Alexander Invariants of Complex Hyperplane Arrangements

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May 20, 1997

Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an affine arrangement of lines in $\mathbb{C}^2$, transverse to infinity, with vertices $\{v_1, \dots, v_s\}$.

Braid monodromy generators: $\{\alpha_1, \dots, \alpha_s\}$

$$\alpha_k = A_{V_k}^{\delta_k} \in P_n$$

- $V_k = \{i_1, \dots, i_{m_k}\}$ if $v_k = H_{i_1} \cap \dots \cap H_{i_{m_k}}$

- A_{V_k} is the pure braid in P_n which performs a full twist on the strands corresponding to V_k

$v = H_1 \cap H_2 \cap H_3$ $\qquad\qquad$ $A_V \in P_3$

- δ_k is a pure braid determined by a *braided wired diagram*, computed directly from a defining polynomial for $\mathcal{A}$.

The complement of $\mathcal{A}$: $M = \mathbb{C}^2 \setminus \bigcup_{i=1}^n H_i$

The group of $\mathcal{A}$: $G = \pi_1(M)$

Braid monodromy presentation of G:

$$\begin{aligned} G &= G(\alpha_1, \dots, \alpha_s) \\ &= \langle t_1, \dots, t_n \mid \alpha_k(t_i) = t_i \rangle \end{aligned}$$

where $i = 1, \dots, m_k - 1$ and $k = 1, \dots, s$.

Here, $\alpha_k \in P_n$ acts on $F_n = \langle t_1, \dots, t_n \rangle$ via the Artin representation, by basis-conjugating automorphisms:

$$\alpha_k(t_i) = z_{k,i} \cdot t_i \cdot z_{k,i}^{-1}$$

Remark. The complement M is homotopy equivalent to $K(G)$, the 2-complex modeled on the above presentation.

Abelianization ab : $\pi_1(M) \to H_1(M) = \mathbb{Z}^n$. Corresponding (maximal abelian) cover:

$$p : M' \to M$$

Alexander module: $A = H_1(M', p^{-1}(*); \mathbb{Z})$

Alexander invariant: $B = H_1(M'; \mathbb{Z})$

- Modules over the group ring $\mathbb{Z}\mathbb{Z}^n$, which may be identified with $\Lambda = \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$

- Depend only on G (and choice of t_i):

  $A = \mathbb{Z}\mathbb{Z}^n \otimes_{\mathbb{Z}G} IG$, where $IG = \ker(\mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z})$ is the augmentation ideal

  $B = G'/G''$, with action of $G/G' = \mathbb{Z}^n$ defined by the (maximal metabelian) extension $1 \to G'/G'' \to G/G'' \to G/G' \to 1$

- Related by Crowell exact sequence $0 \to B \to A \to I \to 0$, where $I = I(\mathbb{Z}^n)$

To find presentations for modules A and B, start with standard free Λ-resolution of $\mathbb{Z}$,

$$0 \to C_n \xrightarrow{d_n} \cdots \to C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

where $C_0 = \Lambda$, $C_1 = \Lambda^n$, $C_k = \bigwedge^k C_1 = \Lambda^{\binom{n}{k}}$, and $d_k(e_J) = \sum_{r=1}^{k}(-1)^{k+r}(t_{j_r} - 1) \cdot e_{J\setminus\{j_r\}}$, where $e_J = e_{j_1} \wedge \cdots \wedge e_{j_k}$ if $J = \{j_1, \ldots, j_k\}$.

Let $\alpha \in P_n$, with $\alpha(t_i) = z_i \cdot t_i \cdot z_i^{-1}$. Consider the group $G = G(\alpha) = \langle t_1, \ldots, t_n \mid [z_i, t_i] = 1\rangle$.

Gassner representation: $\Theta : P_n \to \mathrm{Aut}(C_1)$

$$\Theta(\alpha)(e_i) = \nabla^{\mathrm{ab}}(\alpha(t_i))$$

where $\nabla(w) = \sum_{j=1}^{n} \frac{\partial w}{\partial t_j} e_j$ is the Fox gradient.

Define $\Phi(\alpha) : C_1 \to C_2$ by

$$\Phi(\alpha)(e_i) = \nabla^{\mathrm{ab}}(z_i) \wedge e_i.$$

Let $X = K(G)$, and X' its max abelian cover. Chain map from $C_\bullet(X')$ to standard res:

$$\begin{array}{ccccccccccc}
 & & C_1 & \xrightarrow{\mathrm{id}-\Theta(\alpha)} & C_1 & \xrightarrow{d_1} & C_0 & \xrightarrow{\epsilon} & \mathbb{Z} & \longrightarrow & 0 \\
 & & \big\downarrow \Phi(\alpha) & & \big\downarrow = & & \big\downarrow = & & \big\downarrow = & & \\
C_3 & \xrightarrow{d_3} & C_2 & \xrightarrow{d_2} & C_1 & \xrightarrow{d_1} & C_0 & \xrightarrow{\epsilon} & \mathbb{Z} & \longrightarrow & 0
\end{array}$$

Get presentations for Alexander modules of $G = G(\alpha)$:

$$C_1 \xrightarrow{\mathrm{id}-\Theta(\alpha)} C_1 \to A \to 0$$

$$C_1 \oplus C_3 \xrightarrow{\begin{pmatrix}\Phi(\alpha) & d_3\end{pmatrix}^\top} C_2 \to B \to 0$$

E.g.: $V = \{i_1, \ldots, i_r\}$; $\quad \alpha = A_V \in P_n$

$G(A_V) = \langle t_1, \ldots, t_n \mid [t_V, t_i] = 1, i \in V\rangle$

where $t_V = t_{i_1} \cdots t_{i_r}$

$\Phi_V = \Phi(A_V) : C_1 \to C_2$

$e_i \mapsto (e_{i_1} + t_{i_1}e_{i_2} + \cdots + t_{i_1} \cdots t_{i_{r-1}}e_{i_r}) \wedge e_i$

for $i \in V$

so $\Phi_V(C_1(V)) \subset C_2(V)$,

where $C_k(V) = \mathrm{span}\{e_J \mid J \subset V\}$

Now back to $\mathcal{A}$, with complement $M = M(\mathcal{A})$, vertex sets $L_2(\mathcal{A}) = \{V_1, \dots, V_s\}$, braid monodromy generators $\alpha_k = A_{V_k}^{\delta_k}$, and group $G = G(\alpha_1, \dots, \alpha_s)$. Using above techniques, get presentation for $B = B(\mathcal{A})$:

$$C_1^s \oplus C_3 \xrightarrow{\begin{pmatrix} \Phi_{V_1}^{\delta_1} & \cdots & \Phi_{V_s}^{\delta_s} & d_3 \end{pmatrix}^\top} C_2 \to B \to 0,$$

where

$$\Phi_V^\delta = \Phi(A_V^\delta) = \Theta_2(\delta) \circ \Phi_V \circ \Theta(\delta^{-1}) : C_1 \to C_2$$
$$\Theta_2(\delta) = \Theta(\delta) \wedge \Theta(\delta) : C_2 \to C_2$$

Theorem. *The Alexander invariant of an arrangement $\mathcal{A}$ has presentation*

$$K_1 \xrightarrow{\Delta} K_0 \to B(\mathcal{A}) \to 0,$$

where $K_1 = \bigoplus_{k=1}^s C_1(V_k') \oplus C_3$, $K_0 = C_2$.

This pres. has $\binom{n}{2}$ generators and $b_2(M) + \binom{n}{3}$ relations. If $\mathcal{A}$ is a complexified real arrangement, may further simplify to $\binom{n}{2} - b_2(M)$ generators, and $\binom{n}{3}$ relations.

We now relate the Alexander invariant, $B(\mathcal{A})$, to a *(coarse) combinatorial Alexander invariant*, $B^{\mathsf{c}}(\mathcal{A})$, determined by $L(\mathcal{A})$ (in fact, by the number and multiplicities of the elements of $L_2(\mathcal{A})$).

For $V \in L_2(\mathcal{A})$, consider the "vertex group"

$$\begin{aligned} G_V &= G(\{A_V, A_{i,j} \mid |\{i,j\} \cap V| \leq 1\}) \\ &= \langle t_i \mid [t_V, t_i] = 1 \text{ if } i \in V, [t_j, t_i] = 1 \text{ if } \{i,j\} \not\subset V \rangle \end{aligned}$$

We write down an explicit free resolution of $B_V = B(G_V)$,

$$\cdots \to C_2(V') \wedge C_2 \xrightarrow{\Delta_V^2} C_2(V') \wedge C_1 \xrightarrow{\Delta_V} C_2(V') \to B_V \to 0,$$

and a chain map $\Psi_{V,\bullet} : C_\bullet \to C_2(V') \wedge C_{\bullet-2}$.

Let $B^{\mathsf{c}}(\mathcal{A}) = \oplus_{V \in L_2(\mathcal{A})} B_V$.

By taking direct sums, get a free resolution of B^{c}, and a chain map from the standard resolution to this resolution:

$$\cdots \to L_2 \xrightarrow{D_2} L_1 \xrightarrow{D_1} L_0 \to B^{\mathsf{c}} \to 0$$

$$\Psi_\bullet : C_\bullet \to L_{\bullet-2}$$

Theorem. *There exists a chain map* $\Upsilon_\bullet$ *from the presentation* $K_\bullet \to B(\mathcal{A})$ *to the resolution* $L_\bullet \to B^{\mathsf{c}}(\mathcal{A})$*,*

$$\begin{array}{ccccccccc} & & & & K_1 & \xrightarrow{\Delta} & K_0 & \longrightarrow & B & \longrightarrow & 0 \\ & & & & \Big\downarrow \Upsilon_1 & & \Big\downarrow \Upsilon_0 & & \Big\downarrow \Pi & & \\ \ldots & \longrightarrow & L_2 & \xrightarrow{D_2} & L_1 & \xrightarrow{D_1} & L_0 & \longrightarrow & B^{\mathsf{c}} & \longrightarrow & 0, \end{array}$$

given by $\Upsilon_0 = \Psi_2$*,* $\Upsilon_1(x,y) = \Gamma(x) + \Psi_3(y)$*. Furthermore, the resulting map* $\Pi : B \to B^{\mathsf{c}}$ *is surjective.*

Let $\widehat{B} = \varprojlim B/I^k B$ be the I-adic completion.

Theorem. *The chain map* $\widehat{\Upsilon}_\bullet : \widehat{K}_\bullet \to \widehat{L}_\bullet$ *induces an isomorphism* $\widehat{B} \xrightarrow{\sim} \widehat{B}^{\mathsf{c}}$ *if and only if the map* $\widehat{\Psi}_3 : \widehat{C}_3 \to \widehat{L}_1$ *is surjective.*

Let G be a finitely presented group. LCS:

$$G_1 = G, \ldots, G_{k+1} = [G_k, G], \ldots$$

The *Chen groups* of G are the LCS quotients of the maximal metabelian quotient G/G'':

$$\frac{(G/G'')_k}{(G/G'')_{k+1}}$$

They are finitely generated abelian groups, of rank θ_k. (If ϕ_k is the rank of the k^{th} LCS quotient of G itself, then $\phi_k = \theta_k$ for $k \leq 3$, and $\phi_k \geq \theta_k$ for $k > 3$.)

Assume G/G' is torsion free. The Chen groups of G are determined by the Alexander invariant $B = B(G)$—in fact, by the associated graded module of its completion (Massey):

$$\frac{(G/G'')_k}{(G/G'')_{k+1}} = \frac{I^{k-2}B}{I^{k-1}B} = \frac{\mathfrak{m}^{k-2}\widehat{B}}{\mathfrak{m}^{k-1}\widehat{B}}$$

Starting with a presentation for B, and using a Groebner basis algorithm to find a presentation for $\operatorname{gr}\widehat{B}$, one computes the Chen ranks $\theta_k(G)$ as the coefficients of the Hilbert series for $\operatorname{gr}\widehat{B}$.

Given an arrangement $\mathcal{A}$, let $\theta_k(\mathcal{A}) = \theta_k(G(\mathcal{A}))$ be its Chen ranks. For $k \geq 2$, define the *(coarse) combinatorial Chen ranks* of $\mathcal{A}$ by

$$\theta_k^{\mathsf{C}}(\mathcal{A}) := \sum_{V \in L_2(\mathcal{A})} \theta_k(G_V)$$
$$= (k-1) \cdot \sum_{r \geq 3} c_r \binom{k+r-3}{k}$$

where $c_r = \#\{V \in L_2(\mathcal{A}) \mid |V| = r\}$.

As a corollary to previous theorems, we get a combinatorial lower bound for Chen ranks:

$$\theta_k(\mathcal{A}) \geq \theta_k^{\mathsf{C}}(\mathcal{A})$$

Remark. As shown by Falk, the ranks of the LCS quotients of $G(\mathcal{A})$ are combinatorial, and satisfy analogous lower bounds: $\phi_k(\mathcal{A}) \geq \phi_k^{\mathsf{C}}(\mathcal{A}) = \sum_{V \in L_2(\mathcal{A})} \phi_k(G_V)$.

Let $\widehat{\epsilon} : \widehat{\Lambda} \to \mathbb{Z}$ be the augmentation map. If $\widehat{F} = \widehat{\Lambda}^p$ is a free module, denote its image under $\widehat{\epsilon}$ by $\overline{F} = \mathbb{Z}^p$.

Theorem. *The rank of*

$$\overline{\Psi}_3 : \overline{C}_3 \to \overline{L}_1 = \bigoplus_{V \in L_2(\mathcal{A})} \overline{C}_2(V') \wedge \overline{C}_1$$

is combinatorially determined.

The rank of the third Chen group of $\mathcal{A}$ is given by the combinatorial formula

$$\theta_3(\mathcal{A}) = \operatorname{rank}(\operatorname{coker} \overline{\Psi}_3) + \theta_3^{\mathsf{c}}(\mathcal{A}).$$

If $\overline{\Psi}_3$ is surjective, then $\widehat{B} \cong \widehat{B}^{\mathsf{c}} = \oplus_V \widehat{B}_V$, and the ranks of the Chen groups of $\mathcal{A}$ are given by the combinatorial formula

$$\theta_k(\mathcal{A}) = \theta_k^{\mathsf{c}}(\mathcal{A}).$$

Example: The Pappus 9_3 configurations

$\mathcal{P}_1$: realization of Pappus configuration $(9_3)_1$
$\mathcal{P}_2$: realization of Pappus configuration $(9_3)_2$

The combinatorial distinction between these two arrangements is detected by the maps $\overline{\Psi}_3(\mathcal{P}_k) : \mathbb{Z}^{84} \to \mathbb{Z}^{63}$.

- The map $\overline{\Psi}_3(\mathcal{P}_2)$ is surjective, and so the module $\widehat{B}(\mathcal{P}_2)$ decomposes as a direct sum. We get:

$$\theta_k(\mathcal{P}_2) = 9(k-1) \text{ for } k \geq 2.$$

- The map $\overline{\Psi}_3(\mathcal{P}_1)$ is not surjective, and $\widehat{B}(\mathcal{P}_1)$ does not decompose. We get:

$$\theta_2(\mathcal{P}_1) = 9$$
$$\theta_k(\mathcal{P}_1) = 10(k-1) \text{ for } k \geq 3.$$