acts trivially on $H_1(Y, \mathbb{C})$, we cannot conclude that it acts trivially on $H_1(Y, \mathbb{Z})$.

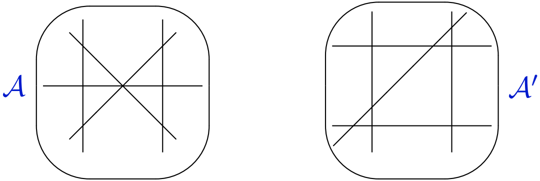
EXAMPLE (COHEN, DENHAM, S. 2003)

Let F_m be the Milnor fiber of the deleted braid arrangement, with a suitable choice of multiplicities m. Then h_* acts trivially on $H_1(F_m, \mathbb{C})$, but not on $H_1(F_m, \mathbb{Z})$, which has torsion subgroup $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ on which the monodromy acts as $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$.

Questions:

1. Is $H_1(F(\mathcal{A}), \mathbb{Z})$ always torsion-free?
2. Suppose $h_*\colon H_1(F, \mathbb{C}) \circlearrowleft$ is trivial (which conjecturally happens precisely when $\beta_2(\mathcal{A}) = \beta_3(\mathcal{A}) = 0$). Is then $H_1(F, \mathbb{Z})$ torsion-free (so that $h_*\colon H_1(F, \mathbb{Z}) \circlearrowleft$ is also trivial)?
3. Suppose $\mathcal{A}$ and $\mathcal{A}'$ have trivial monodromy action (over $\mathbb{Z}$), and suppose $L(\mathcal{A}) \not\cong L(\mathcal{A}')$. Is it true that $\pi_1(F) \not\cong \pi_1(F')$?

A pair of arrangements



- Let $\mathcal{A}$ and $\mathcal{A}'$ be the above pair of arrangements. Both have 2 triple points and 9 double points, yet $L(\mathcal{A}) \not\cong L(\mathcal{A}')$.
- As noted by Rose and Terao (1988), the respective OS-algebras are isomorphic. In fact, as shown by Falk (1993), $U(\mathcal{A}) \simeq U(\mathcal{A}')$.
- Since $L(\mathcal{A}) \not\cong L(\mathcal{A}')$, the corresponding boundary manifolds, $\partial\overline{U}$ and $\partial\overline{U}'$, are not homotopy equivalent, and so $U \not\approx U'$.
- In fact, $\mathcal{V}_1^1(\partial\overline{U})$ consists of 7 codimension-1 subtori in $(\mathbb{C}^*)^{13}$, while $\mathcal{V}_1^1(\partial\overline{U}')$ consists of 8 such subtori.

- The corresponding Milnor fibers, F and F', have trivial algebraic monodromy (over $\mathbb{Z}$); in particular, $b_1(F) = b_1(F') = 5$.
- The boundaries of the Milnor fibers, $\partial\overline{F}$ and $\partial\overline{F}'$, have the same characteristic polynomials for the algebraic monodromy (over $\mathbb{C}$):

$$\delta = \delta' = (t-1)^{13}(t^2+t+1)^2.$$

- $V_1^1(U) = T_P \cup T_Q$, where T_P and T_Q are the 2-dim subtori of $(\mathbb{C}^*)^5$ corresponding to the triple points of $\mathcal{A}$, and similarly for $V_1^1(U')$.
- Thus, $\mathcal{V}_1^1(F) = \pi^*(T_P) \cup \pi^*(T_Q)$ and $\mathcal{V}_1^1(F') = \pi'^*(T_{P'}) \cup \pi'^*(T_{Q'})$.
- But $\mathcal{V}_2^1(F) = \pi^*(T_P) \cap \pi^*(T_Q) = \{\mathbf{1}, \rho, \rho^2\} \cong \mathbb{Z}_3$,

  whereas $\mathcal{V}_2^1(F') = \pi'^*(T_{P'}) \cap \pi'^*(T_{Q'}) = \{\mathbf{1}\}$.
- Thus, $\pi_1(F) \not\cong \pi_1(F')$.

References

[1] S. Papadima, A.I. Suciu, *The Milnor fibration of a hyperplane arrangement: from modular resonance to algebraic monodromy*, Proceedings of the London Mathematical Society **114** (2017), no. 6, 961–1004.

[2] A.I. Suciu, *Hyperplane arrangements and Milnor fibrations*, Annales de la Faculté des Sciences de Toulouse **23** (2014), no. 2, 417–481.

[3] A.I. Suciu, *On the topology of Milnor fibrations of hyperplane arrangements*, Revue Roumaine de Mathématiques Pures et Appliquées **62** (2017), no. 1, 191–215.

[4] A.I. Suciu, *Homotopy types of Milnor fibers of line arrangements*, in preparation.