Tropical bounds for the BNSR invariants

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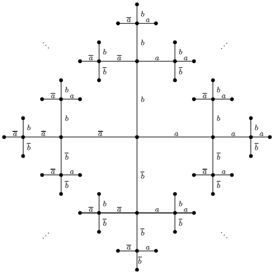
The Bieri–Neumann–Strebel–Renz invariants

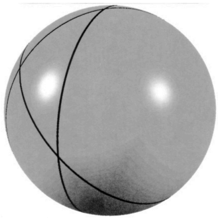
- Let G be a finitely generated group. Let
  - $n = b_1(G) > 0$.
  - $S(G) = S^{n-1}$, the unit sphere in $\mathrm{Hom}(G, \mathbb{R}) = \mathbb{R}^n$.
  - $\mathrm{Cay}(G)$, the Cayley graph of G.



- (Bieri–Neumann–Strebel 1987)

$$\Sigma^1(G) = \{\chi \in S(G) \mid \mathrm{Cay}_\chi(G) \text{ is connected}\}$$

where $\mathrm{Cay}_\chi(G)$ is the induced subgraph of $\mathrm{Cay}(G)$ on vertex set $G_\chi = \{g \in G \mid \chi(g) \geqslant 0\}$.



- (Bieri–Renz 1988) $\Sigma^q(G,\mathbb{Z}) = \{\chi \in S(G) \mid G_\chi \text{ is of type } \mathrm{FP}_q\}$, i.e., there is a projective $\mathbb{Z}G_\chi$-resolution $P_\bullet \to \mathbb{Z}$, with P_i finitely generated for all $i \leqslant q$. Moreover, $\Sigma^1(G,\mathbb{Z}) = -\Sigma^1(G)$.

- Similar definition for $\Sigma^q(G,R)$, where R is any commutative ring.

- The BNSR-invariants of form a descending chain of open subsets,

$$S(G) \supseteq \Sigma^1(G,\mathbb{Z}) \supseteq \Sigma^2(G,\mathbb{Z}) \supseteq \cdots .$$

- The Σ-invariants control the finiteness properties of normal subgroups $N \lhd G$ for which G/N is free abelian:

$$N \text{ is of type } \mathrm{FP}_q \iff S(G,N) \subseteq \Sigma^q(G,\mathbb{Z})$$

  where $S(G,N) = \{\chi \in S(G) \mid \chi(N) = 0\}$.

- In particular: $\ker(\chi\colon G \twoheadrightarrow \mathbb{Z})$ is f.g. $\iff \{\pm\chi\} \subseteq \Sigma^1(G)$.

Novikov–Sikorav homology

- For each $\chi \in S(G)$, the Novikov–Sikorav completion of $\mathbb{Z}G$,
$$\widehat{\mathbb{Z}G}_{\chi} = \Big\{\lambda \in \mathbb{Z}^{G} \mid \{g \in \operatorname{supp}\lambda \mid \chi(g) \geqslant c\} \text{ is finite, } \forall c \in \mathbb{R}\Big\},$$
is a ring containing $\mathbb{Z}G$ as a subring.
- Example: $G = \mathbb{Z} = \langle t \rangle$ and $\chi(t) = 1$. Then
$$\widehat{\mathbb{Z}G}_{\chi} = \mathbb{Z}[[t^{-1}]][t] = \Big\{\sum_{i \leqslant k} n_i t^i \mid n_i \in \mathbb{Z}, \text{ for some } k \in \mathbb{Z}\Big\}.$$
- Now let X be a connected CW-complex with finite q-skeleton, for some $q \geqslant 1$. Write $G := \pi_1(X)$ and $S(X) := S(G)$.
- (Farber–Geoghegan–Schütz 2010)
$$\Sigma^q(X, \mathbb{Z}) = \{\chi \in S(X) \mid H_i(X; \widehat{\mathbb{Z}G}_{-\chi}) = 0, \ \forall\, i \leqslant q\}.$$
- (Bieri 2007) If G is FP_k, then $\Sigma^q(G, \mathbb{Z}) = \Sigma^q(K(G,1), \mathbb{Z})$, $\forall q \leqslant k$.
- $\Sigma^1(G) = -\Sigma^1(G, \mathbb{Z})$ consists of those characters $\chi \in S(G)$ for which both $H_0(G; \widehat{\mathbb{Z}G}_{\chi})$ and $H_1(G; \widehat{\mathbb{Z}G}_{\chi})$ vanish.

Characteristic and resonance varieties

- Let $\mathbb{T}_G := \mathrm{Hom}(G, \mathbb{C}^*)$ be the character group of $G = \pi_1(X)$, which may be identified with $\mathrm{Char}(X) := H^1(X; \mathbb{C}^*)$.

- The *characteristic varieties* of X are the sets

$$\mathcal{V}^i(X) = \{\rho \in \mathbb{T}_G \mid H_i(X; \mathbb{C}_\rho) \neq 0\}.$$

- If X has finite q-skeleton, then $\mathcal{V}^i(X)$ is Zariski closed for all $i \leqslant q$.

- Define similarly $\mathcal{V}^i(X, \Bbbk) \subset H^1(X; \Bbbk^*)$ for any field $\Bbbk$. If $\Bbbk \subset \mathbb{L}$, then $\mathcal{V}^i(X, \Bbbk) = \mathcal{V}^i(X, \mathbb{L}) \cap H^1(X; \Bbbk^*)$ and $\mathcal{V}^i(X, \mathbb{L}) = \mathcal{V}^i(X, \Bbbk) \times_{\Bbbk} \mathbb{L}$.

- Let $\exp\colon \mathbb{C}^n \to (\mathbb{C}^*)^n$. Given a subvariety $W \subset (\mathbb{C}^*)^n$, define its *exponential tangent cone* at $1 \in (\mathbb{C}^*)^n$ as

$$\tau_1(W) = \{z \in \mathbb{C}^n \mid \exp(\lambda z) \in W,\ \forall \lambda \in \mathbb{C}\}.$$

- If $T \cong (\mathbb{C}^*)^r$ is an algebraic subtorus, then $\tau_1(T) = T_1(T) \cong \mathbb{C}^r$.

- (Dimca–Papadima–S. 2009) $\tau_1(W)$ is a finite union of rationally defined linear subspaces.
- For any subfield $\Bbbk \subset \mathbb{C}$, set $\tau_1^{\Bbbk}(W) = \tau_1(W) \cap \Bbbk^n$.
- Let $A = H^*(X;\mathbb{C})$. For each $a \in A^1$, we have that $a^2 = 0$. Thus, there is a cochain complex, $(A, \cdot a)\colon\ A^0 \xrightarrow{a} A^1 \xrightarrow{a} A^2 \longrightarrow \cdots$.
- The *resonance varieties* of X are the homogeneous algebraic sets
$$\mathcal{R}^i(X) = \{a \in A^1 \mid H^i(A, a) \neq 0\}.$$
- Identify $A^1 = H^1(X;\mathbb{C})$ with $\mathbb{C}^n$, where $n = b_1(X)$. The map $\exp\colon H^1(X;\mathbb{C}) \to H^1(X;\mathbb{C}^*)$ has image $\mathbb{T}_G^0 = (\mathbb{C}^*)^n$.
- (DPS 2009)
$$\tau_1(\mathcal{V}^i(X)) \subseteq \mathcal{R}^i(X).$$
- (DPS 2009, DP 2014) If X is a q-formal space, then, for all $i \leqslant q$,
$$\tau_1(\mathcal{V}^i(X)) = \mathcal{R}^i(X).$$

Bounding the Σ-invariants

Theorem (Papadima–S. 2010)

Let X be a connected CW-complex with finite q-skeleton. Then

$$\Sigma^q(X,\mathbb{Z}) \subseteq S\left(\tau_1^{\mathbb{R}}\left(\mathcal{V}^{\leqslant q}(X)\right)\right)^{\mathrm{c}}$$

- Thus, $\Sigma^q(X,\mathbb{Z})$ is contained in the complement of a finite union of rationally defined great subspheres.
- If X is q-formal, then $\Sigma^q(X,\mathbb{Z}) \subseteq S\left(\mathcal{R}^{\leqslant q}(X)\right)^{\mathrm{c}}$.

Example

Let X be a nilmanifold. Then $\Sigma^q(X,\mathbb{Z}) = S(X)$, while $\mathcal{V}^q(X) = \{1\}$, $\forall q$. Thus, $\Sigma^q(X,\mathbb{Z}) = S(\tau_1^{\mathbb{R}}(\mathcal{V}^{\leqslant q}(X)))^{\mathrm{c}}$, for all q.

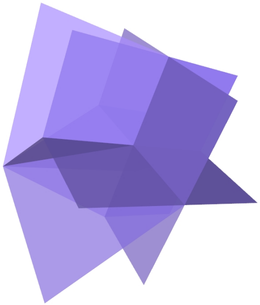
Example

- Let $G_\Gamma = \langle v \in V \mid vw = wv \text{ if } \{v,w\} \in E \rangle$ be the right-angled Artin group associated to a finite simple graph $\Gamma = (V, E)$.
- There is a finite $K(G_\Gamma, 1)$ which is formal.
- $\Sigma^q(G_\Gamma, \mathbb{R}) = S(\mathcal{R}^{\leqslant q}(G_\Gamma, \mathbb{R}))^{\mathrm{c}}$ holds for all q.
- $\Sigma^q(G_\Gamma, \mathbb{Z}) = S(\mathcal{R}^{\leqslant q}(G_\Gamma, \mathbb{R}))^{\mathrm{c}}$, provided the homology groups of certain subcomplexes in the flag complex of Γ are torsion-free.
- This condition is always satisfied in degree $q = 1$, giving $\Sigma^1(G_\Gamma) = S(\mathcal{R}^1(G_\Gamma, \mathbb{R}))^{\mathrm{c}}$.
- Here, $\mathcal{R}^1(G_\Gamma, \mathbb{R})$ is the union of the coordinate subspaces $\mathbb{R}^W \subset \mathbb{R}^V$ for which the induced subgraph Γ_W is disconnected.

Tropical varieties

- Let $\mathbb{K} = \mathbb{C}\{\{t\}\} = \bigcup_{n \geqslant 1} \mathbb{C}((t^{1/n}))$ be the field of Puiseux series over $\mathbb{C}$.
- A non-zero element of $\mathbb{K}$ has the form $c(t) = c_1 t^{a_1} + c_2 t^{a_2} + \cdots$, $c_i \in \mathbb{C}^*$ and $a_1 < a_2 < \cdots$ rationals with a common denominator.
- The (algebraically closed) field $\mathbb{K}$ admits a valuation $v \colon \mathbb{K}^* \to \mathbb{Q}$, given by $v(c(t)) = a_1$.
- Let $v \colon (\mathbb{K}^*)^n \to \mathbb{Q}^n \subset \mathbb{R}^n$ be the n-fold product of the valuation.
- The *tropicalization* of a subvariety $W \subset (\mathbb{K}^*)^n$, denoted $\operatorname{Trop}(W)$, is the closure (in the Euclidean topology) of $v(W)$ in $\mathbb{R}^n$.
- This is a rational polyhedral complex in $\mathbb{R}^n$. For instance, if W is a curve, then $\operatorname{Trop}(W)$ is a graph with rational edge directions.

- If T is an algebraic subtorus of $(\mathbb{K}^*)^n$, then $\mathrm{Trop}(T)$ is the linear subspace $\mathrm{Hom}(\mathbb{K}^*, T) \otimes \mathbb{R} \subset \mathrm{Hom}(\mathbb{K}^*, (\mathbb{K}^*)^n) \otimes \mathbb{R} = \mathbb{R}^n$.
- Moreover, if $z \in (\mathbb{K}^*)^n$, then $\mathrm{Trop}(z \cdot T) = \mathrm{Trop}(T) + v(z)$.
- For a variety $W \subset (\mathbb{C}^*)^n$, we may define its tropicalization by setting $\mathrm{Trop}(W) = \mathrm{Trop}(W \times_{\mathbb{C}} \mathbb{K})$.
- In this case, the tropicalization is a polyhedral fan in $\mathbb{R}^n$.

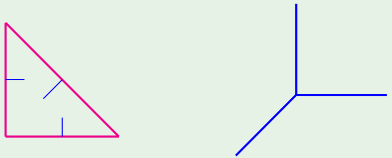


- For a polytope $P \subset \mathbb{R}^n$, let $P^\circ = \{y \in (\mathbb{R}^n)^\vee : x \cdot y \leqslant 1, \forall x \in P\}$ be its polar dual, and set
  - $\mathcal{F}(P)$ the face fan (the set of cones spanned by the faces of P);
  - $\mathcal{N}(P)$ the (inner) normal fan.
- If $0 \in \operatorname{int}(P)$, then $\mathcal{N}(P) = \mathcal{F}(P^\circ)$.
- If $W = V(f)$ is a hypersurface defined by $f = \sum_{\mathrm{u} \in A} a_{\mathrm{u}} \mathrm{t}^{\mathrm{u}} \in \mathbb{C}[\mathrm{t}^{\pm 1}]$, and $\operatorname{Newt}(f) = \operatorname{conv}\{\mathrm{u} \mid a_{\mathrm{u}} \neq 0\} \subset \mathbb{R}^n$, then

$$\operatorname{Trop}(V(f)) = \mathcal{N}(\operatorname{Newt}(f))^{\operatorname{codim}>0}.$$

Example

Let $f = t_1 + t_2 + 1$. Then $\operatorname{Newt}(f) = \operatorname{conv}\{(1,0),(0,1),(0,0)\}$ is a triangle, and so $\operatorname{Trop}(V(f))$ is a tripod.



Tropicalizing the characteristic varieties

- Recall $\mathbb{K} = \mathbb{C}\{\{t\}\}$ comes with a valuation map, $v\colon \mathbb{K}^* \to \mathbb{Q}$.
- Let $\nu_X\colon \operatorname{Char}_{\mathbb{K}}(X) \to \mathbb{Q}^n \subset \mathbb{R}^n$ be the composite

$$H^1(X;\mathbb{K}^*) \xrightarrow{v_*} H^1(X;\mathbb{Q}) \longrightarrow H^1(X;\mathbb{R}).$$

- Given an algebraic subvariety $W \subset H^1(X;\mathbb{C}^*)$ we define its *tropicalization* as the closure in $H^1(X;\mathbb{R}) \cong \mathbb{R}^n$ of the image of $W \times_{\mathbb{C}} \mathbb{K} \subset H^1(X;\mathbb{K}^*)$ under ν_X,

$$\operatorname{Trop}(W) := \overline{\nu_X(W \times_{\mathbb{C}} \mathbb{K})}.$$

- It follows that $\operatorname{Trop}(\mathcal{V}^i(X)) = \overline{\nu_X\big(\mathcal{V}^i(X,\mathbb{K})\big)}$.
- If $W \subset (\mathbb{C}^*)^n$ is an algebraic variety, then $\tau_1^{\mathbb{R}}(W) \subseteq \operatorname{Trop}(W)$.
- Therefore, $\tau_1^{\mathbb{R}}(\mathcal{V}^i(X)) \subseteq \operatorname{Trop}(\mathcal{V}^i(X))$.

A tropical bound for the Σ-invariants

Theorem (PS-2010, S-2021)

Let $\rho\colon \pi_1(X) \to \Bbbk^*$ be a character such that $\rho \in \mathcal{V}^{\leqslant q}(X, \Bbbk)$. Let $v\colon \Bbbk^* \to \mathbb{R}$ be the homomorphism defined by a valuation on $\Bbbk$. If the homomorphism $\chi := v \circ \rho\colon \pi_1(X) \to \mathbb{R}$ is non-zero, then $\chi \notin \Sigma^q(X, \mathbb{Z})$.

Theorem (S-2021)

$$\Sigma^q(X, \mathbb{Z}) \subseteq S(\operatorname{Trop}(\mathcal{V}^{\leqslant q}(X)))^{\mathrm{c}}.$$

Corollary

$$\Sigma^q(X, \mathbb{Z}) \subseteq S(\operatorname{Trop}(\mathcal{V}^{\leqslant q}(X)))^{\mathrm{c}} \subseteq S(\tau_1^{\mathbb{R}}(\mathcal{V}^{\leqslant q}(X)))^{\mathrm{c}}.$$

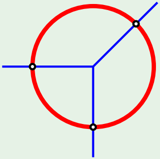
Corollary

If $\mathcal{V}^{\leqslant q}(X)$ contains a component of $\operatorname{Char}(X)$, then $\Sigma^q(X, \mathbb{Z}) = \emptyset$.

Two-generator, one-relator groups

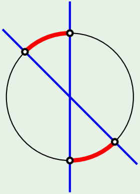
- Let $G = \langle x, y \mid r \rangle$, with $b_1(G) = 2$. In 1987, K. Brown gave a combinatorial algorithm for computing $\Sigma^1(G)$.

Example



- Let $G = \langle a, b \mid b^2(ab^{-1})^2 a^{-2} \rangle$.
- Then $\Sigma^1(G) = S^1 \setminus \{(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (0, -1), (-1, 0)\}$.
- The Alexander polynomial is given by $\Delta_G = a + b + 1$.
- Thus, $\Sigma^1(G) = -S(\mathrm{Trop}(V(\Delta_G)))^{\mathrm{c}}$, though $\tau_1 \mathcal{V}^1(G) = \{0\}$.

Example



- Let $G = \langle a, b \mid a^2ba^{-1}ba^2ba^{-1}b^{-3}a^{-1}ba^2ba^{-1}ba$ $b^{-1}a^{-2}b^{-1}ab^{-1}a^{-2}b^{-1}ab^3ab^{-1}a^{-2}b^{-1}ab^{-1}a^{-1}b \rangle$.
- Then $\Delta_G = (a-1)(ab-1)$, and so $S(\mathrm{Trop}(V(\Delta_G)))$ consists of two pairs of points.
- Yet $\Sigma^1(G)$ consists of two open arcs joining those points.

Compact 3-manifolds

- Let M be a compact, connected, orientable, 3-manifold with empty or toroidal boundary, and assume $b_1(M) \geqslant 2$.
- (C. McMullen 2002) Given $\phi \in H^1(M;\mathbb{Z})$, its *Alexander norm*, $\|\phi\|_A$, is the length of $\phi(\mathrm{Newt}(\Delta_M))$.
- This defines a semi-norm on $H^1(M;\mathbb{Z})$, which extends to $H^1(M;\mathbb{R})$. We let $B_A = \{\phi \in H^1(M;\mathbb{R}) \mid \|\phi\|_A \leqslant 1\}$.

Theorem

(1) $\mathrm{Trop}\left(\mathcal{V}^1(M) \cap \mathrm{Char}(M)^0\right)$ *is the positive-codimension skeleton of* $\mathcal{F}(B_A)$*, the face fan of the unit ball in the Alexander norm.*

(2) (Partly recovers McMullen's theorem) $\Sigma^1(M,\mathbb{Z})$ *is contained in the union of the open cones on the facets of* B_A*.*

Kähler manifolds

- Let M be a compact Kähler manifold. Then M is formal.
- (Beauville, Catanese, Green–Lazarsfeld, Simpson, Arapura, B. Wang) $\mathcal{V}^i(M)$ are finite unions of torsion translates of algebraic subtori of $H^1(M, \mathbb{C}^*)$.

Theorem (Delzant 2010)

$$\Sigma^1(M) = S(M) \setminus \bigcup\nolimits_\alpha S(f_\alpha^*(H^1(C_\alpha; \mathbb{R}))),$$

where the union is taken over those orbifold fibrations $f_\alpha \colon M \to C_\alpha$ *with the property that either* $\chi(C_\alpha) < 0$, *or* $\chi(C_\alpha) = 0$ *and* f_α *has some multiple fiber.*

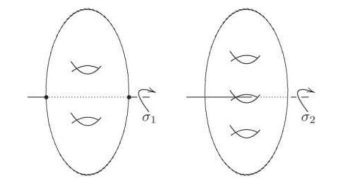
We may recast this result in the tropical setting, as follows.

Corollary

$$\Sigma^1(M) = S(\operatorname{Trop}(\mathcal{V}^1(M))^{\mathrm{c}}.$$

Example (The Catanese–Ciliberto–Mendes Lopes surface)

- Let C_1 be a smooth curve of genus 2 with an elliptic involution σ_1, and C_2 a curve of genus 3 with a free involution σ_2.



- Then $\Sigma_1 = C_1/\sigma_1$ is a curve of genus 1, $\Sigma_2 = C_2/\sigma_2$ is a curve of genus 2, and $M = (C_1 \times C_2)/\sigma_1 \times \sigma_2$ is a smooth, complex projective surface with $H_1(M;\mathbb{Z}) = \mathbb{Z}^6$.
- Projection onto the first coordinate yields an orbifold fibration $f_1 \colon M \to \Sigma_1$ with two multiple fibers, each of multiplicity 2. The other projection defines a smooth fibration $f_2 \colon M \to \Sigma_2$.
- We have $\mathcal{V}^1(M) = \{t \mid t_1 = t_2 = 1\} \cup \{t_4 = t_5 = t_6 = 1,\ t_3 = -1\}$, with the two components obtained by pullback along f_1 and f_2.
- Thus, $\Sigma^1(M) = S^5 \backslash S\big(\{x_3 = \cdots = x_6 = 0\} \cup \{x_1 = x_2 = 0\}\big)$.

Hyperplane arrangements

- Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an (essential, central) arrangement of hyperplanes in $\mathbb{C}^d$.
- Its complement, $M(\mathcal{A}) \subset (\mathbb{C}^*)^d$, is a smooth, quasi-projective Stein manifold; thus, it has the homotopy type of a finite, d-dimensional CW-complex.
- (Orlik–Solomon) The cohomology ring $H^*(M(\mathcal{A}); \mathbb{Z})$ is determined by the intersection lattice $L(\mathcal{A})$.
- (Arapura) The characteristic varieties $\mathcal{V}^i(\mathcal{A}) := \mathcal{V}^i(M(\mathcal{A})) \subset (\mathbb{C}^*)^n$. are unions of translated subtori.
- Consequently, $\operatorname{Trop}(\mathcal{V}^i(\mathcal{A})) = -\operatorname{Trop}(\mathcal{V}^i(\mathcal{A}))$.
- (Denham–S.–Yuzvinsky 2016/17) $M(\mathcal{A})$ is an "abelian duality space"; thus, its jump loci propagate: $\mathcal{V}^1(\mathcal{A}) \subseteq \cdots \subseteq \mathcal{V}^{d-1}(\mathcal{A})$.
- (Arnol'd, Brieskorn) $M(\mathcal{A})$ is formal. Thus, $\tau_1(\mathcal{V}^i(\mathcal{A})) = \mathcal{R}^i(\mathcal{A})$.

Theorem

Let M be the complement of an arrangement of n hyperplanes in $\mathbb{C}^d$. Then, for each $1 \leqslant q \leqslant d-1$:

- $\operatorname{Trop}(\mathcal{V}^q(M))$ *is the union of a subspace arrangement in* $\mathbb{R}^n$.
- $\Sigma^q(M,\mathbb{Z}) \subseteq S(\operatorname{Trop}(\mathcal{V}^q(M)))^{\mathrm{c}}$.

Question (MFO Miniworkshop 2007)

Given an arrangement $\mathcal{A}$, do we have

$$\Sigma^1(M(\mathcal{A})) = S(\mathcal{R}^1(\mathcal{A},\mathbb{R}))^{\mathrm{c}}? \qquad (\star)$$

Example (Koban–McCammond–Meier 2013)

- Let $\mathcal{A}$ be the braid arrangement in $\mathbb{C}^n$, defined by $\prod_{1\leqslant i<j\leqslant n}(z_i - z_j) = 0$. Then $M(\mathcal{A}) = \mathrm{Conf}(n,\mathbb{C}) \simeq K(P_n, 1)$.
- Answer to ($\star$) is yes: $\Sigma^1(M(\mathcal{A}))$ is the complement of the union of $\binom{n}{3} + \binom{n}{4}$ planes in $\mathbb{C}^{\binom{n}{2}}$, intersected with the unit sphere.

Example

- Let $\mathcal{A}$ be the "deleted B_3" arrangement, defined by $z_1 z_2 (z_1^2 - z_2^2)(z_1^2 - z_2^2)(z_2^2 - z_3^2) = 0$.
- (S. 2002) $\mathcal{V}^1(\mathcal{A})$ contains a (1-dimensional) translated torus $\rho \cdot T$.
- Thus, $\mathrm{Trop}(\rho \cdot T) = \mathrm{Trop}(T)$ is a line in $\mathbb{C}^8$ which is *not* contained in $\mathcal{R}^1(\mathcal{A}, \mathbb{R})$. Hence, the answer to ($\star$) is no.

Question (Revised)

$$\Sigma^1(M(\mathcal{A})) = S(\mathrm{Trop}(\mathcal{V}^1(\mathcal{A}))^{\mathrm{c}}? \qquad (\star\star)$$

Exponential tangent cone and tropicalization

Lemma

Let $W \subset (\mathbb{C}^*)^n$ be an algebraic variety. Then $\tau_1^{\mathbb{R}}(W) \subseteq \operatorname{Trop}(W)$.

Sketch of proof.

- Every irreducible component of $\tau_1^{\mathbb{R}}(W)$ is of the form $L \otimes_{\mathbb{Q}} \mathbb{R}$, for some linear subspace $L \subset \mathbb{Q}^n$.
- The complex torus $T := \exp(L \otimes_{\mathbb{Q}} \mathbb{C})$ lies inside W.
- Thus, $\operatorname{Trop}(T) = L \otimes_{\mathbb{Q}} \mathbb{R}$ lies inside $\operatorname{Trop}(W)$. □

A tropical bound for the BNSR invariants

Theorem (PS-2010, S-2021)

Let $\rho\colon \pi_1(X) \to \Bbbk^*$ be a character such that $\rho \in \mathcal{V}^{\leqslant q}(X, \Bbbk)$. Let $v\colon \Bbbk^* \to \mathbb{R}$ be the homomorphism defined by a valuation on $\Bbbk$. If the homomorphism $\chi := v \circ \rho\colon \pi_1(X) \to \mathbb{R}$ is non-zero, then $\chi \notin \Sigma^q(X, \mathbb{Z})$.

Sketch of proof.

- Let $\hat{\Bbbk}$ be the topological completion of $\Bbbk$ with respect to the absolute value $|c| = \exp(-v(c))$. Get a field extension, $\iota\colon \Bbbk \hookrightarrow \hat{\Bbbk}$.
- Let $G = \pi_1(X)$. Extend $\rho\colon G \to \Bbbk^*$ to a ring map, $\bar{\rho}\colon \mathbb{Z}G \to \Bbbk$.
- Since $\chi = v \circ \rho$, we can extend $\bar{\rho}$ to a morphism of topological rings, $\hat{\rho}\colon \widehat{\mathbb{Z}G}_{-\chi} \to \hat{\Bbbk}$, making $\hat{\Bbbk}$ into a $\widehat{\mathbb{Z}G}_{-\chi}$-module, denoted $\hat{\Bbbk}_{\hat{\rho}}$.
- Restricting scalars via the inclusion $\mathbb{Z}G \hookrightarrow \widehat{\mathbb{Z}G}_{-\chi}$ yields the $\mathbb{Z}G$-module $\hat{\Bbbk}_{\iota\circ\rho}$, defined by the character $\iota \circ \rho\colon G \to \hat{\Bbbk}^*$.

- For a ring R, a bounded below chain complex of flat right R-modules K_*, and a left R-module M, there is a (right half-plane, boundedly converging) Künneth spectral sequence,

$$E^2_{ij} = \operatorname{Tor}^R_i(H_j(K), M) \Rightarrow H_{i+j}(K \otimes_R M)\,.$$

- Use ring $R = \widehat{\mathbb{Z}G}_{-\chi}$, chain complex of free R-modules $K_* = C_*(\widetilde{X}, \mathbb{Z}) \otimes_{\mathbb{Z}G} \widehat{\mathbb{Z}G}_{-\chi}$, and R-module $M = \hat{\Bbbk}_{\hat{\rho}}$.
- Now let $\rho \in \mathcal{V}^{\leqslant q}(X, \Bbbk)$, and suppose $\chi = \upsilon \circ \rho \in \Sigma^q(X, \mathbb{Z})$.
- This is equivalent to $H_j(X; \widehat{\mathbb{Z}G}_{-\chi}) = 0$ for all $j \leqslant q$; that is, $H_j(K) = 0$ for $j \leqslant q$. Therefore, $E^2_{ij} = 0$ for $j \leqslant q$.
- Hence, $H_{i+j}(X; \hat{\Bbbk}_{\iota\circ\rho}) = 0$ for $j \leqslant q$, and so $H_j(X; \hat{\Bbbk}_{\iota\circ\rho}) = 0$ for $j \leqslant q$.
- This is equivalent to $\iota \circ \rho \notin \mathcal{V}^{\leqslant q}(X, \hat{\Bbbk})$. Hence, $\rho \notin \mathcal{V}^{\leqslant q}(X, \Bbbk)$, contradicting our hypothesis on ρ.
- Therefore, $\chi \notin \Sigma^q(X, \mathbb{Z})$. □

Theorem (S-2021)

$$\Sigma^q(X,\mathbb{Z}) \subseteq S(\mathrm{Trop}(\mathcal{V}^{\leqslant q}(X)))^{\mathrm{c}}$$

Sketch of proof.

- Let $\rho\colon \pi_1(X) \to \mathbb{K}^*$ and set $\chi = v \circ \rho\colon \pi_1(X) \to \mathbb{Q}$, a rational point on $H^1(X;\mathbb{R})$.
- Suppose $\rho \in \mathcal{V}^{\leqslant q}(X,\mathbb{K}) = \mathcal{V}^{\leqslant q}(X) \times_{\mathbb{C}} \mathbb{K}$.
- Then χ is a rational point on $\mathrm{Trop}(\mathcal{V}^{\leqslant q}(X)) = \overline{\nu_X(\mathcal{V}^{\leqslant q}(X;\mathbb{K}))}$.
- Conversely, all rational points on $\mathrm{Trop}(\mathcal{V}^{\leqslant q}(X))$ are of the form $\nu_X(\rho) = v \circ \rho$, for some $\rho \in \mathcal{V}^{\leqslant q}(X,\mathbb{K})$.
- Finally, assume that $\chi \neq 0$, so that χ represents an (arbitrary) rational point in $S(\mathrm{Trop}(\mathcal{V}^{\leqslant q}(X))$.
- By the previous theorem, $\chi \in \Sigma^q(X,\mathbb{Z})^{\mathrm{c}}$.
- But the rational points are dense in $S(\mathrm{Trop}(\mathcal{V}^{\leqslant q}(X)))$, and $\Sigma^q(X,\mathbb{Z})^{\mathrm{c}}$ is closed in $S(X)$, and so we're done. □

The Alexander polynomial

- Let $H = G_{ab}/\operatorname{tors}(G_{ab})$ be the maximal torsion-free abelian quotient of $G = \pi_1(X)$ and $q\colon X^H \to X$ the respective cover.
- Set $A_X := H_1(X^H; q^{-1}(x_0), \mathbb{Z})$, viewed as a $\mathbb{Z}[H]$-module.
- Let $E_1(A_X) \subseteq \mathbb{Z}[H]$ be the ideal of codimension 1 minors in a presentation for A_X.
- $\Delta_X := \gcd(E_1(A_X)) \in \mathbb{Z}[H]$ is the *Alexander polynomial* of X. It only depends on G, so also write it as Δ_G.
- Suppose $I_H^p \cdot (\Delta_G) \subseteq E_1(A_G)$, for some $p \geqslant 0$. Then

$$\mathcal{V}^1(X) \cap \mathbb{T}_G^0 = \{1\} \cup V(\Delta_G).$$

- This condition is satisfied if G is a 1-relator group, or $G = \pi_1(M)$, where M is a closed, orientable 3-manifold with empty or toroidal boundary (C. McMullen, D. Eisenbud–W. Neumann).

- Let $\mathrm{Newt}(\Delta_G) \subset H_1(G;\mathbb{R})$ be the Newton polytope of Δ_G.
- Given $\phi \in H^1(G;\mathbb{Z}) \cong \mathrm{Hom}(H,\mathbb{Z})$, its *Alexander norm*, $\|\phi\|_A$, is the length of $\phi(\mathrm{Newt}(\Delta_G))$.
- This defines a semi-norm on $H^1(G;\mathbb{R})$, with unit ball

$$B_A = \{\phi \in H^1(G;\mathbb{R}) \mid \|\phi\|_A \leqslant 1\}.$$

- If Δ_G is symmetric (i.e., invariant under $t_i \mapsto t_i^{-1}$), then B_A is, up to a scale factor of $1/2$, the polar dual of the Newton polytope of Δ_G,

$$2B_A = \mathrm{Newt}(\Delta_G)^{\circ}.$$

Proposition

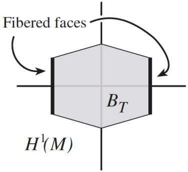
If Δ_G is symmetric and $I_H^p \cdot (\Delta_G) \subseteq E_1(A_G)$, for some $p \geqslant 0$, then

$$\Sigma^1(G) \subseteq \bigcup_{F \text{ an open facet of } B_A} S(F).$$

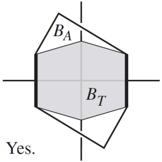
The Thurston and Alexander norms

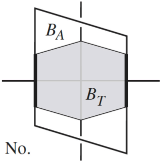
- Let M be a closed, orientable 3-manifold with $b_1(M) > 0$.
- A non-zero class $\phi \in H^1(M;\mathbb{Z}) = \mathrm{Hom}(\pi_1(M),\mathbb{Z})$ is *fibered* if there exists a fibration $p\colon M \to S^1$ such that the induced map $p_*\colon \pi_1(M) \to \pi_1(S^1) = \mathbb{Z}$ coincides with ϕ.
- The *Thurston norm* $\|\phi\|_T$ of a class $\phi \in H^1(M;\mathbb{Z})$ is the infimum of $-\chi(\hat{S})$, where S runs though all the properly embedded, oriented surfaces in M dual to ϕ, and $\hat{S}$ denotes the result of discarding all components of S which are disks or spheres.
- Thurston showed that $\| - \|_T$ defines a seminorm on $H^1(M;\mathbb{Z})$, which can be extended to a continuous seminorm on $H^1(M;\mathbb{R})$.
- The unit norm ball, $B_T = \{\phi \in H^1(M;\mathbb{R}) \mid \|\phi\|_T \leqslant 1\}$, is a rational polyhedron with finitely many sides and symmetric in the origin.

There are facets of B_T, called the *fibered faces* (coming in antipodal pairs), so that a class $\phi \in H^1(M;\mathbb{Z})$ fibers iff it lies in the cone over the interior of a fibered face.



- The BNS invariant of $G = \pi_1(M)$ is the projection onto $S(G)$ of the open fibered faces of B_T; in particular, $\Sigma^1(G) = -\Sigma^1(G)$.
- Under some mild assumptions, McMullen showed that $\|\phi\|_A \leqslant \|\phi\|_T$, leading to an upper bound for $\Sigma_1(G)$ in terms of the Alexander norm ball B_A, explained by Dunfield, as follows:





Theorem

$\Sigma^1(M,\mathbb{Z})$ is contained in the union of the open cones on the facets of B_A.

Reference

A.I. Suciu, *Sigma-invariants and tropical varieties*, Math.Annalen **380** (2021), 1427–1463.