Cohomology jump loci in geometry and topology

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Characteristic varieties

- Let X be a connected, finite-type CW-complex. Then $\pi = \pi_1(X, x_0)$ is a finitely presented group, with $\pi_{ab} \cong H_1(X, \mathbb{Z})$.

- The ring $R = \mathbb{C}[\pi_{ab}]$ is the coordinate ring of the character group, $\mathrm{Char}(X) = \mathrm{Hom}(\pi, \mathbb{C}^*) \cong (\mathbb{C}^*)^n \times \mathrm{Tors}(\pi_{ab})$, where $n = b_1(X)$.

- The *characteristic varieties* of X are the homology jump loci

$$\mathcal{V}^i_s(X) = \{\rho \in \mathrm{Char}(X) \mid \dim_{\mathbb{C}} H_i(X, \mathbb{C}_\rho) \geqslant s\}.$$

- These varieties are homotopy-type invariants of X, with $\mathcal{V}^1_s(X)$ depending only on $\pi = \pi_1(X)$.

- Set $\mathcal{V}^1_1(\pi) := \mathcal{V}^1_1(K(\pi, 1))$; then $\mathcal{V}^1_1(\pi) = \mathcal{V}_1(\pi/\pi'')$.

- The characteristic varieties of a space can be arbitrarily complicated.

Resonance varieties of a CDGA

- Let $A = (A^{\bullet}, \mathrm{d})$ be a commutative, differential graded algebra over a field $\Bbbk$ of characteristic 0. That is:
  - $A = \bigoplus_{i \geqslant 0} A^i$, where A^i are $\Bbbk$-vector spaces.
  - The multiplication $\cdot \colon A^i \otimes A^j \to A^{i+j}$ is graded-commutative, i.e., $ab = (-1)^{|a||b|} ba$ for all homogeneous a and b.
  - The differential $\mathrm{d} \colon A^i \to A^{i+1}$ satisfies the graded Leibniz rule, i.e., $\mathrm{d}(ab) = \mathrm{d}(a)b + (-1)^{|a|} a\,\mathrm{d}(b)$.
- We assume $A^0 = \Bbbk \cdot 1$ and $\dim A^i < \infty$ for all i.
- For each $a \in Z^1(A) \cong H^1(A)$, we build a cochain complex,

$$(A^{\bullet}, \delta_a) \colon \; A^0 \xrightarrow{\delta_a^0} A^1 \xrightarrow{\delta_a^1} A^2 \xrightarrow{\delta_a^2} \cdots,$$

  with differentials $\delta_a^i(u) = a \cdot u + \mathrm{d}(u)$, for all $u \in A^i$.
- The *resonance varieties* of A are the affine varieties

$$\mathcal{R}_s^i(A) = \{a \in H^1(A) \mid \dim_{\Bbbk} H^i(A^{\bullet}, \delta_a) \geqslant s\}.$$

- For a space X as above, set $\mathcal{R}_s^i(X) := \mathcal{R}_s^i((H^{\bullet}(X, \mathbb{C}), \mathrm{d} = 0))$.

Applications of cohomology jump loci

- Obstructions to formality and (quasi-) projectivity
  - Artin groups, RAAGs, and Bestvina–Brady groups
  - Kähler groups and quasi-projective groups
  - 3-manifold groups
- Homology of finite, regular abelian covers
  - Homology of the Milnor fiber of an arrangement
  - Rational homology of smooth, real toric varieties
- Homological and geometric finiteness of regular abelian covers
  - Bieri–Neumann–Strebel–Renz invariants
  - Dwyer–Fried invariants
- Infinitesimal finiteness obstructions
- Resonance varieties and representations of Lie algebras
  - Homological finiteness in the Johnson filtration
- Lower central series and Chen Lie algebras
  - The resonance–Chen ranks formula

ALGEBRAIC MODELS FOR SPACES

- A CDGA map $\varphi\colon A \to B$ is a q-quasi-isomorphism (for some $q \geqslant 1$) if $\varphi^*\colon H^\bullet(A) \to H^\bullet(B)$ is an iso for $\bullet \leqslant q$ and is inj for $\bullet = q+1$.
- Two CDGAs, A and B, are *(q-)equivalent* if there is a zig-zag of (q-)quasi-isomorphisms connecting A to B.
- A is *(q-)formal* if it is (q-)equivalent to $(H^\bullet(A), d=0)$.
- Given any (path-connected) space X, there is an associated Sullivan $\mathbb{Q}$-cdga, $A_{\mathrm{PL}}(X)$, such that $H^\bullet(A_{\mathrm{PL}}(X)) = H^\bullet(X, \mathbb{Q})$.
- An *algebraic (q-)model* (over $\Bbbk$) for X is a $\Bbbk$-cgda (A, d) which is (q-)equivalent to $A_{\mathrm{PL}}(X) \otimes_{\mathbb{Q}} \Bbbk$.
- If M is a smooth manifold, then $\Omega_{\mathrm{dR}}(M)$ is a model for M (over $\mathbb{R}$).
- A space X is 1-formal if and only if $\pi = \pi_1(X)$ is 1-formal, i.e., its Malcev Lie algebra, $\mathfrak{m}(\pi) = \operatorname{Prim}(\widehat{\mathbb{Q}\pi})$, is quadratic.

Tangent cones

- Let $W = V(I)$, a Zariski closed subset of $(\mathbb{C}^*)^n$.
- The *tangent cone* at 1 to W is $\mathrm{TC}_1(W) = V(\mathrm{in}(I))$.
- Let $\exp\colon \mathbb{C}^n \to (\mathbb{C}^*)^n$. The *exponential tangent cone* at 1 to W is

$$\tau_1(W) = \{z \in \mathbb{C}^n \mid \exp(\lambda z) \in W,\ \forall \lambda \in \mathbb{C}\}.$$

- Both tangent cones are homogeneous subvarieties of $\mathbb{C}^n$; are non-empty iff $1 \in W$; depend only on the analytic germ of W at 1; commute with finite unions and arbitrary intersections.
- $\tau_1(W) \subseteq \mathrm{TC}_1(W)$, with $=$ if all irred components of W are subtori, but $\neq$ in general.
- (Dimca–Papadima–S. 2009) $\tau_1(W)$ is a finite union of rationally defined subspaces.

The tangent cone theorem

Let X be a connected CW-complex with finite q-skeleton. Suppose X admits a q-finite q-model A.

Theorem

For all $i \leqslant q$ and all s:

- *(DPS 2009, Dimca–Papadima 2014)* $\mathcal{V}^i_s(X)_{(1)} \cong \mathcal{R}^i_s(A)_{(0)}$.
- *(Budur–Wang 2017) All the irreducible components of $\mathcal{V}^i_s(X)$ passing through the origin of $\mathrm{Char}(X)$ are* algebraic subtori.

Consequently,

$$\tau_1(\mathcal{V}^i_s(X)) = \mathrm{TC}_1(\mathcal{V}^i_s(X)) = \mathcal{R}^i_s(A).$$

Theorem (Papadima–S. 2018)

A f.g. group G admits a 1-finite 1-model if and only if the Malcev Lie algebra $\mathfrak{m}(G)$ is the LCS completion of a finitely presented Lie algebra.

- Examples of spaces having finite-type models include: formal spaces, smooth quasi-projective varieties, compact solvmanifolds, Sasakian manifolds, etc.
- Examples of formal spaces:
  - Compact Kähler manifolds [Deligne–Griffiths–Morgan–Sulivan '75]
  - Complements of complex hyperplane arrangements [Brieskorn '73]
- Examples of 1-formal spaces and groups:
  - Complements of projective hypersurfaces [Kohno '83]
  - Right-angled Artin groups [Notbohm–Ray '05, Papadima–S. '06]
  - Normal projective varieties [Arapura–Dimca–Hain '16]
- Every compact Sasakian $(2n+1)$-manifold is $(n-1)$-formal. [Papadima–S. '18]

Smooth, quasi-projective varieties

Theorem (Arapura 1997, ..., Budur–Wang 2015)

Let X be a smooth, quasi-projective variety. Then each $\mathcal{V}^i_s(X)$ is a finite union of torsion-translated subtori of $\mathrm{Char}(X)$.

The *Alexander polynomial* of a f.p. group π is the Laurent polynomial Δ_π in $\Lambda := \mathbb{C}[\pi_{\mathrm{ab}}/\mathrm{Tors}]$ obtained by taking the gcd of the maximal minors of a presentation matrix for the Λ-module $H_1(\pi, \Lambda)$.

Theorem (Dimca–Papadima–S. 2008)

Let π be a quasi-projective group.

- *If $b_1(\pi) \neq 2$, then the Newton polytope of Δ_π is a line segment.*
- *If π is a Kähler group, then $\Delta_\pi \doteq \mathrm{const}$.*

THEOREM (DIMCA–PAPADIMA–S. 2009)

Let X be a smooth, quasi-projective variety. If X is 1-formal, then the (non-zero) irreducible components of $\mathcal{R}^1_1(X)$ are linear subspaces of $H^1(X,\mathbb{C})$ which intersect pairwise only at 0. Moreover:

- *Each such component L_α is p-isotropic (i.e., the restriction of $\cup_X$ to L_α has rank p), with $\dim L_\alpha \geqslant 2p+2$, for $p = p(\alpha) \in \{0,1\}$.*
- $\mathcal{R}^1_s(X) = \{0\} \cup \bigcup_{\alpha:\dim L_\alpha > s+p(\alpha)} L_\alpha$
- *If X is compact, then X is 1-formal, and each L_α is 1-isotropic.*
- *If $W_1(H^1(X,\mathbb{C})) = 0$, then X is 1-formal, and each L_α is 0-isotropic.*

An analogous result holds for irreducible normal varieties
[Arapura–Dimca–Hain 2016]

Artin groups

- Let $\Gamma = (V, E)$ be a finite, simple graph, and let $\ell\colon E \to \mathbb{Z}_{\geqslant 2}$ be an edge-labeling. The associated *Artin group*:

$$A_{\Gamma,\ell} = \langle v \in V \mid \underbrace{vwv\cdots}_{\ell(e)} = \underbrace{wvw\cdots}_{\ell(e)},\ \text{for } e = \{v,w\} \in E\rangle.$$

- If (Γ, ℓ) is Dynkin diagram of type A_{n-1} with $\ell(\{i, i+1\}) = 3$ and $\ell(\{i,j\}) = 2$ otherwise, then $A_{\Gamma,\ell}$ is the braid group B_n.
- If $\ell(e) = 2$, for all $e \in E$, then

$$A_\Gamma = \langle v \in \mathcal{V} \mid vw = wv \text{ if } \{v,w\} \in E\rangle.$$

  is the *right-angled Artin group* associated to Γ.
- $\Gamma \cong \Gamma' \Leftrightarrow A_\Gamma \cong A_{\Gamma'}$.

  [Kim–Makar-Limanov–Neggers–Roush 80 / Droms 87]

The corresponding *Coxeter group*,

$$W_{\Gamma,\ell} = A_{\Gamma,\ell}/\langle v^2 = 1 \mid v \in V \rangle,$$

fits into exact sequence $1 \to P_{\Gamma,\ell} \to A_{\Gamma,\ell} \to W_{\Gamma,\ell} \to 1$.

Theorem (Brieskorn 1971)

If $W_{\Gamma,\ell}$ is finite, then $A_{\Gamma,\ell}$ is quasi-projective.

Idea: let

- $\mathcal{A}_{\Gamma,\ell} =$ reflection arrangement of type $W_{\Gamma,\ell}$ (over $\mathbb{C}$)
- $X_{\Gamma,\ell} = \mathbb{C}^n \setminus \bigcup_{H \in \mathcal{A}_{\Gamma,\ell}} H$, where $n = |\mathcal{A}_{\Gamma,\ell}|$
- $P_{\Gamma,\ell} = \pi_1(X_{\Gamma,\ell})$

then:

$$A_{\Gamma,\ell} = \pi_1(X_{\Gamma,\ell}/W_{\Gamma,\ell}) = \pi_1(\mathbb{C}^n \setminus \{\delta_{\Gamma,\ell} = 0\})$$

Theorem (Kapovich–Millson 1998)

There exist infinitely many (Γ, ℓ) such that $A_{\Gamma,\ell}$ is not *quasi-projective.*

Theorem (Dimca–Papadima–S. 2009, Arapura–Dimca–Hain 2016)

The following are equivalent:

- $A_\Gamma = \pi_1(X)$, *for some smooth algebraic variety* X.
- $A_\Gamma = \pi_1(X)$, *for some normal algebraic variety* X.
- Γ *is a complete, multipartite graph, i.e.,* $\Gamma = \overline{K}_{n_1} * \cdots * \overline{K}_{n_r}$.
- $A_\Gamma = F_{n_1} \times \cdots \times F_{n_r}$.

Likewise, the following are equivalent:

- $A_\Gamma = \pi_1(X)$, *for some smooth, projective variety* X.
- $A_\Gamma = \pi_1(X)$, *for some normal, projective variety* X.
- $\Gamma = K_{2r}$
- $A_\Gamma = \mathbb{Z}^{2r}$

The quasi-projectivity of arbitrary Artin groups has been further studied by Artal Bartolo, Cogolludo, Matei, and Blasco-García.

3-MANIFOLDS GROUPS

QUESTION (DONALDSON–GOLDMAN 1989)

Which 3-manifold groups are Kähler groups?

Reznikov gave a partial solution in 2002.

THEOREM (DIMCA–S. 2009)

Let π be the fundamental group of a closed 3-manifold. Then π is a Kähler group $\iff \pi$ is a finite subgroup of $O(4)$, acting freely on S^3.

Alternative proofs: Kotschick (2012), Biswas–Mj–Seshadri (2012).

THEOREM (FRIEDL–S. 2014)

Let M be a 3-manifold with non-empty, toroidal boundary. If $\pi_1(M)$ is a Kähler group, then $M \cong S^1 \times S^1 \times I$.

Generalization by Kotschick: If $\pi_1(M)$ is an infinite Kähler group, then $\pi_1(M)$ is a surface group.

Idea of proof of [DS09]:

PROPOSITION

Let M be a closed, orientable 3-manifold. Then:

- $H^1(M, \mathbb{C})$ *is not* 1*-isotropic.*
- *If* $b_1(M)$ *is even, then* $\mathcal{R}^1_1(M) = H^1(M, \mathbb{C})$.

On the other hand, it follows from [DPS 2009] that:

PROPOSITION

Let M be a compact Kähler manifold with $b_1(M) \neq 0$. If $\mathcal{R}^1_1(M) = H^1(M, \mathbb{C})$, then $H^1(M, \mathbb{C})$ is 1-isotropic.

But $\pi = \pi_1(M)$, with M Kähler $\Rightarrow b_1(\pi)$ even.
Thus, if π is both a 3-mfd group and a Kähler group $\Rightarrow b_1(\pi) = 0$.
Using work of Fujiwara (1999) and Reznikov (2002) on Kazhdan's property (T), as well as Perelman (2003) $\Rightarrow G$ finite subgroup of $O(4)$.

THEOREM (S. 2018/2019)

Let M be a closed, orientable, 3-manifold, with intersection form on $H^1(M,\mathbb{C})=\mathbb{C}^n$ given by $\mu_M(a\wedge b\wedge c)=\langle a\cup b\cup c,[M]\rangle$. Then:

- *If* $\operatorname{rank}(\mu_M)=n\geqslant 3$, *then* $\mathcal{R}^1_{n-2}(M)=\mathcal{R}^1_{n-1}(M)=\mathcal{R}^1_n(M)=\{0\}$.
- *If* $n\geqslant 4$, *then* $\dim\mathcal{R}^1_1(M)\geqslant\operatorname{null}(\mu_M)\geqslant 2$.
- *If n is even, then*
  - $\mathcal{R}^1_{2k}(M)=\mathcal{R}^1_{2k+1}(M)$.
  - $\mathcal{R}^1_1(M)=H^1(M,\mathbb{C})$.
  - $\operatorname{TC}_1(\mathcal{V}^1_1(M))=\mathcal{R}^1_1(M)$ *if and only if* $\Delta_M=0$.
- *If n is odd, then*
  - $\mathcal{R}^1_{2k-1}(M)=\mathcal{R}^1_{2k}(M)$.
  - $\mathcal{R}^1_1(M)\neq H^1(M,\mathbb{C})$ *if and only if* μ_M *is "generic".*
  - *If* μ_M *is "generic", then* $\operatorname{TC}_1(\mathcal{V}^1_1(M))=\mathcal{R}^1_1(M)$.

Infinitesimal finiteness obstructions

Theorem

Let X be a connected CW-complex with finite q-skeleton. Suppose X admits a q-finite q-model A. Then, for all $i \leqslant q$ and all s,

- *(Dimca–Papadima 2014)* $\mathcal{V}^i_s(X)_{(1)} \cong \mathcal{R}^i_s(A)_{(0)}$.
  In particular, if X is q-formal, then $\mathcal{V}^i_s(X)_{(1)} \cong \mathcal{R}^i_s(X)_{(0)}$.
- *(Macinic, Papadima, Popescu, S. 2017)* $\mathrm{TC}_0(\mathcal{R}^i_s(A)) \subseteq \mathcal{R}^i_s(X)$.
- *(Budur–Wang 2017) All the irreducible components of* $\mathcal{V}^i_s(X)$ *passing through the origin of* $H^1(X, \mathbb{C}^*)$ *are* algebraic subtori.

Example

Let G be a f.p. group with $G_{\mathrm{ab}} = \mathbb{Z}^n$ and $\mathcal{V}^1_1(G) = \{t \in (\mathbb{C}^*)^n \mid \sum_{i=1}^n t_i = n\}$. Then G admits no 1-finite 1-model.

Theorem (Papadima–S. 2019)

Let X be a space which admits a q-finite q-model. If $\mathcal{M}_q(X)$ is the Sullivan q-minimal model of X, then $b_i(\mathcal{M}_q(X)) < \infty$, for all $i \leqslant q+1$.

Corollary

Let G be a f.g. group. Assume that either G is finitely presented, or G has a 1-finite 1-model. Then $b_2(\mathcal{M}_1(G)) < \infty$.

Example

- Consider the free metabelian group $G = F_n / F_n''$ with $n \geqslant 2$.
- We have $\mathcal{V}^1(G) = \mathcal{V}^1(F_n) = (\mathbb{C}^*)^n$, and so G passes the Budur–Wang test.
- But $b_2(\mathcal{M}_1(G)) = \infty$, and so G admits no 1-finite 1-model (and is not finitely presented).

Finiteness properties for spaces and groups

- A group G has property F_k if it admits a classifying space $K(G,1)$ with finite k-skeleton.
  - F_1: G is finitely generated;
  - F_2: G is finitely presentable.
- G has property FP_k if the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ admits a projective $\mathbb{Z}G$-resolution which is finitely generated in all dimensions up to k.
- The following implications (none of which can be reversed) hold:

$$\begin{aligned} G \text{ is of type } F_k &\Rightarrow G \text{ is of type } FP_k \\ &\Rightarrow H_i(G,\mathbb{Z}) \text{ is finitely generated, for all } i \leqslant k \\ &\Rightarrow b_i(G) < \infty, \text{ for all } i \leqslant k. \end{aligned}$$

- Moreover, $FP_k \,\&\, F_2 \Rightarrow F_k$.

DWYER–FRIED SETS

- For a fixed $r \in \mathbb{N}$, the connected, regular covers $Y \to X$ with group of deck-transformations $\mathbb{Z}^r$ are parametrized by the Grassmannian of r-planes in $H^1(X, \mathbb{Q})$.
- Moving about this variety, and recording when $b_1(Y), \dots, b_i(Y)$ are finite defines subsets $\Omega^i_r(X) \subseteq \mathrm{Gr}_r(H^1(X, \mathbb{Q}))$, which we call the *Dwyer–Fried invariants* of X.
- These sets depend only on the homotopy type of X. Hence, if G is a f.g. group, we may define $\Omega^i_r(G) := \Omega^i_r(K(G, 1))$.

THEOREM

Let G be a f.g. group, and $\nu\colon G \twoheadrightarrow \mathbb{Z}^r$ an epimorphism, with kernel Γ. Suppose $\Omega^k_r(G) = \emptyset$, and Γ is of type F_{k-1}. Then $b_k(\Gamma) = \infty$.

Proof: Set $X = K(G, 1)$; then $X^\nu = K(\Gamma, 1)$. Since Γ is of type F_{k-1}, $b_i(X^\nu) < \infty$ for $i \leqslant k - 1$. But now $\Omega^k_r(X) = \emptyset$ implies $b_k(X^\nu) = \infty$.

Corollary

Let G be a f.g. group, and suppose $\Omega^3_1(G) = \emptyset$. Let $\nu\colon G \twoheadrightarrow \mathbb{Z}$ be an epimorphism. If the group $\Gamma = \ker(\nu)$ is f.p., then $b_3(\Gamma) = \infty$.

Example (The Stallings group)

- Let $Y = S^1 \vee S^1$ and $X = Y \times Y \times Y$. Clearly, X is a classifying space for $G = F_2 \times F_2 \times F_2$.
- Let $\nu\colon G \to \mathbb{Z}$ be the homomorphism taking each standard generator to 1. Set $\Gamma = \ker(\nu)$.
- Stallings (1963) showed that Γ is finitely presented.
- Using a Mayer-Vietoris argument, he also showed that $H_3(\Gamma, \mathbb{Z})$ is not finitely generated.
- Alternate explanation: $\Omega^3_1(X) = \emptyset$. Thus, by the previous Corollary, a stronger statement holds: $b_3(\Gamma)$ is not finite.

KOLLÁR'S QUESTION

QUESTION (J. KOLLÁR 1995)

Given a smooth, projective variety M, is the fundamental group $G = \pi_1(M)$ commensurable, up to finite kernels, with another group, π, admitting a $K(\pi, 1)$ which is a quasi-projective variety?

(Two groups, G_1 and G_2, are said to be *commensurable up to finite kernels* if there is a zig-zag of groups and homomorphisms connecting them, with all arrows of finite kernel and cofinite image.)

THEOREM (DIMCA–PAPADIMA–S. 2009)

For each $k \geqslant 3$, there is a smooth, irreducible, complex projective variety M of complex dimension $k - 1$, such that $\pi_1(M)$ is of type F_{k-1}, but not of type FP_k.

Further examples given by Llosa Isenrich and Bridson (2016–2019).

Duality spaces

Let X be a connected, finite-type CW-complex, and set $\pi = \pi_1(X, x_0)$. Following Bieri and Eckmann (1978), we say that:

- X is a *duality space* of dimension n if $H^i(X, \mathbb{Z}\pi) = 0$ for $i \neq n$ and $H^n(X, \mathbb{Z}\pi) \neq 0$ and torsion-free.
- Let $D = H^n(X, \mathbb{Z}\pi)$ be the dualizing $\mathbb{Z}\pi$-module. Given any $\mathbb{Z}\pi$-module A, we have $H^i(X, A) \cong H_{n-i}(X, D \otimes A)$.
- If $D = \mathbb{Z}$, with trivial $\mathbb{Z}\pi$-action, then X is a Poincaré duality space.
- If $X = K(\pi, 1)$ is a duality space, then π is a *duality group*.

Abelian duality spaces

We introduced in [Denham–S.–Yuzvinsky 2016/17] an analogous notion, by replacing $\pi \rightsquigarrow \pi_{\mathrm{ab}}$.

- X is an *abelian duality space* of dimension n if $H^i(X, \mathbb{Z}\pi_{\mathrm{ab}}) = 0$ for $i \neq n$ and $H^n(X, \mathbb{Z}\pi_{\mathrm{ab}}) \neq 0$ and torsion-free.
- Let $B = H^n(X, \mathbb{Z}\pi_{\mathrm{ab}})$ be the dualizing $\mathbb{Z}\pi_{\mathrm{ab}}$-module. Given any $\mathbb{Z}\pi_{\mathrm{ab}}$-module A, we have $H^i(X, A) \cong H_{n-i}(X, B \otimes A)$.
- The two notions of duality are independent:

Example

- Surface groups of genus at least 2 are not abelian duality groups, though they are (Poincaré) duality groups.
- Let $\pi = \mathbb{Z}^2 * G$, where
$$G = \langle x_1, \dots, x_4 \mid x_1^{-2} x_2 x_1 x_2^{-1}, \dots, x_4^{-2} x_1 x_4 x_1^{-1} \rangle$$
is Higman's acyclic group. Then π is an abelian duality group (of dimension 2), but not a duality group.

Theorem (DSY 2018 (and Liu–Maxim–Wang 2018))

Let X be an abelian duality space of dimension n. Then:

- $b_1(X) \geqslant n-1$.
- $b_i(X) \neq 0$, *for* $0 \leqslant i \leqslant n$ *and* $b_i(X) = 0$ *for* $i > n$.
- $(-1)^n \chi(X) \geqslant 0$.
- *Let* $\rho\colon \pi_1(X) \to \mathbb{C}^*$ *be a character such that* $H^i(X, \mathbb{C}_\rho) \neq 0$, *for some* $i > 0$. *Then* $H^j(X, \mathbb{C}_\rho) \neq 0$, *for all* $i \leqslant j \leqslant n$.

Theorem (Denham–S. 2018)

Let U be a connected, smooth, complex quasi-projective variety of dimension n. Suppose U has a smooth compactification Y for which

- *Components of $Y \backslash U$ form an arrangement of hypersurfaces $\mathcal{A}$;*
- *For each submanifold X in the intersection poset $L(\mathcal{A})$, the complement of the restriction of $\mathcal{A}$ to X is a Stein manifold.*

Then U is both a duality space and an abelian duality space of dimension n.

Linear, elliptic, and toric arrangements

Theorem (Denham–S. 2018)

Suppose that $\mathcal{A}$ is one of the following:

1. *An affine-linear arrangement in $\mathbb{C}^n$, or a hyperplane arrangement in $\mathbb{CP}^n$;*
2. *A non-empty elliptic arrangement in E^n;*
3. *A toric arrangement in $(\mathbb{C}^*)^n$.*

Then the complement $M(\mathcal{A})$ is both a duality space and an abelian duality space of dimension $n-r$, $n+r$, and n, respectively, where r is the corank of the arrangement.

This theorem extends several previous results:

1. Davis, Januszkiewicz, Leary, and Okun (2011);
2. Levin and Varchenko (2012);
3. Davis and Settepanella (2013), Esterov and Takeuchi (2018).

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Happy Birthday, Donu!

Nice 2011