Braids and line arrangements

Old and new

Alex Suciu

Northeastern University

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Institute of the Mathematical Sciences of the Americas
University of Miami

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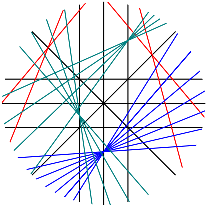

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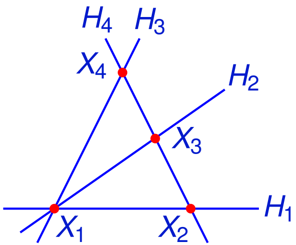
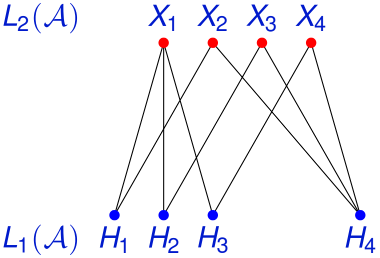
- An *arrangement of hyperplanes* is a finite collection $\mathcal{A}$ of codimension 1 linear (or affine) subspaces in $\mathbb{C}^\ell$.

- *Intersection lattice* $L(\mathcal{A})$: poset of all intersections of $\mathcal{A}$, ordered by reverse inclusion, and ranked by codimension.





- *Complement*: $M(\mathcal{A}) = \mathbb{C}^\ell \setminus \bigcup_{H\in\mathcal{A}} H$. It is a smooth, quasi-projective variety and also a Stein manifold. It has the homotopy type of a finite, connected, ℓ-dimensional CW-complex.

Example (The Boolean arrangement)

- $\mathcal{B}_n$: all coordinate hyperplanes $z_i = 0$ in $\mathbb{C}^n$.
- $L(\mathcal{B}_n)$: Boolean lattice of subsets of $\{0, 1\}^n$.
- $M(\mathcal{B}_n)$: complex algebraic torus $(\mathbb{C}^*)^n \simeq K(\mathbb{Z}^n, 1)$.

Example (The braid arrangement)

- $\mathcal{A}_n$: all diagonal hyperplanes $z_i - z_j = 0$ in $\mathbb{C}^n$.
- $L(\mathcal{A}_n)$: lattice of partitions of $[n] := \{1, \dots, n\}$, ordered by refinement.
- $M(\mathcal{A}_n)$: the (ordered) configuration space of n distinct points in $\mathbb{C}$; it is a classifying space $K(P_n, 1)$ for the pure braid group on n strands, P_n.

Cohomology rings of arrangements

- The homology groups $H_q(M(\mathcal{A}),\mathbb{Z})$ are finitely generated and torsion-free, with ranks given by

$$\textstyle\sum_{q=0}^{\ell} b_q(M(\mathcal{A}))t^q = \sum_{X\in L(\mathcal{A})} \mu(X)(-t)^{\operatorname{rank}(X)},$$

where $\mu\colon L(\mathcal{A})\to\mathbb{Z}$ is the Möbius function, defined by $\mu(\mathbb{C}^\ell)=1$ and $\mu(X)=-\sum_{Y\supsetneq X}\mu(Y)$.

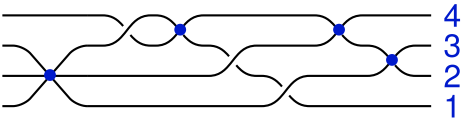
- Let E be the $\mathbb{Z}$-exterior algebra on degree 1 classes e_H dual to the meridians around the hyperplanes $H\in\mathcal{A}$.

- Let $\partial\colon E^*\to E^{*-1}$ be the differential given by $\partial(e_H)=1$, and set $e_{\mathcal{B}}=\prod_{H\in\mathcal{B}} e_H$ for each $\mathcal{B}\subset\mathcal{A}$.

- Building on work of Arnold & Brieskorn, Orlik and Solomon described the cohomology ring of $M(\mathcal{A})$ solely in terms of $L(\mathcal{A})$:

$$H^*(M(\mathcal{A}),\mathbb{Z})\cong E/\big\langle \partial e_{\mathcal{B}} \;\big|\; \operatorname{codim}\big(\textstyle\bigcap_{H\in\mathcal{B}} H\big) < |\mathcal{B}| \,\big\rangle.$$

- Let $\mathcal{A}' = \{H \cap \mathbb{C}^2\}_{H \in \mathcal{A}}$ be a generic planar section of $\mathcal{A}$. Then the arrangement group, $G(\mathcal{A}) = \pi_1(M(\mathcal{A}))$, is isomorphic to $\pi_1(M(\mathcal{A}'))$.
- So let $\mathcal{A}$ be an arrangement of n affine lines in $\mathbb{C}^2$. Taking a generic projection $\mathbb{C}^2 \to \mathbb{C}$ yields the braid monodromy $\alpha = (\alpha_1, \dots, \alpha_s)$, where $s = \#\{\text{multiple points}\}$ and the braids $\alpha_r \in P_n$ can be read off an associated braided wiring diagram,



- The group $G(\mathcal{A})$ has a presentation with meridional generators $x_1, \dots, x_n$ and commutator relators $x_i \alpha_j(x_i)^{-1}$.

Lower central series

- Let G be a group. The *lower central series* $\{\gamma_k(G)\}_{k\geq 1}$ is defined inductively by $\gamma_1(G) = G$ and $\gamma_{k+1}(G) = [G, \gamma_k(G)]$.
- Here, if $H, K < G$, then $[H, K]$ is the subgroup of G generated by $\{[a, b] := aba^{-1}b^{-1} \mid a \in H, b \in K\}$.
- The subgroups $\gamma_k(G)$ are normal; in fact, they are invariant under any automorphism of G. Moreover, $[\gamma_k(G), \gamma_\ell(G)] \subseteq \gamma_{k+\ell}(G)$.
- $\gamma_2(G) = [G, G]$ is the derived subgroup, and so $G/\gamma_2(G) = G_{\mathrm{ab}}$.
- $[\gamma_k(G), \gamma_k(G)] \triangleleft \gamma_{k+1}(G)$, and thus the LCS quotients,

  $$\mathrm{gr}_k(G) := \gamma_k(G)/\gamma_{k+1}(G),$$

  are abelian.
- If G is finitely generated, then so are its LCS quotients. Set $\phi_k(G) := \operatorname{rank} \mathrm{gr}_k(G)$.

Associated graded Lie algebra

- Fix a coefficient ring $\Bbbk$. Given a group G, we let
$$\operatorname{gr}(G, \Bbbk) = \bigoplus_{k \geq 1} \operatorname{gr}_k(G) \otimes \Bbbk.$$
- This is a graded Lie algebra over $\Bbbk$, with Lie bracket $[\ ,\ ] \colon \operatorname{gr}_k \times \operatorname{gr}_\ell \to \operatorname{gr}_{k+\ell}$ induced by the group commutator.
- For $\Bbbk = \mathbb{Z}$, we simply write $\operatorname{gr}(G) = \operatorname{gr}(G, \mathbb{Z})$.
- The construction is functorial.
- If G is finitely generated, so are its associated graded Lie algebras.
- Example: if F_n is the free group of rank n, then
  - $\operatorname{gr}(F_n)$ is the free Lie algebra $\operatorname{Lie}(\mathbb{Z}^n)$.
  - $\operatorname{gr}_k(F_n)$ is free abelian, of rank $\phi_k(F_n) = \frac{1}{k} \sum_{d|k} \mu(d) n^{\frac{k}{d}}$.

Chen Lie algebras

- Let $G^{(i)}$ be the *derived series* of G, starting at $G^{(1)} = G'$, $G^{(2)} = G''$, and defined inductively by $G^{(i+1)} = [G^{(i)}, G^{(i)}]$.
- The quotient groups, $G/G^{(i)}$, are solvable; $G/G' = G_{\mathrm{ab}}$, while G/G'' is the maximal metabelian quotient of G.
- The *i-th Chen Lie algebra* of G is defined as $\operatorname{gr}(G/G^{(i)}, \Bbbk)$. Clearly, this construction is functorial.
- The projection $q_i \colon G \twoheadrightarrow G/G^{(i)}$, induces a surjection $\operatorname{gr}_k(G; \Bbbk) \twoheadrightarrow \operatorname{gr}_k(G/G^{(i)}; \Bbbk)$, which is an iso for $k \leq 2^i - 1$.
- Assuming G is finitely generated, write $\theta_k(G) = \operatorname{rank} \operatorname{gr}_k(G/G'')$ for the *Chen ranks*. We have $\phi_k(G) \geq \theta_k(G)$, with equality for $k \leq 3$.
- Example (K.-T. Chen 1951): $\theta_k(F_n) = (k-1)\binom{n+k-2}{k}$, for $k \geq 2$.

Holonomy Lie algebra

- The *holonomy Lie algebra* of a finitely generated group G over a field $\Bbbk$ is defined as
$$\mathfrak{h}(G,\Bbbk) := \mathrm{Lie}(H_1(G,\Bbbk))/\langle \mathrm{im}(\mu_G^\vee)\rangle,$$
where
  - $\mathbf{L} = \mathrm{Lie}(V)$ the free Lie algebra on the $\Bbbk$-vector space $V = H_1(G;\Bbbk)$, with $\mathbf{L}_1 = V$ and $\mathbf{L}_2 = V \wedge V$.
  - $\mu_G^\vee\colon H_2(G,\Bbbk) \to V \wedge V$ is the dual of the cup product map $\mu_G\colon H^1(G;\Bbbk) \wedge H^1(G;\Bbbk) \to H^2(G;\Bbbk)$.
- Similarly, $\mathfrak{h}(G) = \mathrm{Lie}(H)/\,\mathrm{im}(\mu_G^\vee)$, where $H = H_1(G,\mathbb{Z})/\,\mathrm{Tors}$ and $\mu_G^\vee$ is dual to $\mu_G\colon H^1(G) \wedge H^1(G) \to H^2(G)$.
- By construction, these are (functorially defined) finitely generated graded Lie algebras that admit quadratic presentations.
- For instance, $\mathfrak{h}(F_n) = \mathrm{Lie}(n)$, whereas $\mathfrak{h}(\mathbb{Z}^n) = \mathbb{Z}^n$, concentrated in degree 1.

- If $\mathbb{k}$ is a field or $\mathbb{k} = \mathbb{Z}$, there is a natural, surjective morphism of graded Lie algebras,

$$\mathfrak{h}(G, \mathbb{k}) \twoheadrightarrow \operatorname{gr}(G; \mathbb{k}),$$

  which is an isomorphism in degrees 1 and 2, but not necessarily in higher degrees.

- If G is 1-formal (i.e., its $\mathbb{Q}$-pronilpotent completion is quadratic), then the map $\mathfrak{h}(G, \mathbb{Q}) \twoheadrightarrow \operatorname{gr}(G; \mathbb{Q})$ is an isomorphism.

THEOREM (RYBNIKOV 1998, PORTER–S. 2020)

Suppose G_{ab} is finitely-generated free abelian, and $\mu_G^\vee \colon H_2(G) \to G_{ab} \wedge G_{ab}$ is injective. Then the map $\mathfrak{h}_3(G) \twoheadrightarrow \operatorname{gr}_3(G)$ is an isomorphism.

Holonomy Lie algebras of arrangements

- Let $G = \pi_1(M(\mathcal{A}))$ be an arrangement group.
- Recall that G admits a finite presentation, with generators $\{x_H\}_{H\in\mathcal{A}}$ and commutator-relators.
- The holonomy Lie algebra $\mathfrak{h}(\mathcal{A}) := \mathfrak{h}(G)$ has presentation with generators $\{x_H\}_{H\in\mathcal{A}}$ and relators

$$\Big[x_H, \sum_{H'\in\mathcal{A}\colon H'\supset X} x_{H'}\Big]$$

  for all $X \in L_2(\mathcal{A})$ and all $H \in \mathcal{A}$ with $H \supset X$.
- Clearly, this presentation depends only on $L_{\leq 2}(\mathcal{A})$.
- $\mathfrak{h}_1(\mathcal{A})$ is free abelian of rank $n = |\mathcal{A}|$, with basis $\{x_H\}_{H\in\mathcal{A}}$.
- $\mathfrak{h}_2(\mathcal{A})$ is free abelian of rank $\binom{n}{2} - \sum_{X\in L_2(\mathcal{A})} \mu(X)$, with basis

$$\bigcup_{X\in L_2(\mathcal{A})} \{[x_H, x_{H'}] :\ H, H' \in X \setminus \{\max X\}\}.$$

Lower central series of arrangement groups

- $M(\mathcal{A})$ is formal, and so $G = \pi_1(M(\mathcal{A}))$ is 1-formal.
- Hence, the map $\mathfrak{h}(G, \mathbb{Q}) \twoheadrightarrow \operatorname{gr}(G, \mathbb{Q})$ is an isomorphism.
- Thus, $\operatorname{gr}(G, \mathbb{Q})$ and the LCS ranks $\phi_k(G)$ depend only on $L_{\leq 2}(\mathcal{A})$.
- Explicit combinatorial formulas for the LCS ranks are known in some cases, but not in general.
- (Falk–Randell 1985) If $\mathcal{A}$ is *supersolvable*, with exponents $d_1, \dots, d_\ell$, then $G = F_{d_\ell} \rtimes \cdots \rtimes F_{d_2} \rtimes F_{d_1}$ and

$$\phi_k(G) = \sum_{i=1}^{\ell} \phi_k(F_{d_i}).$$

- The Chen ranks $\theta_k(G) := \operatorname{rank} \operatorname{gr}_k(G/G'')$ are also combinatorially determined [Papadima–S. 2004]. An explicit formula for $k \gg 0$ was conjectured in [S. 2002].

- Let $G/\gamma_k(G)$ be the $(k-1)^{\text{th}}$ nilpotent quotient of $G = G(\mathcal{A})$. Then:
  - $G/\gamma_3(G)$ is determined by $L_{\leq 2}(\mathcal{A})$.
  - $G/\gamma_4(G)$ is *not* determined by $L(\mathcal{A})$ (Rybnikov 1994).
- We have $G_{\text{ab}} \cong \mathbb{Z}^{|\mathcal{A}|}$, and $\mu_G^{\vee}\colon H_2(G) \to G_{\text{ab}} \wedge G_{\text{ab}}$ is injective.
- Hence, $\mathfrak{h}_3(G) \cong \operatorname{gr}_3(G)$.
- (S. 2002) The groups $\operatorname{gr}_k(G)$ may have non-zero torsion for $k \geq 5$. E.g., if $G = G(\text{MacLane})$, then $\operatorname{gr}_5(G) = \mathbb{Z}^{87} \oplus \mathbb{Z}_2^4 \oplus \mathbb{Z}_3$.
- Question (S. 2002): Is that torsion combinatorially determined?
- (Artal Bartolo, Guerville-Ballé, and Viu-Sos 2020): Answer: No!
- There are two arrangements of 13 lines, $\mathcal{A}^{\pm}$, each one with 11 triple points and 2 quintuple points, such that $\operatorname{gr}_k(G^+) \cong \operatorname{gr}_k(G^-)$ for $k \leq 3$, yet $\operatorname{gr}_4(G^+) = \mathbb{Z}^{211} \oplus \mathbb{Z}_2$ and $\operatorname{gr}_4(G^-) = \mathbb{Z}^{211}$.

Decomposable arrangements

- For each flat $X \in L(\mathcal{A})$, let $\mathcal{A}_X := \{H \in \mathcal{A} \mid H \supset X\}$.
- The inclusions $\mathcal{A}_X \subset \mathcal{A}$ give rise to maps $M(\mathcal{A}) \hookrightarrow M(\mathcal{A}_X)$. Restricting to rank 2 flats yields a map

$$j\colon M(\mathcal{A}) \longrightarrow \prod_{X \in L_2(\mathcal{A})} M(\mathcal{A}_X)\,.$$

- The induced homomorphism on fundamental groups, $j_\sharp$, defines a morphism of graded Lie algebras,

$$\mathfrak{h}(j_\sharp)\colon \mathfrak{h}(\mathcal{A}) \longrightarrow \prod_{X \in L_2(\mathcal{A})} \mathfrak{h}(\mathcal{A}_X)\,.$$

Theorem (Papadima–S. 2006)

The map $\mathfrak{h}_k(j_\sharp)$ is a surjection for each $k \geq 3$ and an isomorphism for $k = 2$.

- The arrangement $\mathcal{A}$ is *decomposable* if the map $\mathfrak{h}_3(j_\sharp)$ is an isomorphism.

Theorem (Papadima–S. 2006)

Let $\mathcal{A}$ be a decomposable arrangement, and let $G = G(\mathcal{A})$. Then

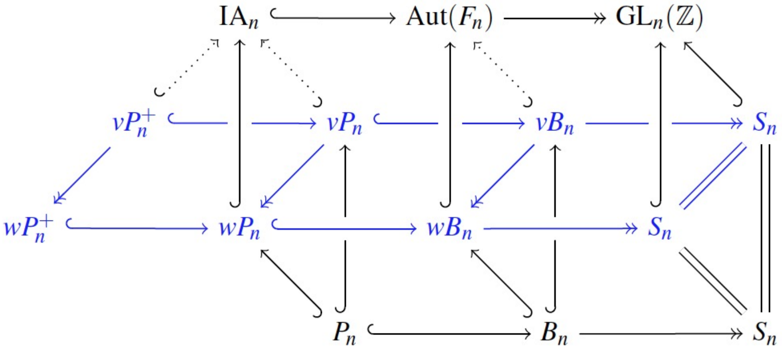
- *The map $\mathfrak{h}'(j_\sharp) \colon \mathfrak{h}'(\mathcal{A}) \to \prod_{X \in L_2(\mathcal{A})} \mathfrak{h}'(\mathcal{A}_X)$ is an isomorphism of graded Lie algebras.*
- *The map $\mathfrak{h}(G) \twoheadrightarrow \operatorname{gr}(G)$ is an isomorphism*
- *For each $k \geq 2$, the group $\operatorname{gr}_k(G)$ is free abelian of rank $\phi_k(G) = \sum_{X \in L_2(\mathcal{A})} \phi_k(F_{\mu(X)})$.*

Theorem (Porter–S. 2020)

Let $\mathcal{A}$ and $\mathcal{B}$ be decomposable arrangements with $L_{\leq 2}(\mathcal{A}) \cong L_{\leq 2}(\mathcal{B})$. Then, for each $k \geq 2$,

$$G(\mathcal{A})/\gamma_k(G(\mathcal{A})) \cong G(\mathcal{B})/\gamma_k(G(\mathcal{B})).$$

Braid-like groups



Artin braid groups

- Let B_n be the group of braids on n strings (under concatenation).
- $B_n = \mathrm{Mod}^1_{0,n}$, the mapping class group of D^2 with n marked points.
- Thus, B_n is a subgroup of $\mathrm{Aut}(F_n)$. In fact:
  $$B_n = \{\beta \in \mathrm{Aut}(F_n) \mid \beta(x_i) = w x_{\tau(i)} w^{-1}, \beta(x_1 \cdots x_n) = x_1 \cdots x_n\},$$
  where $x_1, \dots, x_n$ is a generating set for F_n. s
- Let $P_n = \ker(B_n \twoheadrightarrow S_n)$ be the pure braid group on n strings.
- P_n is a subgroup of $\mathrm{IA}_n = \{\varphi \in \mathrm{Aut}(F_n) \mid \varphi_* = \mathrm{id} \text{ on } H_1(F_n, \mathbb{Z})\}$.
- $P_n = F_{n-1} \rtimes_{\alpha_{n-1}} P_{n-1} = F_{n-1} \rtimes \cdots \rtimes F_2 \rtimes F_1$, where $\alpha_n \colon P_n \subset B_n \hookrightarrow \mathrm{Aut}(F_n)$.
- A classifying space for P_n is the ordered configuration space $\mathrm{Conf}_n(\mathbb{C})$. Thus, $B_n = \pi_1(\mathrm{Conf}_n(\mathbb{C})/S_n)$.

- The set of all permutation-conjugacy automorphisms of F_n forms a subgroup $wB_n < \mathrm{Aut}(F_n)$, called the welded braid group.
- Let $wP_n = \ker(wB_n \twoheadrightarrow S_n) = \mathrm{IA}_n \cap wB_n$ be the pure welded braid group wP_n.
- McCool (1986) gave a finite presentation for wP_n. It is generated by the automorphisms α_{ij} ($1 \le i \ne j \le n$) sending $x_i \mapsto x_j x_i x_j^{-1}$ and $x_k \mapsto x_k$ for $k \ne i$, subject to the relations

$$\begin{aligned} \alpha_{ij}\alpha_{ik}\alpha_{jk} &= \alpha_{jk}\alpha_{ik}\alpha_{ij} && \text{for } i,j,k \text{ distinct,} \\ [\alpha_{ij}, \alpha_{st}] &= 1 && \text{for } i,j,s,t \text{ distinct,} \\ [\alpha_{ik}, \alpha_{jk}] &= 1 && \text{for } i,j,k \text{ distinct.} \end{aligned}$$

- wP_n can be identified with the group of motions of n unknotted, unlinked circles in S^3, and also with the fundamental group of the space of configurations of parallel rings in $\mathbb{R}^3$.

- The upper pure welded braid group (or, upper McCool group) is the subgroup $wP_n^+ < wP_n$ generated by α_{ij} for $i < j$.

- We have: $wP_n^+ \cong F_{n-1} \rtimes \cdots \rtimes F_2 \rtimes F_1$.

- (F. Cohen, Pakhianathan, Vershinin, and Wu, 2007):

$$H^*(wP_n^+, \mathbb{Q}) = \bigwedge_{i<j}(e_{ij}) / \langle e_{ij}(e_{ik} - e_{jk}) \rangle.$$

- (D. Cohen and Pruidze, 2008) This is a Koszul algebra for all n.

- Jensen, McCammond, and Meier, 2006):

$$H^*(wP_n, \mathbb{Q}) = \bigwedge_{i \neq j}(e_{ij}) / \langle e_{ij}e_{ji}, e_{jk}e_{ik} - e_{ij}(e_{ik} - e_{jk}) \rangle.$$

- (Conner and Goetz, 2015) This is not a Koszul algebra for $n \geq 4$.

- For each $n \geq 1$, the groups P_n, wP_n^+, and $\Pi_n := \prod_{i=1}^{n-1} F_i$ have the same Betti numbers and LCS ranks.
- Moreover, for each $n \leq 3$, they are pairwise isomorphic.

THEOREM (S.–WANG 2020)

If G_1 and G_2 are 1-formal (or, more generally, filtered formal), and if $\theta_k(G_1) \neq \theta_k(G_2)$ for some $k \geq 1$, then $\operatorname{gr}(G_1, \mathbb{Q}) \not\cong \operatorname{gr}(G_2, \mathbb{Q})$, as graded Lie algebras.

COROLLARY

For $n \geq 4$, the graded Lie algebras $\operatorname{gr}(P_n, \mathbb{Q})$, $\operatorname{gr}(wP_n^+, \mathbb{Q})$, and $\operatorname{gr}(\Pi_n, \mathbb{Q})$ are pairwise non-isomorphic.

Indeed, these groups are all 1-formal, and:

- $\theta_k(P_n) = (k-1)\binom{n+1}{4}$ for $k \geq 3$. [Cohen–S. 1995]
- $\theta_k(P\Sigma_n^+) = \binom{n+1}{4} + \sum_{i=3}^{k} \binom{n+i-2}{i+1}$ for $k \geq 3$. [S.–Wang 2019]
- $\theta_k(\Pi_n) = (k-1)\binom{k+n-2}{k+1}$ for $k \geq 2$. [Chen, Cohen–S.]

Virtual braid groups

- The virtual braid group vB_n is obtained from wB_n by omitting certain commutation relations.
- Let $vP_n = \ker(vB_n \to S_n)$ be the pure virtual braid group.
- Bardakov (2004) gave a presentation for vP_n, with generators x_{ij} ($1 \le i \ne j \le n$), subject to the relations

$$\begin{aligned} x_{ij}x_{ik}x_{jk} &= x_{jk}x_{ik}x_{ij}, && \text{for } i,j,k \text{ distinct,} \\ [x_{ij}, x_{st}] &= 1, && \text{for } i,j,s,t \text{ distinct.} \end{aligned}$$

- Let vP_n^+ be the subgroup of vP_n generated by x_{ij} for $i < j$. The inclusion $vP_n^+ \hookrightarrow vP_n$ *is* a split injection.
- Bartholdi, Enriquez, Etingof, and Rains (2006) studied vP_n and vP_n^+ as groups arising from the Yang–Baxter equation.
- They constructed classifying spaces by taking quotients of permutahedra by suitable actions of the symmetric groups.

Theorem (Bartholdi–Enriquez–Etingof–Rains 2006, Lee 2013)

For the groups $G_n = vP_n$ and vP_n^+,

- *The cohomology algebra $H^*(G_n, \mathbb{Q})$ is a Koszul algebra.*
- *The maps $\mathfrak{h}(G_n, \mathbb{Q}) \twoheadrightarrow \operatorname{gr}(G_n, \mathbb{Q})$ are isomorphisms, for all n.*

Theorem (S.–Wang 2017)

The LCS ranks of the groups $G_n = vP_n$ and vP_n^+ are given by

$$\phi_k(G_n) = \frac{1}{k} \sum_{d|k} \mu\left(\frac{k}{d}\right) \left[\sum_{m_1+2m_2+\cdots+nm_n=d} (-1)^{s_n} d(m!) \prod_{j=1}^{n} \frac{(b_{n,n-j})^{m_j}}{(m_j)!} \right],$$

where $m_j \geq 0$, $s_n = \sum_{i=1}^{[n/2]} m_{2i}$, $m = \sum_{i=1}^{n} m_i - 1$, and $b_{n,j}$ are the Lah numbers for $G_n = vP_n$ and the Stirling numbers of the second kind for $G_n = vP_n^+$.

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