Betti numbers of abelian covers

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References

- S. Papadima and A. Suciu, *Bieri–Neumann–Strebel–Renz invariants and homology jumping loci*, Proc. London Math. Soc. **100** (2010), no. 3, 795–834.
- A. Suciu, *Resonance varieties and Dwyer–Fried invariants*, to appear in Advanced Studies Pure Math., Kinokuniya, Tokyo, 2011.
- A. Suciu, *Characteristic varieties and Betti numbers of free abelian covers*, preprint 2011.
- A. Suciu, Y. Yang, and G. Zhao, *Homological finiteness of abelian covers*, preprint 2011.

Abelian covers

- Let X be a connected, finite-type CW-complex. We may assume X has a single 0-cell, call it x_0. Set
  - $G = \pi_1(X, x_0)$.
  - $H = G_{ab} = H_1(X, \mathbb{Z})$.
- Let A be a finitely generated abelian group.
- Any epimorphism $\nu\colon G \twoheadrightarrow A$ gives rise to a connected, regular cover, $X^\nu \to X$, with group of deck transformations A.
- Conversely, any conn., regular A-cover $p\colon (Y, y_0) \to (X, x_0)$ yields

$$1 \longrightarrow \pi_1(Y, y_0) \xrightarrow{p_\sharp} \pi_1(X, x_0) \xrightarrow{\nu} A \longrightarrow 1$$

  so that there is an A-equivariant homeomorphism $Y \cong X^\nu$.
- Thus, the set of conn., regular A-covers of X can be identified with

$$\operatorname{Epi}(G, A)/\operatorname{Aut}(A) = \operatorname{Epi}(H, A)/\operatorname{Aut}(A).$$

Theorem (S.–Yang–Zhao 2011)

There is a bijection

$$\mathrm{Epi}(H,A)/\mathrm{Aut}(A) \longleftrightarrow \mathrm{GL}_n(\mathbb{Z}) \times_{\mathrm{P}} \Gamma$$

where $n = \operatorname{rank} H$, $r = \operatorname{rank} A$, *and*

- P *is a parabolic subgroup of* $\mathrm{GL}_n(\mathbb{Z})$.
- $\mathrm{GL}_n(\mathbb{Z})/\mathrm{P} = \mathrm{Gr}_{n-r}(\mathbb{Z}^n)$.
- $\Gamma = \mathrm{Epi}(\mathbb{Z}^{n-r} \oplus \mathrm{Tors}(H), \mathrm{Tors}(A))/\mathrm{Aut}(\mathrm{Tors}(A))$ *is a finite set.*
- $\mathrm{GL}_n(\mathbb{Z}) \times_{\mathrm{P}} \Gamma$ *is the twisted product under the diagonal* P*-action.*

This bijection depends on a choice of splitting $H \cong \overline{H} \oplus \mathrm{Tors}(H)$, and a choice of basis for $\overline{H} = H/\mathrm{Tors}(H)$.

- Simplest situation is when $A = \mathbb{Z}^r$.
- Basic example: the universal cover of the r-torus, $\mathbb{Z}^r \to \mathbb{R}^r \to T^r$.
- All conn., regular $\mathbb{Z}^r$-covers of X arise as pull-backs of this cover:

$$\begin{array}{ccc} X^\nu & \longrightarrow & \mathbb{R}^r \\ \downarrow & & \downarrow \\ X & \xrightarrow{\ f\ } & T^r, \end{array}$$

where $f_\sharp\colon \pi_1(X) \to \pi_1(T^r)$ realizes the epimorphism $\nu\colon G \twoheadrightarrow \mathbb{Z}^r$. (Note: X^ν is the homotopy fiber of f).
- Let $\nu^*\colon \mathbb{Q}^r \to H^1(X, \mathbb{Q}) = \mathbb{Q}^n$. We get:

$$\{\mathbb{Z}^r\text{-covers } X^\nu \to X\} \longleftrightarrow \{r\text{-planes } P_\nu := \operatorname{im}(\nu^*) \text{ in } H^1(X, \mathbb{Q})\}.$$

and so

$$\operatorname{Epi}(H, \mathbb{Z}^r)/\operatorname{Aut}(\mathbb{Z}^r) \cong \operatorname{Gr}_{n-r}(\mathbb{Z}^n) \cong \operatorname{Gr}_r(\mathbb{Q}^n).$$

The Dwyer–Fried sets

Moving about the parameter space for A-covers, and recording how the Betti numbers of those covers vary leads to:

Definition

The *Dwyer–Fried invariants* of X are the subsets

$$\Omega^i_A(X) = \{[\nu] \in \operatorname{Epi}(H, A)/\operatorname{Aut}(A) \mid b_j(X^\nu) < \infty, \text{ for } j \le i\}.$$

where, $X^\nu \to X$ is the cover corresponding to $\nu\colon H \twoheadrightarrow A$.

In particular, when $A = \mathbb{Z}^r$,

$$\Omega^i_r(X) = \{P_\nu \in \operatorname{Gr}_r(H^1(X, \mathbb{Q})) \mid b_j(X^\nu) < \infty \text{ for } j \le i\},$$

with the convention that $\Omega^i_r(X) = \emptyset$ if $r > n = b_1(X)$. For a fixed $r > 0$, get filtration

$$\operatorname{Gr}_r(\mathbb{Q}^n) = \Omega^0_r(X) \supseteq \Omega^1_r(X) \supseteq \Omega^2_r(X) \supseteq \cdots .$$

The Ω-sets are homotopy-type invariants of X:

Lemma

Suppose $X \simeq Y$. For each $r > 0$, there is an isomorphism $\mathrm{Gr}_r(H^1(Y,\mathbb{Q})) \cong \mathrm{Gr}_r(H^1(X,\mathbb{Q}))$ sending each subset $\Omega^i_r(Y)$ bijectively onto $\Omega^i_r(X)$.

Thus, we may extend the definition of the Ω-sets from spaces to groups, by setting

$$\Omega^i_r(G) = \Omega^i_r(K(G,1))$$

Similarly for the sets $\Omega^i_A(X)$.

- Especially manageable situation: $r = n$, where $n = b_1(X) > 0$.
- In this case, $\operatorname{Gr}_n(H^1(X, \mathbb{Q})) = \{\text{pt}\}$.
- This single point corresponds to the maximal free abelian cover, $X^{\alpha} \to X$, where $\alpha\colon G \twoheadrightarrow \overline{G_{\text{ab}}} = \mathbb{Z}^n$.
- The sets $\Omega^i_n(X)$ are then given by

$$\Omega^i_n(X) = \begin{cases} \{\text{pt}\} & \text{if } b_j(X^{\alpha}) < \infty \text{ for } j \le i, \\ \emptyset & \text{otherwise.} \end{cases}$$

Example

Let $X = S^1 \vee S^k$, for some $k > 1$. Then $X^{\alpha} \simeq \bigvee_{j \in \mathbb{Z}} S^k_j$. Thus,

$$\Omega^i_n(X) = \begin{cases} \{\text{pt}\} & \text{for } i < k, \\ \emptyset & \text{for } i \ge k. \end{cases}$$

Comparison diagram

- There is an commutative diagram,

$$\begin{array}{ccc} \Omega^i_A(X) & \hookrightarrow & \operatorname{Epi}(G,A)/\operatorname{Aut} A \cong \mathrm{GL}_n(\mathbb{Z}) \times_{\mathrm{P}} \Gamma \\ \downarrow & & \downarrow \\ \Omega^i_r(X) & \hookrightarrow & \operatorname{Gr}_r(\mathbb{Q}^n) \end{array}$$

- If $\Omega^i_r(X) = \emptyset$, then $\Omega^i_A(X) = \emptyset$.
- The above is a pull-back diagram if and only if:

  If X^ν is a $\mathbb{Z}^r$-cover with finite Betti numbers up to degree i, then any regular $\operatorname{Tors}(A)$-cover of X^ν has the same finiteness property.

Example

Let $X = S^1 \vee \mathbb{RP}^2$. Then $G = \mathbb{Z} * \mathbb{Z}_2$, $G_{\mathrm{ab}} = \mathbb{Z} \oplus \mathbb{Z}_2$, and

$$X^\alpha \simeq \bigvee_{j\in\mathbb{Z}} \mathbb{RP}^2_j,$$

$$X^{\mathrm{ab}} \simeq \bigvee_{j\in\mathbb{Z}} S^1_j \vee \bigvee_{j\in\mathbb{Z}} S^2_j.$$

Thus, $b_1(X^\alpha) = 0$, yet $b_1(X^{\mathrm{ab}}) = \infty$.

Hence, $\Omega^1_1(X) \neq \emptyset$, but $\Omega^1_{\mathbb{Z}\oplus\mathbb{Z}_2}(X) = \emptyset$.

Remark

Finiteness of the Betti numbers of a free abelian cover X^ν does not imply finite-generation of the integral homology groups of X^ν.

E.g., let K be a knot in S^3, with complement $X = S^3 \setminus K$, infinite cyclic cover X^{ab}, and Alexander polynomial $\Delta_K \in \mathbb{Z}[t^{\pm 1}]$. Then

$$H_1(X^{\mathrm{ab}}, \mathbb{Z}) = \mathbb{Z}[t^{\pm 1}]/(\Delta_K).$$

Hence, $H_1(X^{\mathrm{ab}}, \mathbb{Q}) = \mathbb{Q}^d$, where $d = \deg \Delta_K$. Thus,

$$\Omega^1_1(X) = \{\mathrm{pt}\}.$$

But, if Δ_K is not monic, $H_1(X^{\mathrm{ab}}, \mathbb{Z})$ need not be finitely generated.

Example (Milnor 1968)

Let K be the 5_2 knot, with Alex polynomial $\Delta_K = 2t^2 - 3t + 2$. Then $H_1(X^{\mathrm{ab}}, \mathbb{Z}) = \mathbb{Z}[1/2] \oplus \mathbb{Z}[1/2]$ is not f.g., though $H_1(X^{\mathrm{ab}}, \mathbb{Q}) = \mathbb{Q} \oplus \mathbb{Q}$.

Characteristic varieties

- Consider the group of complex-valued characters of G,

$$\widehat{G} = \operatorname{Hom}(G, \mathbb{C}^\times) = H^1(X, \mathbb{C}^\times)$$

- Let $G_{\mathrm{ab}} = G/G' \cong H_1(X, \mathbb{Z})$ be the abelianization of G. The projection $\mathrm{ab}\colon G \to G_{\mathrm{ab}}$ induces an isomorphism $\widehat{G_{\mathrm{ab}}} \xrightarrow{\simeq} \widehat{G}$.
- The identity component, $\widehat{G}^0$, is isomorphic to a complex algebraic torus of dimension $n = \operatorname{rank} G_{\mathrm{ab}}$.
- The other connected components are all isomorphic to $\widehat{G}^0 = (\mathbb{C}^\times)^n$, and are indexed by the finite abelian group $\operatorname{Tors}(G_{\mathrm{ab}})$.
- $\widehat{G}$ parametrizes rank 1 local systems on X:

$$\rho\colon G \to \mathbb{C}^\times \quad \rightsquigarrow \quad \mathcal{L}_\rho$$

  the complex vector space $\mathbb{C}$, viewed as a right module over the group ring $\mathbb{Z}G$ via $a \cdot g = \rho(g)a$, for $g \in G$ and $a \in \mathbb{C}$.

The homology groups of X with coefficients in $\mathcal{L}_\rho$ are defined as

$$H_*(X, \mathcal{L}_\rho) = H_*(\mathcal{L}_\rho \otimes_{\mathbb{Z}G} C_\bullet(\widetilde{X}, \mathbb{Z})),$$

where $C_\bullet(\widetilde{X}, \mathbb{Z})$ is the equivariant chain complex of the universal cover of X.

Definition

The *characteristic varieties* of X are the sets

$$\mathcal{V}^i(X) = \{\rho \in \widehat{G} \mid H_j(X, \mathcal{L}_\rho) \neq 0, \text{ for some } j \leq i\}$$

- Get filtration $\{1\} = \mathcal{V}^0(X) \subseteq \mathcal{V}^1(X) \subseteq \cdots \subseteq \widehat{G}$.
- Each $\mathcal{V}^i(X)$ is a Zariski closed subset of the algebraic group $\widehat{G}$.
- The characteristic varieties are homotopy-type invariants: Suppose $X \simeq X'$. There is then an isomorphism $\widehat{G'} \cong \widehat{G}$, which restricts to isomorphisms $\mathcal{V}^i(X') \cong \mathcal{V}^i(X)$.

The characteristic varieties may be reinterpreted as the support varieties for the Alexander invariants of X.

- Let $X^{\mathrm{ab}} \to X$ be the maximal abelian cover. View $H_*(X^{\mathrm{ab}}, \mathbb{C})$ as a module over $\mathbb{C}[G_{\mathrm{ab}}]$. Then (Papadima–S. 2010),

$$\mathcal{V}^i(X) = V\Big(\operatorname{ann}\Big(\bigoplus_{j\le i} H_j(X^{\mathrm{ab}}, \mathbb{C})\Big)\Big).$$

- Set $\mathcal{W}^i(X) = \mathcal{V}^i(X) \cap \widehat{G}^0$. View $H_*(X^\alpha, \mathbb{C})$ as a module over $\mathbb{C}[G_\alpha] \cong \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$, where $n = b_1(G)$. Then

$$\mathcal{W}^i(X) = V\Big(\operatorname{ann}\Big(\bigoplus_{j\le i} H_j(X^\alpha, \mathbb{C})\Big)\Big).$$

Example

Let $L = (L_1, \dots, L_n)$ be a link in S^3, with complement $X = S^3 \setminus \bigcup_{i=1}^n L_i$ and Alexander polynomial $\Delta_L = \Delta_L(t_1, \dots, t_n)$. Then

$$\mathcal{V}^1(X) = \{z \in (\mathbb{C}^\times)^n \mid \Delta_L(z) = 0\} \cup \{1\}.$$

Computing the Ω-invariants

- Given an epimorphism $\nu\colon G \twoheadrightarrow \mathbb{Z}^r$, let

$$\hat{\nu}\colon \widehat{\mathbb{Z}^r} \hookrightarrow \widehat{G}, \qquad \hat{\nu}(\rho)(g) = \nu(\rho(g))$$

  be the induced monomorphism between character groups.
- Its image, $\mathbb{T}_\nu = \hat{\nu}\big(\widehat{\mathbb{Z}^r}\big)$, is a complex algebraic subtorus of $\widehat{G}$, isomorphic to $(\mathbb{C}^\times)^r$.

Theorem (Dwyer–Fried 1987, Papadima–S. 2010)

Let X be a connected CW-complex with finite k-skeleton, $G = \pi_1(X)$. For an epimorphism $\nu\colon G \twoheadrightarrow \mathbb{Z}^r$, the following are equivalent:

1. *The vector space $\bigoplus_{i=0}^{k} H_i(X^\nu, \mathbb{C})$ is finite-dimensional.*
2. *The algebraic torus $\mathbb{T}_\nu$ intersects the variety $\mathcal{W}^k(X)$ in only finitely many points.*

Let $\exp\colon H^1(X,\mathbb{C}) \to H^1(X,\mathbb{C}^\times)$ be the coefficient homomorphism induced by the homomorphism $\mathbb{C} \to \mathbb{C}^\times$, $z \mapsto e^z$.

Lemma

Let $\nu\colon G \twoheadrightarrow \mathbb{Z}^r$ be an epimorphism. Under the universal coefficient isomorphism $H^1(X,\mathbb{C}^\times) \cong \mathrm{Hom}(G,\mathbb{C}^\times)$, the complex r-torus $\exp(P_\nu \otimes \mathbb{C})$ corresponds to $\mathbb{T}_\nu = \hat{\nu}(\widehat{\mathbb{Z}^r})$.

Thus, we may reinterpret the Ω-invariants, as follows:

Theorem

$$\Omega^i_r(X) = \left\{P \in \mathrm{Gr}_r(H^1(X,\mathbb{Q})) \;\middle|\; \dim\left(\exp(P \otimes \mathbb{C}) \cap \mathcal{W}^i(X)\right) = 0\right\}.$$

More generally,

Theorem (S.–Yang–Zhao 2011)

$$\Omega^i_A(X) = \{\, [\nu] \in \mathrm{Epi}(H, A)/\,\mathrm{Aut}(A) \mid \mathrm{im}(\hat{\nu}) \cap \mathcal{V}^i(X) \textit{ is finite} \,\}.$$

Corollary

Suppose $\mathcal{W}^i(X)$ *is finite. Then* $\Omega^i_r(X) = \mathrm{Gr}_r(H^1(X,\mathbb{Q}))$, $\forall r \le b_1(X)$.

Example

Let M be a nilmanifold. By (Macinic–Papadima 2009): $\mathcal{W}^i(M) = \{1\}$, for all $i \ge 0$. Hence,

$$\Omega^i_r(M) = \mathrm{Gr}_r(\mathbb{Q}^n), \qquad \forall i \ge 0,\ r \le n = b_1(M).$$

Example

Let X be the complement of a knot in S^m, $m \ge 3$. Then

$$\Omega^i_1(X) = \{\mathrm{pt}\}, \qquad \forall i \ge 0.$$

Corollary

Let $n = b_1(X)$. Suppose $\mathcal{W}^i(X)$ is infinite, for some $i > 0$. Then $\Omega^q_n(X) = \emptyset$, for all $q \geq i$.

Example

Let S_g be a Riemann surface of genus $g > 1$. Then

$$\Omega^i_r(S_g) = \emptyset, \quad \text{for all } i, r \geq 1$$
$$\Omega^n_r(S_{g_1} \times \cdots \times S_{g_n}) = \emptyset, \quad \text{for all } r \geq 1$$

Example

Let $Y_m = \bigvee^m S^1$ be a wedge of m circles, $m > 1$. Then

$$\Omega^i_r(Y_m) = \emptyset, \quad \text{for all } i, r \geq 1$$
$$\Omega^n_r(Y_{m_1} \times \cdots \times Y_{m_n}) = \emptyset, \quad \text{for all } r \geq 1$$

Tangent cones

Let $W = V(I)$ be a Zariski closed subset in $(\mathbb{C}^\times)^n$.

Definition

- The *tangent cone* at 1 to W:

$$\mathrm{TC}_1(W) = V(\mathrm{in}(I))$$

- The *exponential tangent cone* at 1 to W:

$$\tau_1(W) = \{z \in \mathbb{C}^n \mid \exp(\lambda z) \in W, \ \forall \lambda \in \mathbb{C}\}$$

Both types of tangent cones

- are homogeneous subvarieties of $\mathbb{C}^n$;
- are non-empty iff $1 \in W$;
- depend only on the analytic germ of W at 1;
- commute with finite unions.

Moreover,

- τ_1 commutes with (arbitrary) intersections;
- $\tau_1(W) \subseteq \mathrm{TC}_1(W)$
  - $=$ if all irred components of W are subtori
  - $\neq$ in general
- (Dimca–Papadima–S. 2009) $\tau_1(W)$ is a finite union of rationally defined linear subspaces of $\mathbb{C}^n$.

Characteristic subspace arrangements

Set $n = b_1(X)$, and identify $H^1(X, \mathbb{C}) = \mathbb{C}^n$ and $H^1(X, \mathbb{C}^\times)^0 = (\mathbb{C}^\times)^n$.

Definition

The *i-th characteristic arrangement* of X, denoted $\mathcal{C}_i(X)$, is the subspace arrangement in $H^1(X, \mathbb{Q})$ whose complexified union is the exponential tangent cone to $\mathcal{W}^i(X)$:

$$\tau_1(\mathcal{W}^i(X)) = \bigcup_{L \in \mathcal{C}_i(X)} L \otimes \mathbb{C}.$$

- We get a sequence $\mathcal{C}_0(X), \mathcal{C}_1(X), \dots$ of rational subspace arrangements, all lying in $H^1(X, \mathbb{Q}) = \mathbb{Q}^n$.
- The arrangements $\mathcal{C}_i(X)$ depend only on the homotopy type of X.

Theorem

$$\Omega^i_r(X) \subseteq \Big(\bigcup_{L \in \mathcal{C}_i(X)} \{ P \in \mathrm{Gr}_r(H^1(X,\mathbb{Q})) \mid P \cap L \neq \{0\} \} \Big)^{\complement}.$$

Proof.

Fix an r-plane $P \in \mathrm{Gr}_r(H^1(X,\mathbb{Q}))$, and let $T = \exp(P \otimes \mathbb{C})$. Then:

$$\begin{aligned}
P \in \Omega^i_r(X) &\iff T \cap \mathcal{W}^i(X) \text{ is finite} \\
&\implies \tau_1(T \cap \mathcal{W}^i(X)) = \{0\} \\
&\iff (P \otimes \mathbb{C}) \cap \tau_1(\mathcal{W}^i(X)) = \{0\} \\
&\iff P \cap L = \{0\}, \text{ for each } L \in \mathcal{C}_i(X),
\end{aligned}$$

□

- For "straight" spaces, the inclusion holds as an equality.
- If $r = 1$, the inclusion always holds as an equality.
- In general, though, the inclusion is strict. E.g., there exist finitely presented groups G for which $\Omega^1_2(G)$ is *not* open.

Example

Let $G = \langle x_1, x_2, x_3 \mid [x_1^2, x_2],\ [x_1, x_3],\ x_1[x_2, x_3]x_1^{-1}[x_2, x_3]\rangle$. Then $G_{\mathrm{ab}} = \mathbb{Z}^3$, and

$$\mathcal{V}^1(G) = \{1\} \cup \{t \in (\mathbb{C}^\times)^3 \mid t_1 = -1\}.$$

Let $T = (\mathbb{C}^\times)^2$ be an algebraic 2-torus in $(\mathbb{C}^\times)^3$. Then

$$T \cap \mathcal{V}^1(G) = \begin{cases} \{1\} & \text{if } T = \{t_1 = 1\} \\ \mathbb{C}^\times & \text{otherwise} \end{cases}$$

Thus, $\Omega^1_2(G)$ consists of a single point in $\mathrm{Gr}_2(H^1(G, \mathbb{Q})) = \mathbb{QP}^2$, and so it's not open.

Example

- Let C_1 be a curve of genus 2 with an elliptic involution σ_1.
- Let C_2 be a curve of genus 3 with a free involution σ_2.

Then

- $\Sigma_1 = C_1/\sigma_1$ is a curve of genus 1.
- $\Sigma_2 = C_2/\sigma_2$ is a curve of genus 2.

We let $\mathbb{Z}_2$ act freely on the product $C_1 \times C_2$ via the involution $\sigma_1 \times \sigma_2$. The quotient space,

$$M = (C_1 \times C_2)/\mathbb{Z}_2,$$

is a smooth, minimal, complex projective surface of general type with $p_g(M) = q(M) = 3$, and $K_M^2 = 8$. The projection $\mathrm{pr}_2 \colon C_1 \times C_2 \to C_2$ induces a smooth fibration,

$$C_1 \to M \to \Sigma_2.$$

Example (Continued)

Let $\pi = \pi_1(M)$. Then $\pi_{ab} = \mathbb{Z}^6$, $\widehat{\pi} = (\mathbb{C}^\times)^6$, and

$$\mathcal{V}^1(\pi) = \{t \mid t_1 = t_2 = 1\} \cup \{t_4 = t_5 = t_6 = 1,\ t_3 = -1\}.$$

It follows that $\Omega_2^1(\pi)$ consists of a single point in $\mathrm{Gr}_2(\mathbb{Q}^6)$, corresponding to the plane spanned by the vectors e_1 and e_2.

In particular, $\Omega_2^1(\pi)$ is not open.

Special Schubert varieties

- Let V be a homogeneous variety in $\Bbbk^n$. The set
$$\sigma_r(V) = \{P \in \mathrm{Gr}_r(\Bbbk^n) \mid P \cap V \neq \{0\}\}$$
is a Zariski closed subset of $\mathrm{Gr}_r(\Bbbk^n)$, called the *variety of incident r-planes* to V.
- When V is a a linear subspace $L \subset \Bbbk^n$, the variety $\sigma_r(L)$ is called the *special Schubert variety* defined by L.
- If L has codimension d in $\Bbbk^n$, then $\sigma_r(L)$ has codimension $d - r + 1$ in $\mathrm{Gr}_r(\Bbbk^n)$.

Example

The Grassmannian $\mathrm{Gr}_2(\Bbbk^4)$ is the hypersurface in $\mathbb{P}(\Bbbk^6)$ with equation $p_{12}p_{34} - p_{13}p_{24} + p_{23}p_{14} = 0$. Let L be a plane in $\Bbbk^4$, represented as the row space of a 2×4 matrix. Then $\sigma_2(L)$ is the 3-fold in $\mathrm{Gr}_2(\Bbbk^4)$ cut out by the hyperplane
$$p_{12}L_{34} - p_{13}L_{24} - p_{23}L_{14} + p_{14}L_{23} - p_{24}L_{13} + p_{34}L_{12} = 0.$$

Theorem

$$\Omega^i_r(X) \subseteq \mathrm{Gr}_r\left(H^1(X,\mathbb{Q})\right) \setminus \Big(\bigcup_{L\in\mathcal{C}_i(X)} \sigma_r(L)\Big).$$

Thus, each set $\Omega^i_r(X)$ is contained in the complement of a Zariski closed subset of $\mathrm{Gr}_r(H^1(X,\mathbb{Q}))$: the union of the special Schubert varieties corresponding to the subspaces comprising $\mathcal{C}_i(X)$.

Corollary

Suppose $\mathcal{C}_i(X)$ contains a subspace of codimension d. Then $\Omega^i_r(X) = \emptyset$, for all $r \geq d+1$.

Corollary

Let X^α be the maximal free abelian cover of X. If $\tau_1(\mathcal{W}^1(X)) \neq \{0\}$, then $b_1(X^\alpha) = \infty$.

The Aomoto complex

Consider the cohomology algebra $A = H^*(X, \mathbb{C})$, with product operation given by the cup product of cohomology classes.

For each $a \in A^1$, we have $a^2 = 0$, by graded-commutativity of the cup product.

Definition

The *Aomoto complex* of A (with respect to $a \in A^1$) is the cochain complex of finite-dimensional, complex vector spaces,

$$(A, a)\colon \; A^0 \xrightarrow{a} A^1 \xrightarrow{a} A^2 \xrightarrow{a} \cdots \xrightarrow{a} A^k \,,$$

with differentials given by left-multiplication by a.

Alternative interpretation: Pick a basis $\{e_1, \dots, e_n\}$ for $A^1 = H^1(X, \mathbb{C})$, and let $\{x_1, \dots, x_n\}$ be the Kronecker dual basis for $A_1 = H_1(X, \mathbb{C})$. Identify $\mathrm{Sym}(A_1)$ with $S = \mathbb{C}[x_1, \dots, x_n]$.

Definition

The *universal Aomoto complex* of A is the cochain complex of free S-modules,

$$: \quad \cdots \longrightarrow A^i \otimes_{\mathbb{C}} S \xrightarrow{d^i} A^{i+1} \otimes_{\mathbb{C}} S \xrightarrow{d^{i+1}} A^{i+2} \otimes_{\mathbb{C}} S \longrightarrow \cdots,$$

where the differentials are defined by $d^i(u \otimes 1) = \sum_{j=1}^n e_j u \otimes x_j$ for $u \in A^i$, and then extended by S-linearity.

Lemma

The evaluation of the universal Aomoto complex at an element $a \in A^1$ coincides with the Aomoto complex (A, a).

Let X be a connected, finite-type CW-complex.

The CW-structure on X is *minimal* if the number of i-cells of X equals the Betti number $b_i(X)$, for every $i \geq 0$.

Equivalently, all boundary maps in $C_\bullet(X, \mathbb{Z})$ are zero.

Theorem (Papadima–S. 2010)

If X is a minimal CW-complex, the linearization of the cochain complex $C^\bullet(X^{\mathrm{ab}}, \mathbb{C})$ coincides with the universal Aomoto complex of $H^*(X, \mathbb{C})$.

Resonance varieties

Definition

The *resonance varieties* of X are the sets

$$\mathcal{R}^i(X) = \{a \in A^1 \mid H^j(A, \cdot a) \neq 0, \text{ for some } j \leq i\},$$

defined for all integers $0 \leq i \leq k$.

- Get filtration
  $\{0\} = \mathcal{R}^0(X) \subseteq \mathcal{R}^1(X) \subseteq \cdots \subseteq \mathcal{R}^k(X) \subseteq H^1(X, \mathbb{C}) = \mathbb{C}^n$.
- Each $\mathcal{R}^i(X)$ is a homogeneous algebraic subvariety of $\mathbb{C}^n$.
- These varieties are homotopy-type invariants of X:
  If $X \simeq Y$, there is an isomorphism $H^1(Y, \mathbb{C}) \cong H^1(X, \mathbb{C})$ which restricts to isomorphisms $\mathcal{R}^i(Y) \cong \mathcal{R}^i(X)$, for all $i \geq 0$.
- (Libgober 2002) $\mathrm{TC}_1(\mathcal{W}^i(X)) \subseteq \mathcal{R}^i(X)$.

Straight spaces

Let X be a connected CW-complex with finite k-skeleton.

Definition

We say X is k*-straight* if the following conditions hold, for each $i \leq k$:

1. All positive-dimensional components of $\mathcal{W}^i(X)$ are algebraic subtori.
2. $\mathrm{TC}_1(\mathcal{W}^i(X)) = \mathcal{R}^i(X)$.

If X is k-straight for all $k \geq 1$, we say X is a *straight space*.

- The k-straightness property depends only on the homotopy type of a space.
- Hence, we may declare a group G to be k-straight if there is a $K(G, 1)$ which is k-straight; in particular, G must be of type F_k.
- X is 1-straight if and only if $\pi_1(X)$ is 1-straight.

Example

- Let $f \in \mathbb{Z}[t]$ with $f(1) = 0$. Then $X_f = (S^1 \vee S^2) \cup_f e^3$ is minimal.
- $\mathcal{W}^1(X_f) = \{1\}$, $\mathcal{W}^2(X_f) = V(f)$: finite subsets of $H^1(X, \mathbb{C}^\times) = \mathbb{C}^\times$.
- $\mathcal{R}^1(X_f) = \{0\}$, and

$$\mathcal{R}^2(X_f) = \begin{cases} \{0\}, & \text{if } f'(1) \neq 0, \\ \mathbb{C}, & \text{otherwise.} \end{cases}$$

- Therefore, X_f is always 1-straight, but

$$X_f \text{ is 2-straight} \iff f'(1) \neq 0.$$

Proposition

For each $k \geq 2$, there is a minimal CW-complex which has the integral homology of $S^1 \times S^k$ and which is $(k-1)$-straight, but not k-straight.

Alternate description of straightness:

Proposition

The space X is k-straight if and only if the following equalities hold, for all $i \leq k$:

$$\mathcal{W}^i(X) = \left(\bigcup_{L \in \mathcal{C}_i(X)} \exp(L \otimes \mathbb{C}) \right) \cup Z_i$$

$$\mathcal{R}^i(X) = \bigcup_{L \in \mathcal{C}_i(X)} L \otimes \mathbb{C}$$

for some finite (algebraic) subsets $Z_i \subset H^1(X, \mathbb{C}^\times)^0$.

Corollary

Let X be a k-straight space. Then, for all $i \leq k$,

1. $\tau_1(\mathcal{W}^i(X)) = \mathrm{TC}_1(\mathcal{W}^i(X)) = \mathcal{R}^i(X)$.
2. $\mathcal{R}^i(X, \mathbb{Q}) = \bigcup_{L \in \mathcal{C}_i(X)} L$.

In particular, the resonance varieties $\mathcal{R}^i(X)$ are unions of rationally defined subspaces.

Example

Let G be the group with generators x_1, x_2, x_3, x_4 and relators $r_1 = [x_1, x_2]$, $r_2 = [x_1, x_4][x_2^{-2}, x_3]$, $r_3 = [x_1^{-1}, x_3][x_2, x_4]$. Then

$$\mathcal{R}^1(G) = \{z \in \mathbb{C}^4 \mid z_1^2 - 2z_2^2 = 0\},$$

which splits into two linear subspaces defined over $\mathbb{R}$, but not over $\mathbb{Q}$. Thus, G is not 1-straight.

Ω-invariants of straight spaces

Theorem

Suppose X is k-straight. Then, for all $i \le k$ and $r \ge 1$,

$$\Omega^i_r(X) = \mathrm{Gr}_r(H^1(X,\mathbb{Q})) \setminus \sigma_r(\mathcal{R}^i(X,\mathbb{Q})).$$

In particular, if all components of $\mathcal{R}^i(X)$ have the same codimension r, then $\Omega^i_r(X)$ is the complement of the Chow divisor of $\mathcal{R}^i(X,\mathbb{Q})$.

Corollary

Let X be k-straight space, with $b_1(X) = n$. Then each set $\Omega^i_r(X)$ is the complement of a finite union of special Schubert varieties in $\mathrm{Gr}_r(\mathbb{Q}^n)$. In particular, $\Omega^i_r(X)$ is a Zariski open set in $\mathrm{Gr}_r(\mathbb{Q}^n)$.

Example

- Let $L = (L_1, L_2)$ be a 2-component link in S^3, with $\operatorname{lk}(L_1, L_2) = 1$, and Alexander polynomial $\Delta_L(t_1, t_2) = t_1 + t_1^{-1} - 1$.

- Let X be the complement of L. Then $\mathcal{W}^1(X) \subset (\mathbb{C}^\times)^2$ is given by

$$\mathcal{W}^1(X) = \{1\} \cup \{t \mid t_1 = e^{\pi i/3}\} \cup \{t \mid t_1 = e^{-\pi i/3}\}$$

  Hence, X is not 1-straight.

- Since $\mathcal{W}^1(X)$ is infinite, we have

$$\Omega_2^1(X) = \emptyset.$$

- On the other hand, $\cup_X$ is non-trivial, and so $\mathcal{R}^1(X, \mathbb{Q}) = \{0\}$. Hence,

$$\sigma_2(\mathcal{R}^1(X, \mathbb{Q}))^{\complement} = \{\mathrm{pt}\}.$$