GEOMETRIC AND HOMOLOGICAL FINITENESS PROPERTIES

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1. **Finiteness properties of spaces and groups.** A recurring theme in topology is to determine the geometric and homological finiteness properties of spaces and groups. For instance, one would like to decide whether a path-connected space X is homotopy equivalent to a CW-complex with finite k-skeleton. In this spirit, a group G is said to have property F_k if it admits a classifying space $K(G,1)$ with finite k-skeleton; property F_1 simply says that G is finitely generated, while property F_2 says that G is finitely presentable. The group G is said to have property FP_k if the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ admits a projective $\mathbb{Z}G$-resolution which is finitely generated in all dimensions up to k. If G is of type F_k then it is of type FP_k; the converse does not hold in general, but properties FP_k and F_2 together imply property F_k.

In [1], Bieri, Neumann, and Strebel associated to every finitely generated group G a subset $\Sigma^1(G)$ of the unit sphere $S(G)$ in the real vector space $\mathrm{Hom}(G,\mathbb{R})$. This "geometric" invariant of the group G is cut out of the sphere by open cones, and is independent of a finite generating set for G. Shortly after, Bieri and Renz introduced a nested family of higher-order invariants, $\{\Sigma^i(G,\mathbb{Z})\}_{i\geq 1}$, which record the finiteness properties of normal subgroups of G with abelian quotients. In [8], Farber, Geoghegan and Schütz further extended these definitions: to each connected, finite-type CW-complex X, they assign a sequence of invariants, $\{\Sigma^i(X,\mathbb{Z})\}_{i\geq 1}$, living in the unit sphere $S(X)\subset H^1(X,\mathbb{R})$. The sphere $S(X)$ can be thought of as parametrizing all free abelian covers of X, while the Σ-invariants (which are again open subsets), keep track of the geometric finiteness properties of those covers.

Another tack was taken by Dwyer and Fried in [7]. Instead of looking at all free abelian covers of X at once, they fix the rank, say r, of the deck-transformation group, and view the resulting covers as being parametrized by the rational Grassmannian $\mathrm{Gr}_r(H^1(X,\mathbb{Q}))$. Inside this Grassmannian, they consider the subsets $\Omega^i_r(X)$, consisting of all covers for which the Betti numbers up to degree i are finite, and show how to determine these sets in terms of the support varieties of the relevant Alexander invariants of X. Unlike the Σ-invariants, though, the Ω-invariants need not be open subsets, see [7, 21].

The Dwyer–Fried sets depend only on the homotopy type of X. Hence, if G is a finitely generated group, we may define $\Omega^i_r(G) := \Omega^i_r(K(G,1))$. Let now $\nu\colon G \twoheadrightarrow \mathbb{Z}^r$ be an epimorphism. As shown in [21], the following holds: If $\Omega^k_r(G)=\emptyset$ and $\Gamma := \ker(\nu)$ is of type F_{k-1}, then $b_k(\Gamma)=\infty$. To see how this works in a concrete example, let $Y=S^1\vee S^1$; then $X=Y\times Y\times Y$ is a classifying space for $G=F_2\times F_2\times F_2$. Let $\nu\colon G\to\mathbb{Z}$ be the homomorphism taking each standard generator to 1. Stallings showed in [18] that the group $\Gamma=\ker(\nu)$ is finitely presented, and that $H_3(\Gamma,\mathbb{Z})$ is not finitely generated. Using our machinery, we compute that $\Omega^3_1(X)=\emptyset$; and so, by the above, a stronger statement holds: $b_3(\Gamma)$ is not finite.

Theorem 1 ([6])**.** *For each $k\geq 3$, there is a smooth, irreducible, complex projective variety M of complex dimension $k-1$, such that $\pi_1(M)$ is of type F_{k-1}, but not of type FP_k.*

This theorem answers in the negative a question of Kollár [12]. Some of the arguments that go into the proof are streamlined in [21]. Further examples of projective groups with exotic finiteness properties can be found in recent work of Llosa Isenrich and Bridson [13, 14, 2].

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2. **Bounds on the Σ- and Ω-invariants.** Let $\widehat{G} = \mathrm{Hom}(G, \mathbb{C}^*) = H^1(X, \mathbb{C}^*)$ be the character group of $G = \pi_1(X)$. The *characteristic varieties* of X are the sets

$$\mathcal{V}^i(X) = \{\rho \in \widehat{G} \mid H_i(X, \mathbb{C}_\rho) \neq 0\}.$$

If the CW-complex X has finite k-skeleton, then $\mathcal{V}^i(X)$ is a Zariski closed subset of the algebraic group $\widehat{G}$, for each $i \leq k$. The varieties $\mathcal{V}^i(X)$ are homotopy-type invariants of X; moreover, $\mathcal{V}^1(X)$ depends only on $G = \pi_1(X)$. If we set $\mathcal{V}^i(G) := \mathcal{V}^i(K(G,1))$, then $\mathcal{V}^1(G) = \mathcal{V}^1(G/G'')$.

Let $\exp\colon H^1(X, \mathbb{C}) \to H^1(X, \mathbb{C}^*)$ be the coefficient homomorphism induced by $\mathbb{C} \to \mathbb{C}^*$, $z \mapsto e^z$. Given a Zariski closed subset $W \subset H^1(X, \mathbb{C}^*)$, let $\tau_1(W)$ be the 'exponential tangent cone' to W, i.e., the set of $z \in H^1(X, \mathbb{C})$ for which $\exp(\lambda z) \in W$, for all $\lambda \in \mathbb{C}$. As shown in [5], this set is a finite union of rationally defined linear subspaces. Furthermore, put $\tau_1^{\Bbbk}(W) = \tau_1(W) \cap H^1(X, \Bbbk)$ for $\Bbbk = \mathbb{Q}$ or $\mathbb{R}$, and write $\mathcal{W}^i(X) = \bigcup_{j \leq i} \mathcal{V}^j(X)$.

Theorem 2 ([16])**.** $\Sigma^i(X, \mathbb{Z}) \subseteq S(X) \setminus S(\tau_1^{\mathbb{R}}(\mathcal{W}^i(X)))$.

For $i = 1$, equality obtains for all right-angled Artin groups [16], and pure braid groups [9]. In general, though, the above inclusion is strict, even in the case of complements of hyperplane arrangements [20].

Given a homogeneous variety $V \subset \Bbbk^n$, the locus of r-planes in $\Bbbk^n$ intersecting V non-trivially, $\sigma_r(V)$, is a Zariski closed subset of the Grassmannian $\mathrm{Gr}_r(\Bbbk^n)$.

Theorem 3 ([19, 21])**.** $\Omega^i_r(X) \subseteq \mathrm{Gr}_r(H^1(X, \mathbb{Q})) \setminus \sigma_r\big(\tau_1^{\mathbb{Q}}(\mathcal{W}^i(X))\big)$.

Furthermore, if the upper bound for the Σ^i-invariants is attained, then the upper bound for the Ω^i_r-invariants is also attained, for all r, see [20].

3. **Infinitesimal finiteness obstructions.** Let A be commutative differential graded $\mathbb{C}$-algebra (for short, a CDGA). We say that A is *q-finite* if it is connected (i.e., $A^0 = \mathbb{C} \cdot 1$) and $\sum_{i \leq q} \dim A^i < \infty$. Two CDGAs A and B have the same q-type (written $A \simeq_q B$) if there is a zig-zag of CDGA maps connecting A and B, with each such map inducing isomorphisms in homology up to degree q and a monomorphism in degree $q+1$. Every CDGA A with $H^0(A) = \mathbb{C}$ admits a *q-minimal model*, $\mathcal{M}_q(A)$, unique up to isomorphism; see [23].

A *q-model* for a space X is a CDGA A with the same q-type as Sullivan's CDGA of piecewise polynomial, complex-valued forms on X [23]. Examples of spaces having finite-type models include formal spaces (such as compact Kähler manifolds, hyperplane arrangement complements, etc), smooth quasi-projective varieties, compact solvmanifolds, and Sasakian manifolds.

For each $a \in Z^1(A) \cong H^1(A)$, we construct a cochain complex, $(A^\bullet, \delta_a)$, with differentials $\delta^i_a\colon A^i \to A^{i+1}$, $u \mapsto a \cdot u + \mathrm{d}\, u$. The *resonance varieties* of A are the sets

$$\mathcal{R}^i(A) = \{a \in H^1(A) \mid H^i(A^\bullet, \delta_a) \neq 0\}.$$

These sets are Zariski closed, for all $i \leq q$. If X is a connected, finite-type CW-complex, we obtain the usual resonance varieties by setting $\mathcal{R}^i(X) := \mathcal{R}^i(H^\bullet(X, \mathbb{C}))$. The following theorem summarizes several results relating the analytic germs of the characteristic and resonance varieties of a space and its (q-finite) model at the respective basepoints.

Theorem 4. *Let X be a connected CW-complex with finite q-skeleton which admits a q-finite q-model A. Then, for all $i \leq q$:*

- [4] $\mathcal{V}^i(X)_{(1)} \cong \mathcal{R}^i(A)_{(0)}$. *In particular, if X is q-formal, then $\mathcal{V}^i(X)_{(1)} \cong \mathcal{R}^i(X)_{(0)}$.*
- [15] $\mathrm{TC}_0(\mathcal{R}^i(A)) \subseteq \mathcal{R}^i(X)$.
- [3] *All the irreducible components of $\mathcal{V}^i(X)$ passing through the identity are* algebraic subtori *of* $\pi_1(X)\widehat{\ }$.

The result of Budur and Wang [3] yields a powerful obstruction for the existence of (partially) finite models for spaces and groups. For instance, if G is a finitely presented group with $G_{\rm ab} = \mathbb{Z}^n$ and $\mathcal{V}^1(G) = \{t \in (\mathbb{C}^*)^n \mid \sum_{i=1}^n t_i = n\}$, then G admits no 1-finite 1-model. In a recent preprint with S. Papadima, we provide a completely different obstruction.

Theorem 5 ([17]). *Suppose X is $(q+1)$ finite, or X admits a q-finite q-model. Then $b_i(\mathcal{M}_q(X)) < \infty$, for all $i \le q+1$.*

Corollary 6 ([17]). *Let G be a finitely generated group. Assume that either G is finitely presented, or G has a 1-finite 1-model. Then $b_2(\mathcal{M}_1(G)) < \infty$.*

For instance, let $G = F_n/F_n''$ be the free metabelian group of rank $n \ge 2$. Then $\mathcal{V}^1(G) = \mathcal{V}^1(F_n) = (\mathbb{C}^*)^n$, and so G passes the Budur–Wang test. Yet $b_2(\mathcal{M}_1(G)) = \infty$, and so, by Corollary 6, this group admits no 1-finite 1-model, and no finite presentation. More generally, we have the following result.

Theorem 7 ([17]). *Let G be a finitely generated group which has a free, non-cyclic quotient. Then G/G'' is not finitely presentable, and does not admit a 1-finite 1-model.*

We also reinterpret the condition that a group G admits a 1-finite 1-model in terms of the Malcev Lie algebra $\mathfrak{m}(G)$ associated to G. This pronilpotent Lie algebra may be defined as the set of primitive elements in the completion of the group algebra $\mathbb{Q}G$ with respect to the filtration by powers of the augmentation ideal; see for instance [22] and reference therein.

Theorem 8 ([17]). *A finitely generated group G admits a 1-finite 1-model if and only if $\mathfrak{m}(G)$ is the lower central series completion of a finitely presented Lie algebra.*

4. **RFRp groups, finiteness, and largeness.** In recent work with T. Koberda, we modify Agol's celebrated definition of RFRS groups, as follows. Let G be a finitely generated group and let p be a prime. We say that G is *residually finite rationally* p if there exists a descending sequence of subgroups $\{G_i\}_{i\ge 0}$ such that $G_0 = G$; $G_{i+1} \triangleleft G_i$; $\bigcap_{i\ge 0} G_i = \{1\}$; G_i/G_{i+1} is an elementary abelian p-group; and $\ker(G_i \to H_1(G_i,\mathbb{Q})) < G_{i+1}$. The class of RFR$p$ groups is closed under taking subgroups, finite direct products, and finite free products. Such groups are residually finite, torsion-free, and residually torsion-free polycyclic.

Theorem 9 ([11]). *Let G be a finitely presented group which is non-abelian and RFRp for infinitely many primes p. Then G is bi-orderable; the maximal k-step solvable quotients $G/G^{(k)}$ are not finitely presented, for any $k \ge 2$; and $\Sigma^1(G)^{\complement} \neq \emptyset$.*

Surface groups and right-angled Artin groups are RFRp, for all p, but finite groups and non-abelian nilpotent groups are *not* RFRp, for any p. We show in [11] that a large class of groups occurring at the interface between complex algebraic geometry and low-dimensional topology enjoy the RFRp property. More precisely, let $\mathcal{C}$ be an algebraic curve in $\mathbb{C}^2$, with boundary manifold M. Suppose that each irreducible component of $\mathcal{C}$ is smooth and transverse to the line at infinity, and all singularities of $\mathcal{C}$ are of type A. Then $\pi_1(M)$ is RFRp, for all p.

A finitely generated group G is said to be *large* if there is a finite-index subgroup $H < G$ which surjects onto a free, non-cyclic group. As shown in [10], a finitely presented group G is large if and only if there exists a finite-index subgroup $K < G$ such that $\mathcal{V}^1(K)$ has infinitely many torsion points.

Theorem 10 ([11]). *Let G be a finitely presented group which is non-abelian and RFRp for infinitely many primes p. Then G is large.*

The following result from [17] (based on foundational work of Arapura) gives a geometric interpretation of largeness within the class of quasi-projective groups.

Proposition 11 ([17]). *Let X be a quasi-projective manifold. Then $\pi_1(X)$ is large if and only if there is a finite cover $Y \to X$ and a regular, surjective map from Y to a smooth curve C with $\chi(C) < 0$, so that the generic fiber is connected.*

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