Resonance varieties

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Introduction. One of the most fruitful ideas to arise from the theory of hyperplanes arrangements is that of turning the cohomology ring of a space into a family of cochain complexes, parametrized by the cohomology group in degree one, and extracting certain varieties from these data, as the loci where the cohomology of those cochain complexes jumps.

What makes these "resonance" varieties especially useful is their close connection with a different kind of jumping loci: the "characteristic" varieties, which record the jumps in homology with coefficients in rank 1 local systems. The geometry of these varieties is intimately related to the formality, (quasi-) projectivity, and homological finiteness properties of the fundamental group, and controls to a large extent the Betti numbers of finite abelian covers. For more on this, we refer to [8, 9, 10], and references therein.

I will present here an abstraction of the first resonance variety of a group, based on recent work with Stefan Papadima [6, 7]. This point of view leads to a new stratification of the Grassmannian, and a host of new questions.

Resonance schemes and Koszul modules. Let V be a finite-dimensional complex vector space, and let $K \subset V \wedge V$ be a subspace. The *resonance variety* associated to these data, $\mathcal{R} = \mathcal{R}(V, K)$, is the set of elements a in the dual vector space V^* for which there is an element $b \in V^*$, not proportional to a, such that $a \wedge b$ belongs to the orthogonal complement $K^\perp \subseteq V^* \wedge V^*$; we also declare that $0 \in \mathcal{R}$. It is readily seen that $\mathcal{R}$ is a conical, Zariski-closed subset of the affine space V^*. For instance, if $K = 0$ and if $\dim V > 1$, then $\mathcal{R} = V^*$; at the other extreme, if $K = V \wedge V$, then $\mathcal{R} = 0$.

The resonance variety comes endowed with a natural scheme structure: its defining ideal is the annihilator of the *Koszul module*, $\mathcal{B} = \mathcal{B}(V, K)$. This is a graded module over the symmetric algebra $S = \mathrm{Sym}(V)$, with presentation matrix $\delta_3 \oplus (\mathrm{id}_S \otimes \iota)$, where $\delta_3 \colon S \otimes \bigwedge^3 V \to S \otimes \bigwedge^2 V$ is the third Koszul differential, and $\iota \colon K \to V \wedge V$ is the inclusion map.

Here is an alternate point of view. Let $A = A(V, K)$ be the quadratic algebra defined as the quotient of the exterior algebra $E = \bigwedge V^*$ by the ideal generated by $K^\perp$. Then $\mathcal{R}$ is the set of points $a \in A^1$ where the first Betti number of the cochain complex $A^0 \xrightarrow{a} A^1 \xrightarrow{a} A^2$ jumps. Using results from [2, 4, 5], we may reinterpret the graded pieces of the Koszul module in terms of the linear strand in an appropriate Tor module: $\mathcal{B}_q^* \cong \mathrm{Tor}^E_{q+1}(A, \mathbb{C})_{q+2}$.

Groups and resonance. The main example I have in mind is as follows. Let G be a finitely generated group. The resonance variety of G is then defined as $\mathcal{R}(G) = \mathcal{R}(V, K)$, where $V^* = H^1(G, \mathbb{C})$ and $K^\perp$ is the kernel of the cup-product map $\cup_G \colon V^* \wedge V^* \to H^2(G, \mathbb{C})$.

Rationally, every resonance variety arises in this fashion. More precisely, let V be an n-dimensional $\mathbb{C}$-vector space, and suppose $K \subseteq V \wedge V$ is a linear subspace, defined over $\mathbb{Q}$. Then, as shown in [7], there is a finitely presented, commutator-relators group G with $V^* = H^1(G, \mathbb{C})$ and $K^\perp = \ker(\cup_G)$.

For instance, suppose G_Γ is a right-angled Artin group associated to a finite simple graph Γ on vertex set V. As shown in [5], the resonance variety $\mathcal{R}(G_\Gamma) \subset \mathbb{C}^{\mathsf{V}}$ is the union of all coordinate subspaces $\mathbb{C}^{\mathsf{W}}$ corresponding to subsets $\mathsf{W} \subset \mathsf{V}$ for which the induced graph Γ_{W} is disconnected. Moreover, the Hilbert series $\sum_{q\geq 0} \dim \mathcal{B}_q t^{q+2}$ equals $Q_\Gamma(t/(1-t))$, where $Q_\Gamma(t)$ is the "cut polynomial" of Γ, with coefficient of t^k equal to $\sum_{\mathsf{W}\subset\mathsf{V}\colon |\mathsf{W}|=k} \tilde{b}_0(\Gamma_{\mathsf{W}})$, where $\tilde{b}_0(\Gamma_{\mathsf{W}})$ is one less than the number of components of the induced subgraph on W.

A stratification of the Grassmannian. Now fix $n = \dim V$ and $m = \dim K$. Then K can be viewed as a point in the Grassmannian of m-planes in $V \wedge V$. Moving about this Grassmannian and recording the way the resonance scheme $\mathcal{R}(V, K)$ varies defines a stratification of $\mathbb{G} = \mathrm{Gr}_m(V \wedge V)$.

For instance, consider the "generic" stratum $U = U(n, m)$, consisting of those planes $K \in \mathbb{G}$ for which $\mathcal{R}(V, K) = 0$. Clearly, K belongs to U if and only if the plane $\mathbb{P}(K^\perp) \subset \mathbb{P}(V^* \wedge V^*)$ misses the image of $\mathrm{Gr}_2(V^*)$ under the Plücker embedding. Thus, U is a Zariski open subset of $\mathbb{G}$. Moreover, as noted in [7], this set is non-empty if and only if $m \geq 2n - 3$, in which case there is an integer $q = q(n, m)$ such that $\mathcal{B}_q(V, K) = 0$, for every $K \in U$.

The geometry of the non-generic strata is being studied in joint work with Eric Babson [1]. A key ingredient in this study is the Fulton–MacPherson compactification [3] of the configuration space of $\binom{n}{2} - m$ distinct points in $\mathrm{Gr}_2(\binom{n}{2})$.

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