The equivariant spectral sequence and cohomology with local coefficients

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(joint work with Stefan Papadima)

In his pioneering work from the late 1940s, J.H.C. Whitehead established the category of CW-complexes as the natural framework for much of homotopy theory. A key role in this theory is played by the cellular chain complex of the universal cover of a connected CW-complex, which in turn is tightly connected to (co-)homology with local coefficients. In [8], we revisit these classical topics, drawing much of the motivation from recent work on the topology of complements of complex hyperplane arrangements, and the study of cohomology jumping loci.

A spectral sequence. Let X be a connected CW-complex, π its fundamental group, and $\Bbbk\pi$ the group ring over a coefficient ring $\Bbbk$. The cellular chain complex of the universal cover, $C_\bullet(\widetilde{X},\Bbbk)$, is a chain complex of left $\Bbbk\pi$-modules, and so it is filtered by the powers of the augmentation ideal. We investigate the spectral sequence associated to this filtration, with coefficients in an arbitrary right $\Bbbk\pi$-module M. To start with, we identify the d^1 differential.

Theorem 1. *There is a second-quadrant spectral sequence,* $\{E^r(X,M), d^r\}_{r\geq 1}$*, with* $E^1_{-p,p+q}(X,M) = H_q(X, \operatorname{gr}^p(M))$*. If* $\Bbbk$ *is a field, or* $\Bbbk = \mathbb{Z}$ *and* $H_*(X,\mathbb{Z})$ *is torsion-free, then* $E^1_{-p,p+q}(X,M) = \operatorname{gr}^p(M)\otimes_{\Bbbk} H_q(X,\Bbbk)$*, and the* d^1 *differential decomposes as*

$$\begin{array}{ccc} \operatorname{gr}^p(M)\otimes_{\Bbbk} H_q \xrightarrow{\ \operatorname{id}\otimes\nabla_X\ } & \operatorname{gr}^p(M)\otimes_{\Bbbk}(H_1\otimes_{\Bbbk} H_{q-1}) & \\ & \big\downarrow\cong & \\ & (\operatorname{gr}^p(M)\otimes_{\Bbbk}\operatorname{gr}^1(\Bbbk\pi))\otimes_{\Bbbk} H_{q-1} & \xrightarrow{\ \operatorname{gr}(\mu_M)\otimes\operatorname{id}\ } \operatorname{gr}^{p+1}(M)\otimes_{\Bbbk} H_{q-1}\,, \end{array}$$

where ∇_X *is the comultiplication map on* $H_* = H_*(X,\Bbbk)$*, and* $\mu_M\colon M\otimes_{\Bbbk}\Bbbk\pi \to M$ *is the multiplication map of the module* M*.*

Under fairly general assumptions, $E^\bullet(X,M)$ has an E^∞ term. In general, though, $E^\bullet(X,\Bbbk\pi)$ does not converge, even if X has only finitely many cells.

Base change. To obtain more structure in the spectral sequence, we restrict to a special situation. Suppose $\nu\colon \pi \twoheadrightarrow G$ is an epimorphism onto a group G; then the group ring $\Bbbk G$ becomes a right $\Bbbk\pi$-module, via extension of scalars. The resulting spectral sequence, $E^\bullet(X,\Bbbk G_\nu)$, is a spectral sequence in the category of left $\operatorname{gr}_J(\Bbbk G)$-modules, where J is the augmentation ideal of $\Bbbk G$.

Now let G be an abelian group. Assuming X is of finite type and $\Bbbk$ is a field, the spectral sequence $E^\bullet(X,\Bbbk G_\nu)$ does converge, and computes the J-adic completion of $H_*(X,\Bbbk G_\nu) = H_*(Y,\Bbbk)$, where $Y \to X$ is the Galois G-cover defined by ν. As a particular case, we recover in dual form a result of A. Reznikov [9] on the mod p cohomology of cyclic p-covers of aspherical complexes.

Monodromy action. Let X be a connected, finite-type CW-complex. Suppose $\nu\colon \pi_1(X) \twoheadrightarrow \mathbb{Z}$ is an epimorphism, and $\Bbbk$ is a field. Let $(H^*(X,\Bbbk), \cdot\nu_{\Bbbk})$ be the cochain complex defined by left-multiplication by $\nu_{\Bbbk} \in H^1(X,\Bbbk)$, the cohomology class corresponding to ν.

Theorem 2. *For each $q \geq 0$, the $\mathrm{gr}_J(\Bbbk\mathbb{Z})$-module structure on $E^\infty(X, \Bbbk\mathbb{Z}_\nu)$ determines P_0^q and P_{t-1}^q, the free and $(t-1)$-primary parts of $H_q(X, \Bbbk\mathbb{Z}_\nu)$, viewed as a module over $\Bbbk\mathbb{Z} = \Bbbk[t^{\pm 1}]$. Moreover, the monodromy action of $\mathbb{Z}$ on $P_0^j \oplus P_{t-1}^j$ is trivial for all $j \leq q$ if and only if $H^j(H^*(X,\Bbbk), \cdot\nu_{\Bbbk}) = 0$, for all $j \leq q$.*

Particularly interesting is the case of a smooth manifold X fibering over the circle, with $\nu = p_*\colon \pi \twoheadrightarrow \mathbb{Z}$ the homomorphism induced by the projection map, $p\colon X \to S^1$. The homology of the resulting infinite cyclic cover was studied by J. Milnor in [7]. This led to another spectral sequence, introduced by M. Farber, and further developed by S.P. Novikov, see [6]. The Farber-Novikov spectral sequence has (E_1, d_1)-page dual to our $(E^1(X, \Bbbk\mathbb{Z}_\nu), d_\nu^1)$-page, and higher differentials given by certain Massey products. Their spectral sequence, though, converges to the free part of $H_*(X, \Bbbk\mathbb{Z}_\nu)$, and thus misses the information on the $(t-1)$-primary part captured by the equivariant spectral sequence.

Formality and Jordan blocks. As an application of our machinery, we develop a new 1-formality obstruction for groups, based on the interplay of two ingredients: the connection between the formality property (in the sense of D. Sullivan) and the cohomology jumping loci, established in [4], and the connection between the monodromy action and the Aomoto complex, established in Theorem 2.

Theorem 3. *Let N be the kernel of an epimorphism $\nu\colon \pi \twoheadrightarrow \mathbb{Z}$. Suppose π is 1-formal, and $b_1(N, \mathbb{C}) < \infty$. Then the eigenvalue 1 of the monodromy action of $\mathbb{Z}$ on $H_1(N, \mathbb{C})$ has only 1×1 Jordan blocks.*

Given a reduced polynomial function $f\colon (\mathbb{C}^2, \mathbf{0}) \to (\mathbb{C}, 0)$, there are two standard fibrations associated with it. The above result helps explain the radically different properties of these two fibrations.

- The Milnor fibration, $S_\epsilon^3 \setminus K \to S^1$, has total space the complement of the link at the origin. As shown in [5], this space is formal. Theorem 3 allows us then to recover the well-known fact that the algebraic monodromy has no Jordan blocks of size greater than 1 for the eigenvalue $\lambda = 1$.
- The fibration $f^{-1}(D_\epsilon^*) \to D_\epsilon^*$ is obtained by restricting f to the preimage of a small punctured disk around 0. As pointed out by Alex Dimca at the Oberwolfach Mini-Workshop, the algebraic monodromy of this fibration can have larger Jordan blocks for $\lambda = 1$, see [1]. In such a situation, the total space, $f^{-1}(D_\epsilon^*)$, is non-formal, by Theorem 3.

Bounds on twisted cohomology ranks. Our approach yields readily computable upper bounds on the ranks of the cohomology groups of a space, with coefficients in a prime-power order, rank one local system.

Theorem 4. *Let X be a connected, finite-type CW-complex, and let $\rho\colon \pi_1(X) \to \mathbb{C}^{\times}$ be a character given by $\rho(g) = \zeta^{\nu(g)}$, where $\nu\colon \pi \to \mathbb{Z}$ is a homomorphism, and ζ is a root of unity of order a power of a prime p. Then, for all $q \geq 0$,*

$$\dim_{\mathbb{C}} H^q(X, {}_\rho\mathbb{C}) \leq \dim_{\mathbb{F}_p} H^q(X, \mathbb{F}_p).$$

If, moreover, $H_*(X, \mathbb{Z})$ is torsion-free,

$$\dim_{\mathbb{C}} H^q(X, {}_\rho\mathbb{C}) \leq \dim_{\mathbb{F}_p} H^q(H^*(X, \mathbb{F}_p), \nu_{\mathbb{F}_p}).$$

Neither of these inequalities can be sharpened further. Indeed, we give examples showing that both the prime-power hypothesis on the order of ζ, and the torsion-free hypothesis on $H_*(X, \mathbb{Z})$ are really necessary. The second inequality above generalizes a result of D. Cohen and P. Orlik [2], valid only for complements of complex hyperplane arrangements.

Minimality and linearization. Suppose now X has a *minimal* cell structure, i.e., the number of q-cells of X coincides with the (rational) Betti number $b_q(X)$, for every $q \geq 0$; in particular, $H_*(X, \mathbb{Z})$ is torsion-free. Let $\Bbbk = \mathbb{Z}$, or a field. Pick a basis $\{e_1, \dots, e_n\}$ for $H_1 = H_1(X, \Bbbk)$, and identify the symmetric algebra on H_1 with the polynomial ring $S = \Bbbk[e_1, \dots, e_n]$.

Theorem 5. *Under the above assumptions, the linearization of the equivariant cochain complex of the universal abelian cover of X coincides with the universal Aomoto complex, $(H^*(X, \Bbbk) \otimes_{\Bbbk} S, D)$, with differentials $D(\alpha \otimes 1) = \sum_{i=1}^{n} e_i^* \cdot \alpha \otimes e_i$.*

This theorem generalizes results from [2] and [3], and answers a question posed by M. Yoshinaga in [10].

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