Representation varieties and cohomology jump loci

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OVERVIEW

- The study of analytic germs of representation varieties and cohomology jump loci is a basic problem in deformation theory with homological constraints.
- Building on work of Goldman–Millson [1988], it was shown by Dimca–Papadima [2014] that the germs at the origin of those loci are isomorphic to the germs at the origin of infinitesimal jump loci of a CDGA that is a finite model for the space in question.
- Budur and Wang [2015] have extended this result away from the origin, by developing a theory of differential graded Lie algebra modules which control the corresponding deformation problem.

- Work of Papadima–S [2017] reveals a surprising connection between $\mathrm{SL}_2(\mathbb{C})$ representation varieties of arrangement groups and the monodromy action on the homology of Milnor fibers of hyperplane arrangements.
- On the other hand, the universality theorem of Kapovich and Millson [1998] shows that $\mathrm{SL}_2(\mathbb{C})$ representation varieties of Artin groups may have arbitrarily bad singularities away from 1.
- This lead us to focus on germs at the origin of such varieties, and look for explicit descriptions via infinitesimal CDGA methods.

REPRESENTATION VARIETIES

- Let π be a finitely generated group.
- G be a $\Bbbk$-linear algebraic group.
- The set $\mathrm{Hom}(\pi, G)$ has a natural structure of an affine variety, called the *G-representation variety of π*.
- Every homomorphism $\varphi\colon \pi \to \pi'$ induces an algebraic morphism, $\varphi^!\colon \mathrm{Hom}(\pi', G) \to \mathrm{Hom}(\pi, G)$.
- Example: $\mathrm{Hom}(F_n, G) = G^n$.
- $\mathrm{Hom}(\mathbb{Z}^2, \mathrm{GL}_k(\mathbb{C}))$ is irreducible, but not much else is known about the varieties of commuting matrices, $\mathrm{Hom}(\mathbb{Z}^n, \mathrm{GL}_k(\mathbb{C}))$.
- The varieties $\mathrm{Hom}(\pi_1(\Sigma_g), G)$ are connected if $G = \mathrm{SL}_k(\mathbb{C})$, and irreducible if $G = \mathrm{GL}_k(\mathbb{C})$.

COHOMOLOGY JUMP LOCI

- Let (X, x) be a pointed, path-connected space, and assume $\pi = \pi_1(X, x)$ is finitely generated.
- $\operatorname{Hom}(\pi, G)$ is a parameter space for finite-dimensional local systems on X of type G.
- The *characteristic varieties* of X with respect to a representation $\iota\colon G \to \operatorname{GL}(V)$ are the sets

$$\mathcal{V}^i_r(X, \iota) = \{\rho \in \operatorname{Hom}(\pi, G) \mid \dim_{\mathbb{C}} H^i(X, V_{\iota\circ\rho}) \geqslant r\}.$$

- For all $i \geqslant 0$, these sets form a descending filtration of $\operatorname{Hom}(\pi, G)$.
- The pairs $\big(\operatorname{Hom}(\pi, G), \mathcal{V}^i_r(X, \iota)\big)$ depend only on the homotopy type of X and on the representation ι.
- If X is a finite-type CW-complex, and ι is a rational representation, then the sets $\mathcal{V}^i_r(X, \iota)$ are closed subvarieties of $\operatorname{Hom}(\pi, G)$.

FLAT CONNECTIONS

- The infinitesimal analogue of the G-representation variety is

$$F(A, \mathfrak{g}),$$

the set of $\mathfrak{g}$-valued flat connections on a commutative, differential graded $\mathbb{C}$-algebra $(A^{\bullet}, d)$, where $\mathfrak{g}$ is a Lie algebra.

- This set consists of all elements $\omega \in A^1 \otimes \mathfrak{g}$ which satisfy the Maurer–Cartan equation,

$$d\omega + \frac{1}{2}[\omega, \omega] = 0.$$

- If A^1 and $\mathfrak{g}$ are finite dimensional, then $F(A, \mathfrak{g})$ is a Zariski-closed subset of the affine space $A^1 \otimes \mathfrak{g}$.

INFINITESIMAL COHOMOLOGY JUMP LOCI

- For each $\omega \in \mathcal{F}(A, \mathfrak{g})$, we turn $A \otimes V$ into a cochain complex,

$$(A \otimes V, d_\omega)\colon\ A^0 \otimes V \xrightarrow{d_\omega} A^1 \otimes V \xrightarrow{d_\omega} A^2 \otimes V \xrightarrow{d_\omega} \cdots,$$

  using as differential the covariant derivative $d_\omega = d \otimes \mathrm{id}_V + \mathrm{ad}_\omega$. (The flatness condition on ω insures that $d_\omega^2 = 0$.)
- The *resonance varieties* of the CDGA $(A^\bullet, d)$ with respect to a representation $\theta\colon \mathfrak{g} \to \mathfrak{gl}(V)$ are the sets

$$\mathcal{R}^i_r(A, \theta) = \{\omega \in \mathcal{F}(A, \mathfrak{g}) \mid \dim_{\mathbb{C}} H^i(A \otimes V, d_\omega) \geqslant r\}.$$

- For each $i \geqslant 0$, these sets form a descending filtration of $\mathcal{F}(A, \mathfrak{g})$.
- If A, $\mathfrak{g}$, and V are all finite-dimensional, the sets $\mathcal{R}^i_r(A, \theta)$ are closed subvarieties of $\mathcal{F}(A, \mathfrak{g})$.

ALGEBRAIC MODELS FOR SPACES

- From now on, X will be a connected space having the homotopy type of a finite CW-complex.
- Let $A_{\mathrm{PL}}(X)$ be the Sullivan CDGA of piecewise polynomial $\mathbb{C}$-forms on X. Then $H^{\bullet}(A_{\mathrm{PL}}(X)) \cong H^{\bullet}(X, \mathbb{C})$.
- A CDGA (A, d) is a *model* for X if it may be connected by a zig-zag of quasi-isomorphisms to $A_{\mathrm{PL}}(X)$.
- A is a *finite* model if $\dim_{\mathbb{C}} A < \infty$ and A is connected.
- X is *formal* if $(H^{\bullet}(X, \mathbb{C}), \mathrm{d} = 0)$ is a (finite) model for X.
  - E.g.: Compact Kähler manifolds, complements of hyperplane arrangments.
- Thus, if X is formal, then $H^{\bullet}(X, \mathbb{C})$ is a finite model for X.
  - Converse not true. E.g.: all nilmanifolds, solvmanifolds, Sasakian manifolds, smooth quasi-projective varieties, etc, admit finite models, but many are non-formal.

GERMS OF JUMP LOCI

THEOREM (DIMCA–PAPADIMA 2014)

Suppose X admits a finite CDGA model A. Let $\iota\colon G \to \mathrm{GL}(V)$ be a rational representation, and $\theta\colon \mathfrak{g} \to \mathfrak{gl}(V)$ its tangential representation. There is then an analytic isomorphism of germs,

$$\mathcal{F}(A, \mathfrak{g})_{(0)} \xrightarrow{\simeq} \mathrm{Hom}(\pi_1(X), G)_{(1)},$$

restricting to isomorphisms $\mathcal{R}^i_r(A, \theta)_{(0)} \xrightarrow{\simeq} \mathcal{V}^i_r(X, \iota)_{(1)}$ for all i, r.

Rank 1 case:

- For $G = \mathbb{C}^*$, the representation variety $\mathrm{Hom}(\pi, \mathbb{C}^*) = H^1(X, \mathbb{C}^*)$ is the character group of $\pi = \pi_1(X)$.
- For $\iota\colon \mathbb{C}^* \xrightarrow{\simeq} \mathrm{GL}_1(\mathbb{C})$ and $V = \mathbb{C}$, we get the usual characteristic varieties, $\mathcal{V}^i_r(X)$
- For $\mathfrak{g} = \mathbb{C}$, we have $\mathcal{F}(A, \mathfrak{g}) \cong H^1(A)$. Also, for $\theta = \mathrm{id}_{\mathbb{C}}$, we get the usual resonance varieties $\mathcal{R}^i_r(A)$.

- The local analytic isomorphism $H^1(A)_{(0)} \xrightarrow{\simeq} \mathrm{Hom}(\pi_1(X), \mathbb{C}^*)_{(1)}$ is induced by the exponential map $H^1(X, \mathbb{C}) \to H^1(X, \mathbb{C}^*)$.

THEOREM (DIMCA–PAPADIMA 2014, MACINIC–PAPADIMA–POPESCU–S. 2017)

If (A, d) is a finite CDGA such that $A_{\mathrm{PL}}(X) \simeq A$, then

$$\mathrm{TC}_0(\mathcal{R}^i_r(A)) \subseteq \mathcal{R}^i_r(H^\bullet(A)).$$

Moreover, if (A, d) is rationally defined, with positive weights, and $A_{\mathrm{PL}}(X) \simeq A$ over $\mathbb{Q}$, then each $\mathcal{R}^i_r(A)$ is a finite union of rationally defined linear subspaces of $H^1(A)$, and $\mathcal{R}^i_r(A) \subseteq \mathcal{R}^i_r(H^\bullet(A))$.

THEOREM (BUDUR–WANG 2017)

If X admits a finite CDGA model A, then all the components of the characteristic varieties $\mathcal{V}^i_r(X)$ passing through 1 are algebraic subtori.

QUASI-KÄHLER MANIFOLDS AND ADMISSIBLE MAPS

- Let M be a quasi-Kähler manifold, that is, the complement of a normal crossing divisor D in a compact, connected Kähler manifold $\overline{M}$.
- Arapura [1997]: there is a finite set $\mathcal{E}(M)$ of equivalence classes of 'admissible' maps, up to reparametrization in the target.
- Each such map, $f\colon M \to M_f$, is regular and surjective, its generic fiber is connected, and M_f is a smooth complex curve with $\chi(M_f) < 0$. Let $f_\sharp\colon \pi \to \pi_f$ be the induced homomorphism on π_1.
- Let $f_0\colon M \to K(\pi_{\text{abf}}, 1)$ be a classifying map for the projection $\text{abf}\colon \pi \to \pi_{\text{abf}}$ onto the maximal, torsion-free abelian quotient.
- Set $E(M) := \mathcal{E}(M) \cup \{f_0\}$.

RANK 1 JUMP LOCI OF QUASI-PROJ MANIFOLDS

THEOREM (ARAPURA 1997)

The correspondence $f \rightsquigarrow f^*(H^1(M_f,\mathbb{C}^*))$ gives a bijection between the set $\mathcal{E}(M)$ and the set of positive-dimensional irreducible components of $\mathcal{V}^1_1(M)$ passing through the identity of the character group $H^1(M,\mathbb{C}^*)$.

THEOREM (BUDUR–WANG 2015)

If M is a smooth quasi-projective variety, then all components of the characteristic varieties $\mathcal{V}^i_r(M)$ are torsion-translated algebraic subtori.

THEOREM (DIMCA–PAPADIMA 2014)

Let A be a finite CDGA model with positive weights for M. Then the set $\mathcal{E}(M)$ is in bijection with the set of positive-dimensional, irreducible components of $\mathcal{R}^1_1(A) \subseteq H^1(A) = H^1(M,\mathbb{C})$ via the correspondence $f \rightsquigarrow f^!(H^1(M_f,\mathbb{C}))$.

ORLIK–SOLOMON MODELS

- Now let M be a smooth, quasi-projective variety. Then M admits a 'convenient' compactification, $\overline{M} = M \cup D$, where $\overline{M}$ is a smooth projective variety, and D is a union of smooth hypersurfaces, intersecting locally like hyperplanes.
- For such a compactification, every element of $\mathcal{E}(M)$ is represented by an admissible map, $f\colon M \to M_f$, which is induced by a regular morphism of pairs, $\bar{f}\colon (\overline{M}, D) \to (\overline{M}_f, D_f)$.
- Work of Morgan, as recently sharpened by Dupont, associates to these data a bigraded, rationally defined CDGA, $A = \mathrm{OS}(\overline{M}, D)$, called the *Orlik–Solomon model* of M.
- This CDGA is a finite model of M, which is functorial with respect to regular morphisms of pairs $(\overline{M}, D)$ as above.

PULLBACKS AND TRANSVERSALITY

- If $f\colon M \to M_f$ is an admissible map, we let $\Phi_f\colon A_f \to A$ be the induced map between OS models, and $\Phi_f^!\colon \mathcal{F}(A_f, \mathfrak{g}) \to \mathcal{F}(A, \mathfrak{g})$ the induced morphism between varieties of flat connections.

THEOREM

Let M be a quasi-Kähler manifold, and let $f, g \in \mathcal{E}(M)$ be two distinct admissible maps.

- *If M is a smooth, quasi-projective variety, then*

$$\Phi_f^! \mathcal{F}(A_f, \mathfrak{g}) \cap \Phi_g^! \mathcal{F}(A_g, \mathfrak{g}) = \{0\}.$$

- *If M is either a compact, connected Kähler manifold or the complement of a complex hyperplane arrangement, then*

$$f_\sharp^! \operatorname{Hom}(\pi_f, G)_{(1)} \cap g_\sharp^! \operatorname{Hom}(\pi_g, G)_{(1)} = \{1\}.$$

- Let G be a complex linear algebraic group, let $\iota\colon G \to \mathrm{GL}(V)$ be a rational representation, and let $\theta\colon \mathfrak{g} \to \mathfrak{gl}(V)$ be its tangential representation. For all $r \geqslant 0$, we have inclusions

$$\mathcal{V}^1_r(\pi,\iota) \supseteq \bigcup_{f\in E(M)} f^!_\sharp \mathcal{V}^1_r(\pi_f,\iota),$$

- For $r = 0$ and 1, these inclusions are equivalent to the two inclusions

$$\mathrm{Hom}(\pi,G) \supseteq \mathrm{abf}^! \,\mathrm{Hom}(\pi_{\mathrm{abf}},G) \cup \bigcup_{f\in \mathcal{E}(M)} f^!_\sharp \,\mathrm{Hom}(\pi_f,G), \qquad (\star)$$

$$\mathcal{V}^1_1(\pi,\iota) \supseteq \mathrm{abf}^! \,\mathcal{V}^1_1(\pi_{\mathrm{abf}},\iota) \cup \bigcup_{f\in \mathcal{E}(M)} f^!_\sharp \,\mathrm{Hom}(\pi_f,G). \qquad (\star\star)$$

PULLBACKS AND INCLUSIONS

We also have infinitesimal counterparts of inclusions (⋆) and (⋆⋆):

$$\mathcal{F}(A,\mathfrak{g}) \supseteq \mathcal{F}^1(A,\mathfrak{g}) \cup \bigcup_{f\in\mathcal{E}(M)} \Phi_f^! \mathcal{F}(A_f,\mathfrak{g}), \qquad (\dagger)$$

$$\mathcal{R}^1_1(A,\theta) \supseteq \Pi(A,\theta) \cup \bigcup_{f\in\mathcal{E}(M)} \Phi_f^! \mathcal{F}(A_f,\mathfrak{g}), \qquad (\ddagger)$$

where

$$\mathcal{F}^1(A,\mathfrak{g}) = \{\eta\otimes g \in A^1\otimes\mathfrak{g} \mid d\eta = 0\},$$

$$\Pi(A,\theta) = \{\eta\otimes g \in \mathcal{F}^1(A,\mathfrak{g}) \mid \det\theta(g) = 0\}.$$

PULLBACKS AND EQUALITIES

THEOREM A

Let M be quasi-projective manifold with $b_1(M) > 0$. For an arbitrary rational representation of $G = \mathrm{SL}_2(\mathbb{C})$ or its standard Borel subgroup $\mathrm{Sol}_2(\mathbb{C})$, the following statements are equivalent.

- *Inclusion $(\star)$ becomes an equality near 1.*
- *Both $(\star)$ and $(\star\star)$ become equalities near 1.*
- *Inclusion $(\dagger)$ is an equality, for some convenient compactification of M.*
- *Both $(\dagger)$ and $(\ddagger)$ are equalities, for any convenient compactification of M.*

IRREDUCIBLE DECOMPOSITIONS

THEOREM B

Suppose the equivalent properties from Theorem A are satisfied.

- *If $b_1(M_f) \neq b_1(M)$ for all $f \in \mathcal{E}(M)$, then we have the following decompositions into irreducible components of analytic germs:*

$$\mathrm{Hom}(\pi, G)_{(1)} = \mathrm{abf}^! \,\mathrm{Hom}(\pi_{\mathrm{abf}}, G)_{(1)} \cup \bigcup_{f \in \mathcal{E}(M)} f_\sharp^! \,\mathrm{Hom}(\pi_f, G)_{(1)},$$

$$\mathcal{V}^1_1(\pi, \iota)_{(1)} = \mathrm{abf}^! \,\mathcal{V}^1_1(\pi_{\mathrm{abf}}, \iota)_{(1)} \cup \bigcup_{f \in \mathcal{E}(M)} f_\sharp^! \,\mathrm{Hom}(\pi_f, G)_{(1)},$$

$$\mathcal{F}(A, \mathfrak{g})_{(0)} = \mathcal{F}^1(A, \mathfrak{g})_{(0)} \cup \bigcup_{f \in \mathcal{E}(M)} \Phi_f^! \mathcal{F}(A_f, \mathfrak{g})_{(0)},$$

$$\mathcal{R}^1_1(A, \theta)_{(0)} = \Pi(A, \theta)_{(0)} \cup \bigcup_{f \in \mathcal{E}(M)} \Phi_f^! \mathcal{F}(A_f, \mathfrak{g})_{(0)}.$$

- *If $b_1(M_f) = b_1(M)$ for some $f \in \mathcal{E}(M)$, then we have the following equalities of irreducible germs:*

$$\operatorname{Hom}(\pi, G)_{(1)} = f_\sharp^! \operatorname{Hom}(\pi_f, G)_{(1)}, \quad \mathcal{V}_1^1(\pi, \iota)_{(1)} = f_\sharp^! \operatorname{Hom}(\pi_f, G)_{(1)},$$

$$\mathcal{F}(A, \mathfrak{g})_{(0)} = \Phi_f^! \mathcal{F}(A_f, \mathfrak{g})_{(0)}, \quad \mathcal{R}_1^1(A, \theta)_{(0)} = \Phi_f^! \mathcal{F}(A_f, \mathfrak{g})_{(0)}.$$

- *For any two distinct admissible maps $f, g \in \mathcal{E}(M)$,*

$$f_\sharp^! \operatorname{Hom}(\pi_f, G)_{(1)} \cap g_\sharp^! \operatorname{Hom}(\pi_g, G)_{(1)} = \{1\}.$$

Under our assumptions, this theorem gives a local, more precise and simple, classification for representations of π into $\mathrm{SL}(2, \mathbb{C})$, when compared to the global, more sophisticated classification obtained by Corlette–Simpson [2008] and Loray–Pereira–Touzet [2016].

APPLICATIONS

THEOREM

Suppose M is a smooth, quasi-projective variety satisfying one of the following hypotheses.

- *M is projective.*
- *$W_1 H^1(M) = 0$.*
- *$M = F_\Gamma(\Sigma_g)$ is a graphic configuration space of a smooth curve.*
- *$\mathcal{R}^1_1(H^\bullet(M)) = \{0\}$.*
- *$M = S \backslash \{0\}$, where S is a quasi-homogeneous affine surface having a normal, isolated singularity at 0.*

Then, for $G = \mathrm{SL}_2(\mathbb{C})$ or $\mathrm{Sol}_2(\mathbb{C})$, the equivalent properties from Theorem A are satisfied, and thus, the conclusions of Theorem B hold.

RANK GREATER THAN 2

- Let $M = S \setminus \{0\}$, where S is a quasi-homogeneous affine surface having a normal, isolated singularity at 0.
- There is a positive weight $\mathbb{C}^\times$-action on M with finite isotropy groups.
- $M/\mathbb{C}^\times = \Sigma_g$, where $g = \frac{1}{2} b_1(M)$. We will assume that $g > 0$.
- The canonical projection, $f\colon M \to M/\mathbb{C}^\times = M_f$, is an admissible map. Furthermore, $\mathcal{E}(M) = \emptyset$ if $g = 1$, and $\mathcal{E}(M) = \{f\}$ if $g > 1$.

THEOREM

If $G = \mathrm{SL}_n(\mathbb{C})$ with $n \geqslant 3$, then

$$\mathrm{Hom}(\pi, G)_{(1)} \supsetneqq \mathrm{abf}^! \,\mathrm{Hom}(\pi_{\mathrm{abf}}, G)_{(1)} \cup \bigcup_{f \in \mathcal{E}(M)} f_\sharp^! \,\mathrm{Hom}(\pi_f, G)_{(1)}.$$

DEPTH GREATER THAN 1

THEOREM

Let M be a connected, compact Kähler manifold, or the complement of a complex hyperplane arrangement, and let $\iota\colon G \to \mathrm{GL}(V)$ be a rational representation of $G = \mathrm{SL}_2(\mathbb{C})$ or $\mathrm{Sol}_2(\mathbb{C})$. Suppose there is an admissible map $f\colon M \to M_f$ such that $b_1(M) > b_1(M_f)$. Then

$$\mathcal{V}^1_1(\pi,\iota)_{(1)} = \bigcup_{f\in E(M)} f^!_\sharp \mathcal{V}^1_1(\pi_f,\iota)_{(1)},$$

Nevertheless, if there is $0 \neq v \in V^G$, there is then an $r > 1$ such that

$$\mathcal{V}^1_r(\pi,\iota)_{(1)} \supsetneqq \bigcup_{f\in E(M)} f^!_\sharp \mathcal{V}^1_r(\pi_f,\iota)_{(1)}.$$

EXAMPLE

Let $M = \Sigma_g \times N$, where Σ_g is a smooth projective curve of genus $g > 1$ and N is a projective manifold with $b_1(N) > 0$. Then the projection $f\colon M \to \Sigma_g$ defines an element $f \in \mathcal{E}(M)$ with $b_1(M) > b_1(\Sigma_g)$.

EXAMPLE

Let $\mathcal{A}$ be an arrangement of lines in $\mathbb{CP}^2$, with an intersection point of multiplicity $k \geqslant 3$. There is then a pencil $f\colon M(A) \to M(\mathcal{B})$, where $\mathcal{B}$ consists of k points in $\mathbb{CP}^1$. If $\mathcal{A}$ is not itself a pencil of lines, then $b_1(M(\mathcal{A})) > b_1(M(\mathcal{B}))$.

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