Jumping loci and finiteness properties of groups

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(joint work with Alexandru Dimca, Stefan Papadima)

This is an extended abstract of a talk given on the first day of the Mini-Workshop. In the first part, we give a quick overview of characteristic and resonance varieties. In the second part, we describe recent work [11], relating the cohomology jumping loci of a group to the homological finiteness properties of a related group.

1. Cohomology jumping loci

Characteristic varieties. Let X be a connected CW-complex with finitely many cells in each dimension, and G its fundamental group. The characteristic varieties of X are the jumping loci for cohomology with coefficients in rank 1 local systems:

$$V^i_k(X) = \{\rho \in \operatorname{Hom}(G, \mathbb{C}^*) \mid \dim H^i(X, \mathbb{C}_\rho) \geq k\}.$$

These varieties emerged from the work of Novikov [22] on Morse theory for closed 1-forms on manifolds. It turns out that $V^1_k(X)$ is the zero locus of the annihilator of the k-th exterior power of the complexified Alexander invariant of G; thus, we may write $V_k(G) := V^1_k(X)$. For example, if X is a knot complement, $V_k(G)$ is the set of roots of the Alexander polynomial with multiplicity at least k.

One may compute the first Betti number of a finite abelian regular cover, $Y \to X$, by counting torsion points of a certain order on $\operatorname{Hom}(G, \mathbb{C}^*)$, according to their depth in the filtration $\{V_k(G)\}$, see Libgober [17]. One may also obtain information on the torsion in $H_1(Y, \mathbb{Z})$ by considering characteristic varieties over suitable Galois fields, see [21]. This approach gives a practical algorithm for computing the homology of the Milnor fiber F of a central arrangement in $\mathbb{C}^3$, leading to examples of multi-arrangements with torsion in $H_1(F, \mathbb{Z})$, see [4].

Foundational results on the structure of the cohomology support loci for local systems on smooth, quasi-projective algebraic varieties were obtained by Beauville [2], Green–Lazarsfeld [14], Simpson [28], and ultimately Arapura [1]: if G is the fundamental group of such a variety, then $V_1(G)$ is a union of (possibly translated) subtori of $\operatorname{Hom}(G, \mathbb{C}^*)$. The characteristic varieties of arrangement groups have been studied by, among others, Cohen–Suciu [6], Libgober–Yuzvinsky [20], and Libgober [18]. As noted in [30, 31], translated subtori do occur in this setting; for an in-depth explanation of this phenomenon, see Dimca [7, 8].

Resonance varieties. Consider now the cohomology algebra $H^*(X, \mathbb{C})$. Right-multiplication by a class $a \in H^1(X, \mathbb{C})$ yields a cochain complex $(H^*(X, \mathbb{C}), \cdot a)$. The *resonance varieties* of X are the jumping loci for the homology of this complex:

$$R^i_k(X) = \{a \in H^1(X, \mathbb{C}) \mid \dim H^i(H^*(X, \mathbb{C}), \cdot a) \geq k\}.$$

These varieties were first defined by Falk [12] in the case when X is the complement of a complex hyperplane arrangement. In this setting, a purely combinatorial description of $R^1_k(X)$ was given by Falk [12], Libgober–Yuzvinsky [20], Falk–Yuzvinsky [13], and Pereira–Yuzvinsky [26].

The varieties $R_k(G) := R^1_k(X)$ depend only on $G = \pi_1(X)$. In [30], two conjectures were made, expressing (under some conditions) the lower central series ranks and the Chen ranks of an arrangement group G solely in terms of the dimensions of the components of $R_1(G)$. For recent progress in this direction, see [23, 27].

The tangent cone formula. If G is a finitely presented group G, the tangent cone to $V_k(G)$ at the origin, $TC_1(V_k(G))$, is contained in $R_k(G)$, see Libgober [19]. In general, though, the inclusion is strict, see [21, 9]. Now suppose G is a 1-*formal* group, in the sense of Quillen and Sullivan; that is, the Malcev Lie algebra of G is quadratic. Then, as shown in [9], equality holds:

$$TC_1(V_k(G)) = R_k(G).$$

This extends previous results from [6, 18], valid only for arrangement groups. It is also known that $TC_1(V^i_k(X)) = R^i_k(X)$, for all $i \geq 1$, in the case when X is the complement of a complex hyperplane arrangement, see Cohen–Orlik [5]. A generalization to arbitrary formal spaces is expected.

2. Non-finiteness properties of projective groups

In [29], Stallings constructed the first example of a finitely presented group G with $H_3(G, \mathbb{Z})$ infinitely generated; such a group is of type F_2 but not FP_3. It turns out that Stallings' group is isomorphic to the fundamental group of the complement of a complex hyperplane arrangement, see [23].

More generally, to every finite simple graph Γ, with flag complex $\Delta(\Gamma)$, Bestvina and Brady associate in [3] a group N_Γ and show that N_Γ is finitely presented if and only if $\pi_1(\Delta(\Gamma)) = 0$, while N_Γ is of type FP_{n+1} if and only if $\widetilde{H}_{\leq n}(\Delta(\Gamma), \mathbb{Z}) = 0$.

In joint work with Dimca and Papadima [10], we determine precisely which Bestvina-Brady groups N_Γ occur as fundamental groups of smooth quasi-projective varieties. (The proof uses previous work on the jumping loci of right-angled Artin groups [24, 9] and Bestvina-Brady groups [25].) This classification yields examples of quasi-projective groups which are not commensurable, even up to finite kernels, to the fundamental group of an aspherical, quasi-projective variety.

In [11] we go further, and construct smooth, complex *projective* varieties whose fundamental groups have exotic homological finiteness properties.

Theorem 1 ([11]). *For each $n \geq 2$, there is an n-dimensional, smooth, irreducible, complex projective variety M such that:*

1. *The homotopy groups $\pi_i(M)$ vanish for $2 \leq i \leq n-1$, while $\pi_n(M) \neq 0$.*
2. *The universal cover $\widetilde{M}$ is a Stein manifold.*
3. *The group $\pi_1(M)$ is of type F_n, but not of type FP_{n+1}.*
4. *The group $\pi_1(M)$ is not commensurable (up to finite kernels) to any group having a classifying space of finite type.*

Theorem 1 provides a negative answer to the following question raised by Kollár in [16]: Is a projective group G commensurable (up to finite kernels) with another group G', admitting a $K(G', 1)$ which is a quasi-projective variety?

Theorem 1 also sheds light on the following question of Johnson and Rees [15]: Are fundamental groups of compact Kähler manifolds Poincaré duality groups of even cohomological dimension? In [32], Toledo answered this question, by producing examples of smooth projective varieties M with $\pi_1(M)$ of *odd* cohomological dimension. Our results show that fundamental groups of smooth projective varieties need not be Poincaré duality groups of *any* cohomological dimension: their Betti numbers need not be finite.

A key point in our approach is a theorem connecting the characteristic varieties of a group G to the homological finiteness properties of some of its normal subgroups N.

Theorem 2 ([11])**.** *Let G be a finitely generated group. Suppose $\nu\colon G \to \mathbb{Z}^m$ is a non-trivial homomorphism, and set $N = \ker(\nu)$. If $V_1^r(G) = \mathrm{Hom}(G, \mathbb{C}^*)$ for some integer $r \geq 1$, then:*

(1) $\dim_{\mathbb{C}} H_{\leq r}(N, \mathbb{C}) = \infty$.
(2) *N is not commensurable (up to finite kernels) to any group of type FP_r.*

The proof of Theorem 2 depends on the following lemma. Let $\mathbb{T} = \mathrm{Hom}(\mathbb{Z}^m, \mathbb{C}^*)$ be the character torus of $\mathbb{Z}^m$, and let $\Lambda = \mathbb{C}\mathbb{Z}^m$ be its coordinate ring. Let A be a Λ-module which is finite-dimensional as a $\mathbb{C}$-vector space. Then, for each $j \geq 0$, the set $A_j := \{\rho \in \mathbb{T} \mid \mathrm{Tor}_j^{\Lambda}(\mathbb{C}_\rho, A) = 0\}$ is a Zariski open, non-empty subset of the algebraic torus $\mathbb{T}$.

To obtain our examples, we start with an elliptic curve E and take 2-fold branched covers $f_j\colon C_j \to E$ ($1 \leq j \leq r$ and $r \geq 3$), so that each curve C_j has genus at least 2. Setting $X = \prod_{j=1}^r C_j$, we see that X is a smooth, projective variety, whose universal cover is a contractible, Stein manifold. Moreover, $V_1^r(\pi_1(X)) = \mathrm{Hom}(\pi_1(X), \mathbb{C}^*)$.

Using the group law on E, define a map $h\colon X \to E$ by $h = \sum_{j=1}^r f_j$. Let M be the smooth fiber of h. Under certain assumptions on the branched covers f_j, we show that M is connected and h has only isolated singularities. A complex Morse-theoretic argument shows that the induced homomorphism, $\nu = h_\sharp\colon \pi_1(X) \to \pi_1(E)$, is surjective, with kernel N isomorphic to $\pi_1(M)$. Applying Theorem 2 to this situation (with $n = r - 1$) finishes the proof of Theorem 1.

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