Milnor fibrations of arrangements with trivial algebraic monodromy

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1. **Descending series and graded Lie algebras.** Among all the descending series of subgroups associated to a group G, the most prominent are the lower central series, $\{\gamma_k(G)\}_{k\geqslant 1}$, and the derived series, $\{G^{(r)}\}_{r\geqslant 0}$. Following Stallings [9], we also consider the rational and mod-p versions of these series. All these series start at G, and obey the following recursion formulas:

$$\gamma_{k+1}(G) = [G, \gamma_k(G)] \qquad G^{(r)} = [G^{(r-1)}, G^{(r-1)}]\,, \tag{1}$$

$$\gamma^{\mathbb{Q}}_{k+1}(G) = \sqrt{[G, \gamma^{\mathbb{Q}}_k(G)]} \qquad G^{(r)}_{\mathbb{Q}} = \sqrt{\big[G^{(r-1)}_{\mathbb{Q}}, G^{(r-1)}_{\mathbb{Q}}\big]}\,, \tag{2}$$

$$\gamma^{p}_{k+1}(G) = (\gamma^p_k G)^p [G, \gamma^p_k G] \qquad G^{(r)}_{p} = (G^{(r-1)}_p)^p\big[G^{(r-1)}_{p}, G^{(r-1)}_{p}\big]\,. \tag{3}$$

Each one of the series on the left forms an *N-series* for G, that is, a descending filtration, $N = \{N_k\}_{k\geqslant 1}$, of subgroups such that $N_1 = G$ and $[N_k, N_\ell] \subseteq N_{k+\ell}$. Therefore, each subgroup N_k is normal; moreover, each quotient N_k/N_{k+1} lies in the center of G/N_{k+1}, and thus is an abelian group. The direct sum of these quotients, $\operatorname{gr}^N(G) := \bigoplus_{k\geqslant 1} N_k/N_{k+1}$, acquires the structure of a graded Lie algebra. When N is one of the aforementioned N-series, the corresponding associated graded Lie algebra is denoted by $\operatorname{gr}(G)$, $\operatorname{gr}^{\mathbb{Q}}(G)$, and $\operatorname{gr}^p(G)$, respectively.

2. **Alexander invariants and characteristic varieties.** Let $G' = [G, G]$ and $G'' = [G', G']$ be the first two terms in the derived series of G. The *Alexander invariant* of G is the abelian group $B(G) := G'/G''$, viewed as a $\mathbb{Z}[G_{\mathrm{ab}}]$-module. The module structure is induced from conjugation in the maximal metabelian quotient, G/G''; that is, $gG'\cdot xG'' = gxg^{-1}G''$ for $g \in G$ and $x \in G'$. In like fashion, we define the rational and mod-p Alexander invariants as $B_{\mathbb{Q}}(G) := G'_{\mathbb{Q}}/G''_{\mathbb{Q}}$ and $B_p(G) := G'_p/G''_p$, viewed as modules over $\mathbb{Z}[G_{\mathrm{abf}}]$ and $\mathbb{Z}[H_1(G, \mathbb{Z}_p)]$, respectively.

Assume now that G is finitely generated. Then the group of complex-valued characters, $\mathbb{T}_G := \operatorname{Hom}(G, \mathbb{C}^*)$, is a complex algebraic group, with identity component $\mathbb{T}^0_G \cong (\mathbb{C}^*)^n$, where $n = \operatorname{rank} G_{\mathrm{ab}}$. The (depth k) *characteristic varieties* of G are the algebraic subsets $\mathscr{V}_k(G) \subseteq \mathbb{T}_G$ consisting of those characters $\rho\colon G \to \mathbb{C}^*$ for which $\dim_{\mathbb{C}} H^1(G, \mathbb{C}_\rho) \geqslant k$. The set $\mathscr{V}_1(G)$ coincides, at least away from the identity $1 \in \mathbb{T}_G$, with the zero locus of the annihilator ideal of $B(G)\otimes\mathbb{C}$. Likewise, $\mathscr{W}_1(G) := \mathscr{V}_1(G) \cap \mathbb{T}^0_G$ coincides, away from 1, with $V(\operatorname{ann}(B_{\mathbb{Q}}(G)\otimes\mathbb{C}))$.

3. **Split extensions.** Given a split extension of groups, $G = K \rtimes_\varphi Q$, we consider a certain series of normal subgroups of K. This series, $L = \{L_n\}_{n\geqslant 1}$, was recently introduced by Guaschi and Pereiro in [5], who showed that $\gamma_n(G) = L_n \rtimes_\varphi \gamma_n(Q)$ for all $n \geqslant 1$. In [13], we prove that the series L is, in fact, an N-series, and recover their result. As a corollary, we show that $\operatorname{gr}(G)$ splits as a semidirect product of graded Lie algebras, $\operatorname{gr}^L(G) \rtimes_{\bar\varphi} \operatorname{gr}(Q)$.

In the case when Q acts trivially on the abelianization K_{ab}, we show that $L_n = \gamma_n(K)$ for all $n \geqslant 1$. As a corollary, we recover a well-known theorem of Falk and Randell [4]. If, moreover, Q is abelian and the inclusion $\iota\colon K \to G$ induces an injection $\iota_{\mathrm{ab}}\colon K_{\mathrm{ab}} \to G_{\mathrm{ab}}$ (this always happens if $Q = \mathbb{Z}$), we prove in [14] that:

(a) The map $\mathrm{gr}(\iota)\colon \mathrm{gr}_{>1}(K) \to \mathrm{gr}_{>1}(G)$ is an isomorphism.
(b) The map $B(\iota)\colon B(K) \to B(G)$ is a $\mathbb{Z}[K_{\mathrm{ab}}]$-linear isomorphism.
(c) The map $\iota^*\colon \mathbb{T}_G \twoheadrightarrow \mathbb{T}_K$ restricts to a surjection $\iota^*\colon \mathscr{V}_1(G) \twoheadrightarrow \mathscr{V}_1(K)$.

In the case when G is a right-angled Artin group and K is the corresponding Bestvina–Brady group, this recovers the main results from [7].

For the rational lower central series we start by showing that $\gamma_n^{\mathbb{Q}}(G) = \sqrt[G]{\gamma_n(G)}$, where $\sqrt[G]{S} \coloneqq \{g \in G \mid g^k \in S \text{ for some } k > 0\}$ for $S \subseteq G$. Work of Massuyeau [6] now implies that $\{\gamma^{\mathbb{Q}}(G)\}_{n\geqslant 1}$ and $\{L_n\}_{n\geqslant 1}$ are N-series for G and K, respectively. We then show in [13] that $\gamma_n^{\mathbb{Q}}(G) = \sqrt[K]{L_n} \rtimes_{\varphi} \gamma_n^{\mathbb{Q}}(Q)$ for all $n \geqslant 1$, and thus $\mathrm{gr}^{\mathbb{Q}}(G) = \mathrm{gr}^{\sqrt[K]{L}}(G) \rtimes_{\bar{\varphi}} \mathrm{gr}^{\mathbb{Q}}(Q)$. In the case when Q acts trivially on the torsion-free abelianization $K_{\mathrm{abf}} = \mathrm{gr}_1^{\mathbb{Q}}(K)$, we show that $\sqrt[K]{L_n} = \gamma_n^{\mathbb{Q}}(K)$ for all $n \geqslant 1$, and thus $\mathrm{gr}^{\mathbb{Q}}(G) = \mathrm{gr}^{\mathbb{Q}}(K) \rtimes_{\bar{\varphi}} \mathrm{gr}^{\mathbb{Q}}(Q)$. If, moreover, Q is torsion-free abelian and the map $\iota\colon K \hookrightarrow G$ induces an injection $K_{\mathrm{abf}} \hookrightarrow G_{\mathrm{abf}}$, we prove in [14] that:

(a′) The map $\mathrm{gr}(\iota)\colon \mathrm{gr}^{\mathbb{Q}}_{>1}(K) \to \mathrm{gr}^{\mathbb{Q}}_{>1}(G)$ is an isomorphism.
(b′) The map $B(\iota)\colon B^{\mathbb{Q}}(K) \to B^{\mathbb{Q}}(G)$ is a $\mathbb{Z}[K_{\mathrm{abf}}]$-linear isomorphism.
(c′) The map $\iota^*\colon \mathbb{T}^0_G \twoheadrightarrow \mathbb{T}^0_K$ restricts to a surjection $\iota^*\colon \mathscr{W}_1(G) \twoheadrightarrow \mathscr{W}_1(K)$.

Though a parallel theory in characteristic p is not yet fully developed, some of the above results do have analogues over $\mathbb{Z}_p$. For instance, if $G = K \rtimes_{\varphi} Q$ is a split extension with Q acting trivially on $H_1(K, \mathbb{Z}_p)$, it is shown in [2] that $\gamma_n^p(G) = \gamma_n^p(K) \rtimes_{\varphi} \gamma_n^p(Q)$ for all $n \geqslant 1$; therefore, $\mathrm{gr}^p(G) = \mathrm{gr}^p(K) \rtimes_{\bar{\varphi}} \mathrm{gr}^p(Q)$.

4. **Milnor fibrations of arrangements.** A construction due to Milnor associates to each homogeneous polynomial $f \in \mathbb{C}[z_0, \dots, z_d]$ a fiber bundle, with base space $\mathbb{C}^*$, total space the complement $M = \mathbb{C}^{d+1} \backslash \{f = 0\}$, and projection map $f\colon M \to \mathbb{C}^*$. The Milnor fiber $F = f^{-1}(1)$ is a Stein manifold, and thus has the homotopy type of a finite CW-complex of dimension d. The monodromy of the fibration, $h\colon F \to F$, is given by $h(z) = e^{2\pi i/n} z$, where $n = \deg f$. If the polynomial f has an isolated singularity at the origin, then F is homotopy equivalent to a bouquet of d-spheres, whose number can be determined by algebraic means. In general, though, it is a hard problem to compute the homology groups of the Milnor fiber, even in the case when f completely factors into distinct linear forms.

This situation is best described by a hyperplane arrangement, that is, a finite collection, $\mathscr{A}$, of codimension-1 linear subspaces in $\mathbb{C}^{d+1}$, for some $d > 0$. Choosing a linear form f_H with kernel H for each hyperplane $H \in \mathscr{A}$, we obtain a homogeneous polynomial, $f = \prod_{H \in \mathscr{A}} f_H$. The long exact sequence in homotopy of the Milnor fibration $F \to M \to \mathbb{C}^*$ yields a split extension at the level of fundamental groups, $G = K \rtimes_{\varphi} \mathbb{Z}$, where $G = \pi_1(M)$, $K = \pi_1(F)$, and $\varphi(1) = h_*\colon K \to K$. In general, the monodromy action of $\mathbb{Z}$ on K_{ab} is highly non-trivial, and the determination of $b_1(F) = \operatorname{rank} K_{\mathrm{ab}}$ is far from known, except in some cases, see for instance [8, 11, 12]. It is also known that $H_*(F, \mathbb{Z})$ may have non-trivial torsion (see [3]), and that such torsion can, in fact, occur even in $H_1(F, \mathbb{Z})$ (see [16]).

Finally, it is known that the ranks of the groups $\operatorname{gr}_k(G)$ are determined by the intersection lattice, yet $\operatorname{gr}_k(G)$ may have torsion (as noted in [10]), and such torsion is not necessarily combinatorially determined (see [1]).

In forthcoming work, [15], we use the general theory described above to study Milnor fibrations of arrangements for which the monodromy $h\colon F \to F$ acts trivially on either $H_1(F,\mathbb{Z})$, or $H_1(F,\mathbb{Z})/\mathrm{Tors}$, or $H_1(F,\mathbb{Z}_p)$ for some prime p.

In the first case, we have by (a) and (c) that the inclusion $F \hookrightarrow M$ induces an isomorphism $\operatorname{gr}_{>1}(\pi_1(F)) \cong \operatorname{gr}_{>1}(\pi_1(M))$ and a surjection $\mathscr{V}_1(M) \twoheadrightarrow \mathscr{V}_1(F)$. Nevertheless, examples from [12] show that the map $\mathscr{V}_2(M) \to \mathscr{V}_2(F)$ may not be surjective. In fact, there are pairs of arrangements $\mathscr{A}$ and $\mathscr{A}'$ with trivial monodromy in first homology such that $M \simeq M'$ and yet $F \not\simeq F'$, with the difference picked up by the depth 2 characteristic varieties.

In the second case, we have by (a′) and (c′) an isomorphism $\operatorname{gr}^{\mathbb{Q}}_{>1}(\pi_1(F)) \cong \operatorname{gr}^{\mathbb{Q}}_{>1}(\pi_1(M))$ and a surjection $\mathscr{W}_1(M) \twoheadrightarrow \mathscr{W}_1(F)$. We illustrate this phenomenon with Yoshinaga's icosidodecahedral arrangement from [16]. Trying to better understand this example and those from [12] has motivated much of this work.

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