Arrangements and lower central series

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N-series

- An *N-series* for a group G is a descending filtration $G = K_1 \geqslant \cdots \geqslant K_n \geqslant \cdots$ such that $[K_m, K_n] \subseteq K_{m+n}$, $\forall m, n \geqslant 1$.
- In particular, $\kappa = \{K_n\}_{n\geqslant 1}$ is a *central series*, i.e., $[G, K_n] \subseteq K_{n+1}$.
- Thus, it is also a *normal series*, i.e., $K_n \lhd G$.
- Consequently, each quotient K_n/K_{n+1} lies in the center of G/K_{n+1}, and thus is an abelian group.
- If all those quotients are torsion-free, κ is called an *N_0-series*.
- *Associated graded Lie algebra*:

$$\operatorname{gr}^{\kappa}(G) = \bigoplus_{n\geqslant 1} K_n/K_{n+1},$$

with addition induced by $\cdot \colon G \times G \to G$, and Lie bracket $[\ ,\ ]\colon \operatorname{gr}_m \times \operatorname{gr}_n \to \operatorname{gr}_{m+n}$ induced by $[x, y] := xyx^{-1}y^{-1}$.

Lower central series

- The *lower central series*, $\gamma(G) = \{\gamma_n(G)\}_{n\geqslant 1}$ is defined inductively by $\gamma_1(G) = G$, $\gamma_2(G) = G'$, and $\gamma_{n+1}(G) = [G, \gamma_n(G)]$.
- It is an N-series, and the fastest descending central series for G.
- If $\varphi\colon G \to H$ is a homomorphism, then $\varphi(\gamma_n(G)) \subseteq \gamma_n(H)$.
- $\mathrm{gr}(G) := \mathrm{gr}^{\gamma}(G)$ is generated by $\mathrm{gr}_1(G) = G_{\mathrm{ab}}$.
- If $b_1(G) < \infty$, the *LCS ranks* of G are $\phi_n(G) := \dim_{\mathbb{Q}} \mathrm{gr}_n(G) \otimes \mathbb{Q}$.
- For each N-series κ, there is a morphism $\mathrm{gr}(G) \to \mathrm{gr}^{\kappa}(G)$.
- $\Gamma_n := G/\gamma_n(G)$ is the maximal $(n-1)$-step nilpotent quotient of G.
- $G/\gamma_2(F) = G_{\mathrm{ab}}$, while $G/\gamma_3(G) \leftrightarrow H^{\leqslant 2}(G, \mathbb{Z})$.
- G is residually nilpotent $\iff \gamma_\omega(G) := \bigcap_{n\geqslant 1} \gamma_n(G)$ is trivial.

Split exact sequences

- A short exact sequence of groups,

$$1 \longrightarrow K \xrightarrow{\iota} G \xrightarrow{\pi} Q \longrightarrow 1 \qquad (*)$$

  yields representations $\varphi\colon Q \to \mathrm{Out}(K)$ and $\bar{\varphi}\colon Q \to \mathrm{Aut}(K_{\mathrm{ab}})$.

- If (*) admits a splitting, $\sigma\colon Q \to G$, then $G = K \rtimes_{\varphi} Q$, where $\varphi\colon Q \to \mathrm{Aut}(K)$, $x \mapsto$ conjugation by $\sigma(x)$.

- (*) is *ab-exact* if $0 \longrightarrow K_{\mathrm{ab}} \xrightarrow{\iota_{\mathrm{ab}}} G_{\mathrm{ab}} \xrightarrow{\pi_{\mathrm{ab}}} Q_{\mathrm{ab}} \longrightarrow 0$ is also exact; equivalently, Q acts trivially on K_{ab} and ι_{ab} is injective.

THEOREM (FALK-RANDELL (1985/88))

Let $G = K \rtimes_{\varphi} Q$. If Q acts trivially on K_{ab}, then

- $\gamma_n(G) = \gamma_n(K) \rtimes_{\varphi} \gamma_n(Q)$, *for all* $n \geqslant 1$.
- $\mathrm{gr}(G) = \mathrm{gr}(K) \rtimes_{\tilde{\varphi}} \mathrm{gr}(Q)$, *where* $\tilde{\varphi}\colon \mathrm{gr}(Q) \to \mathrm{Der}(\mathrm{gr}(K))$.
- *If K and Q are residually nilpotent, then G is residually nilpotent.*

- For a split extension $G = K \rtimes_\varphi Q$, Guaschi and de Miranda e Pereiro define a sequence $L = \{L_n\}_{n\geqslant 1}$ of subgroups of K by
$$L_1 = K, \quad L_{n+1} = \langle [K, L_n], [K, \gamma_n(Q)], [L_n, Q] \rangle.$$

THEOREM (GUASCHI–PEREIRO 2020)

- $\varphi\colon Q \to \mathrm{Aut}(K)$ *restricts to* $\varphi\colon \gamma_n(Q) \to \mathrm{Aut}(L_n)$.
- $\gamma_n(G) = L_n \rtimes_\varphi \gamma_n(Q)$.

LEMMA

L is an N-series for K.

THEOREM

$\mathrm{gr}(G) = \mathrm{gr}^L(K) \rtimes_{\tilde\varphi} \mathrm{gr}(Q)$, *where* $\tilde\varphi\colon\ \mathrm{gr}(Q) \to \mathrm{Der}(\mathrm{gr}(K))$.

REMARK

If Q acts trivially on K_{ab}, then $L = \gamma(K)$. So these results generalize those of Falk and Randell.

Isolators

- The *isolator* in G of a subset $S \subseteq G$ is the subset

$$\sqrt{S} := \sqrt[G]{S} = \{g \in G \mid g^m \in S \text{ for some } m \in \mathbb{N}\}$$

- Clearly, $S \subseteq \sqrt{S}$ and $\sqrt{\sqrt{S}} = \sqrt{S}$. Also, if $\varphi\colon G \to H$ is a homomorphism, and $\varphi(S) \subseteq T$, then $\varphi(\sqrt[G]{S}) \subseteq \sqrt[H]{T}$.
- The isolator of a subgroup of G need not be a subgroup; for instance, $\sqrt[G]{\{1\}} = \operatorname{Tors}(G)$, which is not a subgroup in general (although it is if G is nilpotent).
- If $N \lhd G$ is a normal subgroup, then $\sqrt[G]{N} = \pi^{-1}(\operatorname{Tors}(G/N))$, where $\pi\colon G \twoheadrightarrow G/N$, and so $\sqrt[G]{N}/N \cong \operatorname{Tors}(G/N)$.

Proposition (Massuyeau 2007)

Suppose $\kappa = \{K_n\}_{n\geqslant 1}$ is an N-series for G. Then $\sqrt{\kappa} := \{\sqrt{K_n}\}_{n\geqslant 1}$ is an N_0-series for G.

The rational lower central series

- The *rational lower central series*, $\gamma^{\mathbb{Q}}(G)$, is defined by $\gamma_1^{\mathbb{Q}}(G) = G$ and $\gamma_{n+1}^{\mathbb{Q}}(G) = \sqrt{[G, \gamma_n^{\mathbb{Q}}(G)]}$. (Stallings, 1965)
- $\gamma_n^{\mathbb{Q}}(G) = \sqrt{\gamma_n(G)}$ for all $n \geqslant 1$.
- Hence, $\gamma^{\mathbb{Q}}(G)$ is an N_0-series (since $\gamma(G)$ is an N-series).
- $G/\gamma_n^{\mathbb{Q}}(G) = \Gamma_n/\operatorname{Tors}(\Gamma_n)$ is the maximal torsion-free $(n-1)$-step nilpotent quotient of G; in particular, $G/\gamma_2^{\mathbb{Q}}(G) = G_{\text{abf}}$.
- Associated graded Lie algebra: $\operatorname{gr}^{\mathbb{Q}}(G) = \bigoplus_{n\geqslant 1} \gamma_n^{\mathbb{Q}}(G)/\gamma_{n+1}^{\mathbb{Q}}(G)$.
- G is residually torsion-free nilpotent (RTFN) iff $\gamma_\omega^{\mathbb{Q}}(G) = \{1\}$.

PROPOSITION (BASS & LUBOTZKY, 1994)

- $\operatorname{gr}(G) \to \operatorname{gr}^{\mathbb{Q}}(G)$ *has torsion kernel and cokernel in each degree.*
- $\operatorname{gr}(G) \otimes \mathbb{Q} \to \operatorname{gr}^{\mathbb{Q}}(G) \otimes \mathbb{Q}$ *is an isomorphism.*

- Thus, if $b_1(G) < \infty$, then $\phi_n^{\mathbb{Q}}(G) = \phi_n(G)$

Split extensions

- Let $G = K \rtimes_{\varphi} Q$. Since L is an N-series, $\sqrt{L}$ is an N_0-series for K.

Theorem

- $\varphi\colon Q \to \operatorname{Aut}(K)$ *restricts to* $\varphi\colon \sqrt[Q]{\gamma_n(Q)} \to \operatorname{Aut}\big(\sqrt[K]{L_n}\big)$.
- $\sqrt[G]{\gamma_n(G)} = \sqrt[K]{L_n} \rtimes_{\varphi} \sqrt[Q]{\gamma_n(Q)}$.
- $\operatorname{gr}^{\mathbb{Q}}(G) \cong \operatorname{gr}^{\sqrt{L}}(K) \rtimes_{\tilde{\varphi}} \operatorname{gr}^{\mathbb{Q}}(Q)$.

Theorem

Suppose Q acts trivially on $K_{\rm abf} := H_1(K,\mathbb{Z})/\operatorname{Tors}$. Then

- $\sqrt[K]{L_n} = \sqrt[K]{\gamma_n(K)}$ *for all* n.
- $\sqrt[G]{\gamma_n(G)} = \sqrt[K]{\gamma_n(K)} \rtimes_{\varphi} \sqrt[Q]{\gamma_n(Q)}$.
- $\operatorname{gr}^{\mathbb{Q}}(Q) \cong \operatorname{gr}^{\mathbb{Q}}(K) \rtimes_{\tilde{\varphi}} \operatorname{gr}^{\mathbb{Q}}(Q)$.

Corollary

Let $G = K \rtimes Q$ be a split extension of RTFN groups. If Q acts trivially on $K_{\rm abf}$, then G is also RTFN.

Alexander invariants and Chen ranks

- The *Chen Lie algebra* of G is $\operatorname{gr}(G/G'')$, where $G'' = (G')'$.
- If $b_1(G) < \infty$, the *Chen ranks* of G are defined as $\theta_n(G) := \dim_{\mathbb{Q}} \operatorname{gr}_n(G/G'') \otimes \mathbb{Q}$.
- $\theta_n(G) \leqslant \phi_n(G)$, with equality for $n \leqslant 3$.
- *Alexander invariant*: $B(G) := G'/G''$, viewed as a $\mathbb{Z}[G_{\mathrm{ab}}]$-module via $gG' \cdot xG'' = gxg^{-1}G''$ for $g \in G$ and $x \in G'$.
- (Massey) $I^n B(G) = \gamma_{n+2}(G/G'')$, where I is the augmentation ideal of $\mathbb{Z}[G_{\mathrm{ab}}]$, and hence $\operatorname{gr}_n(B(G)) \cong \operatorname{gr}_{n+2}(G/G'')$, for all $n \geqslant 0$.
- If $b_1(G) < \infty$, then $\operatorname{Hilb}(\operatorname{gr}(B(G) \otimes \mathbb{Q}), t) = \sum_{n \geqslant 0} \theta_{n+2}(G) t^n$.

Theorem

Suppose $1 \to K \xrightarrow{\iota} G \to Q \to 1$ *is an* ab*-exact sequence of groups, and* Q *is abelian. Then,*

- *The induced map on Alexander invariants,* $B(\iota)\colon B(K) \to B(G)$*, factors through a* $\mathbb{Z}[K_{\mathrm{ab}}]$*-linear isomorphism,* $B(K) \to B(G)_{\iota}$*.*
- *If* G_{ab} *is finitely generated, then* $\theta_n(K) \leqslant \theta_n(G)$ *for all* $n \geqslant 1$*.*
- *If the sequence is split exact, then* ι *induces isos of graded Lie algebras,* $\mathrm{gr}_{\geqslant 2}(K) \xrightarrow{\simeq} \mathrm{gr}_{\geqslant 2}(G)$ *and* $\mathrm{gr}_{\geqslant 2}(K/K'') \xrightarrow{\simeq} \mathrm{gr}_{\geqslant 2}(G/G'')$*.*
  - *Consequently, if* $b_1(G) < \infty$*, then* $\phi_n(K) = \phi_n(G)$ *and* $\theta_n(K) = \theta_n(G)$ *for all* $n \geqslant 2$*.*

The rational Alexander invariant

- Let $B_{\mathbb{Q}}(G) := G'_{\mathbb{Q}}/G''_{\mathbb{Q}}$, viewed as a module over $\mathbb{Z}G_{\mathrm{abf}}$, where $G''_{\mathbb{Q}} = (G'_{\mathbb{Q}})'_{\mathbb{Q}} = \sqrt{[G'_{\mathbb{Q}}, G'_{\mathbb{Q}}]}$.
- $I^n(B_{\mathbb{Q}}(G) \otimes \mathbb{Q}) = \gamma^{\mathbb{Q}}_{n+2}(G/G''_{\mathbb{Q}}) \otimes \mathbb{Q}$, where $I = I_{\mathbb{Q}}(G_{\mathrm{abf}})$.
- Hence, $\mathrm{gr}_n(B_{\mathbb{Q}}(G) \otimes \mathbb{Q}) \cong \mathrm{gr}_{n+2}(G/G''_{\mathbb{Q}}) \otimes \mathbb{Q}$, for all $n \geqslant 0$.

Theorem

Let $1 \to K \xrightarrow{\iota} G \to Q \to 1$ be an abf-exact sequence and suppose Q is torsion-free abelian. Then,

- *The map ι induces a $\mathbb{Z}[K_{\mathrm{abf}}]$-linear isomorphism, $B_{\mathbb{Q}}(K) \to B_{\mathbb{Q}}(G)_{\iota}$.*
- *If G_{abf} is finitely generated, then $\theta_n(K) \leqslant \theta_n(G)$ for all $n \geqslant 1$.*
- *If the sequence is split exact, then ι induces isos of graded Lie algebras, $\mathrm{gr}^{\mathbb{Q}}_{\geqslant 2}(K) \xrightarrow{\simeq} \mathrm{gr}^{\mathbb{Q}}_{\geqslant 2}(G)$ and $\mathrm{gr}^{\mathbb{Q}}_{\geqslant 2}(K/K'') \xrightarrow{\simeq} \mathrm{gr}^{\mathbb{Q}}_{\geqslant 2}(G/G'')$.*
  - *Consequently, if $b_1(G) < \infty$, then $\phi_n(K) = \phi_n(G)$ and $\theta_n(K) = \theta_n(G)$ for all $n \geqslant 2$.*

Characteristic varieties

- Let G be a finitely generated group. Then $\mathbb{T}_G := \operatorname{Hom}(G, \mathbb{C}^*)$ is an algebraic group, with identity 1 the trivial character, $g \mapsto 1$.
- Clearly, $\mathbb{T}_G = \mathbb{T}_{G_{\mathrm{ab}}}$ and $\mathbb{T}^0_G = \mathbb{T}_{G_{\mathrm{abf}}}$.
- Characteristic varieties: $\mathcal{V}_k(G) := \{\rho \in \mathbb{T}_G \mid \dim H^1(G, \mathbb{C}_\rho) \geqslant k\}$.
- Set $\mathcal{W}_k(G) := \mathcal{V}_k(G) \cap \mathbb{T}^0_G$.
- For each $k \geqslant 1$, we have

$$\mathcal{V}_k(G) = V\big(\operatorname{ann}(\textstyle\bigwedge^k B(G) \otimes \mathbb{C})\big)$$

$$\mathcal{W}_k(G) = V\big(\operatorname{ann}(\textstyle\bigwedge^k B_{\mathbb{Q}}(G) \otimes \mathbb{C})\big),$$

at least away from $1 \in \mathbb{T}^0_G$.

Theorem

Let $1 \to K \xrightarrow{\iota} G \to Q \to 1$ *be an exact sequence of f.g. groups.*

- *If the sequence is* ab*-exact and* Q *is abelian, then* $\iota^* \colon \mathbb{T}_G \to \mathbb{T}_K$ *restricts to maps* $\iota^* \colon \mathcal{V}_k(G) \to \mathcal{V}_k(K)$ *for all* $k \geqslant 1$*; furthermore,* $\iota^* \colon \mathcal{V}_1(G) \to \mathcal{V}_1(K)$ *is a surjection.*
- *If the sequence is* abf*-exact and* Q *is torsion-free abelian, then* $\iota^* \colon \mathbb{T}^0_G \twoheadrightarrow \mathbb{T}^0_K$ *restricts to maps* $\iota^* \colon \mathcal{W}_k(G) \to \mathcal{W}_k(K)$ *for all* $k \geqslant 1$*; furthermore,* $\iota^* \colon \mathcal{W}_1(G) \to \mathcal{W}_1(K)$ *is a surjection.*

Holonomy Lie algebra

- Assume G_{abf} is finitely generated, and let $\mathbb{L} = \text{Lie}(G_{\text{abf}})$ be the free Lie algebra on G_{abf}, so that $\mathbb{L}_1 = G_{\text{abf}}$ and $\mathbb{L}_2 = G_{\text{abf}} \wedge G_{\text{abf}}$.
- The *holonomy Lie algebra* of G is $\mathfrak{h}(G) := \text{Lie}(G_{\text{abf}})/(\text{im}(\cup_G^\vee))$, where $\cup_G^\vee\colon H^2(G)^\vee \to (H^1(G) \wedge H^1(G))^\vee \cong G_{\text{abf}} \wedge G_{\text{abf}}$.
- There is a natural epimorphism $\mathfrak{h}(G) \twoheadrightarrow \text{gr}(G)$, which induces epimorphisms $\mathfrak{h}(G)/\mathfrak{h}(G)'' \twoheadrightarrow \text{gr}(G/G'')$.
- Let $\bar{\theta}_n(G) := \text{rank}\,(\mathfrak{h}(G)/\mathfrak{h}(G)'')_n$. Then: $\bar{\theta}_n(G) \geqslant \theta_n(G)$, $\forall n \geqslant 1$.
- If $b_1(G) < \infty$, we may also define $\mathfrak{h}(G;\mathbb{Q})$. If G_{abf} is finitely generated, $\mathfrak{h}(G;\mathbb{Q}) = \mathfrak{h}(G) \otimes \mathbb{Q}$.
- The *infinitesimal Alexander invariant* is $\mathfrak{B}(G) := \mathfrak{h}(G)'/\mathfrak{h}(G)''$, viewed as a graded module over $\text{Sym}(G_{\text{abf}})$ via $g \cdot \bar{x} = \overline{[g,x]}$ for $g \in \mathfrak{h}/\mathfrak{h}' = G_{\text{abf}}$ and $x \in \mathfrak{h}'$.
- If $b_1(G) < \infty$, then $\bar{\theta}_n(G) = \dim_{\mathbb{Q}} \mathfrak{B}_{n-2}(G;\mathbb{Q})$, for all $n \geqslant 2$.

Resonance varieties

- Let G be a group with $b_1(G) < \infty$. Let $H^* = H^*(G;\mathbb{C})$.
- For each $a \in H^1$, left-multiplication by a yields a cochain complex,

$$(H,\delta_a)\colon\ H^0 \xrightarrow{\delta_a^0} H^1 \xrightarrow{\delta_a^1} H^2.$$

- The *resonance varieties* of G:

$$\mathcal{R}_k(G) := \{a \in H^1 \mid \dim_{\mathbb{C}} H^1(H,\delta_a) \geqslant k\}.$$

- They are homogeneous algebraic subvarieties of the affine space $H^1 \cong \mathbb{C}^{b_1(G)}$. Note: $0 \in \mathcal{R}_k(G)$ iff $b_1(G) \geqslant k$.
- $\mathcal{R}_k(G)$ contains every isotropic subspace of H^1 of dimension $\leqslant k+1$; moreover, $\mathcal{R}_1(G)$ is the union of all isotropic planes in H^1.
- $\mathcal{R}_k(G) = V(\operatorname{ann}(\bigwedge^k \mathfrak{B}(G;\mathbb{C}))$, away from 0

Theorem

Let $1 \to K \xrightarrow{\iota} G \to Q \to 1$ *be an exact sequence of f.g. groups. Suppose that either*

- *The sequence is split exact,* G *is graded formal,* Q *is abelian, and* Q *acts trivially on* $H_1(K;\mathbb{Q})$.
- *The sequence if* ab*-exact,* G *and* K *are* 1*-formal, and* Q *is abelian.*
- *The sequence if* abf*-exact,* G *and* K *are* 1*-formal, and* Q *is torsion-free abelian.*

Then $\iota^* \colon H^1(G,\mathbb{C}) \twoheadrightarrow H^1(K,\mathbb{C})$ *restricts to maps* $\iota^* \colon \mathcal{R}_k(G) \twoheadrightarrow \mathcal{R}_k(K)$ *for all* $k \geqslant 1$*; furthermore,* $\iota^* \colon \mathcal{R}_1(G) \twoheadrightarrow \mathcal{R}_1(K)$ *is surjective.*

Corollary

With hypothesis as above, suppose that $\mathcal{R}_1(G) \subseteq \{0\}$*. Then*

- $\mathcal{R}_1(K) \subseteq \{0\}$.
- $\bar{\theta}_n(K) \leqslant \bar{\theta}_n(G)$ *for all* $n \geqslant 1$.
- $\bar{\theta}_n(G) = 0$ *for* $n \gg 0$ *and* $\bar{\theta}_n(K) = 0$ *for* $n \gg 0$.

The complement of a hyperplane arrangement

- Let $\mathcal{A}$ be a central arrangement of m hyperplanes in $\mathbb{C}^d$. For each $H \in \mathcal{A}$ let α_H be a linear form with $\ker(\alpha_H) = H$; set $f = \prod_{H\in\mathcal{A}} \alpha_H$.
- The complement, $M(\mathcal{A}) := \mathbb{C}^d \setminus \bigcup_{H\in\mathcal{A}} H$, is a Stein manifold, and so it has the homotopy type of a (connected) d-dimensional CW-complex.
- In fact, $M = M(\mathcal{A})$ has a minimal cell structure. Consequently, $H_*(M,\mathbb{Z})$ is torsion-free (and finitely generated).
- In particular, $H_1(M,\mathbb{Z}) = \mathbb{Z}^m$, generated by meridians $\{x_H\}_{H\in\mathcal{A}}$.
- The cohomology ring $H^*(M,\mathbb{Z})$ is determined solely by the intersection lattice, $L(\mathcal{A})$.
- The quasi-projective variety M admits a *pure* mixed Hodge structure, and so M is $\mathbb{Q}$-formal (albeit not $\mathbb{Z}_p$-formal, in general).

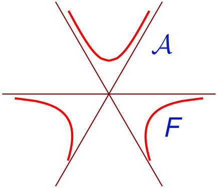
Fundamental groups of arrangements

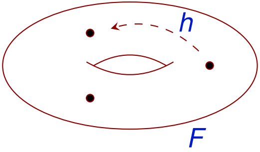
- For an arrangement $\mathcal{A}$, the group $G = \pi_1(M(\mathcal{A}))$ admits a finite presentation, with generators $\{x_H\}_{H\in\mathcal{A}}$ and commutator-relators.
- $\mathcal{V}_k(M)$ is a finite union of torsion-translated subtori of $\mathbb{T}_G = (\mathbb{C}^*)^m$.
- $G/\gamma_2(G)$ and $G/\gamma_3(G)$ are determined by $L_{\leqslant 2}(\mathcal{A})$.
- $G/\gamma_4(G)$—and thus G—is not necessarily determined by $L_{\leqslant 2}(\mathcal{A})$.
- If $\mathcal{A}$ is decomposable, though, *all* nilpotent quotients are combinatorially determined [Porter–S.]
- Since M is formal, G is 1-formal, i.e., its pronilpotent completion, $\mathfrak{m}(G)$, is quadratic.
- Hence, $\mathrm{gr}(G) \otimes \mathbb{Q} = \mathrm{gr}(\mathfrak{m}(G))$ is determined by $L_{\leqslant 2}(\mathcal{A})$.
- Let $\mathfrak{h}(G)$ be the holonomy Lie algebra associated to $H^{\leqslant 2}(G, \mathbb{Z})$.

- Then $\mathfrak{h}(G) \twoheadrightarrow \mathrm{gr}(G)$ (always), and $\mathfrak{h}(G) \otimes \mathbb{Q} \xrightarrow{\simeq} \mathrm{gr}(G) \otimes \mathbb{Q}$ (since G is 1-formal).
- An explicit combinatorial formula is lacking in general for the LCS ranks $\phi_n(G)$, although such formulas are known when
  - $\mathcal{A}$ is supersolvable $\Rightarrow H^*(M, \mathbb{Q})$ is Koszul
  - $\mathcal{A}$ is decomposable ($\mathrm{gr}_3(G)$ is as predicted by $\mu\colon L_2(\mathcal{A}) \to \mathbb{Z}$)
  - $\mathcal{A}$ is a graphic arrangement

  and in some more cases just for $\phi_3(G)$.
- $\mathrm{gr}_n(G)$ may have torsion (at least for $n \geqslant 4$), but the torsion is not necessarily determined by $L_{\leqslant 2}(\mathcal{A})$.
- The map $\mathfrak{h}_3(G) \to \mathrm{gr}_3(G)$ is an isomorphism [Porter–S.], but it is not known whether $\mathfrak{h}_3(G)$ is torsion-free.
- The Chen ranks $\theta_n(G)$ are also combinatorially determined.

The Milnor fibration





- The map $f\colon \mathbb{C}^d \to \mathbb{C}$ restricts to a smooth fibration, $f\colon M \to \mathbb{C}^*$, called the *Milnor fibration* of $\mathcal{A}$.
- The *Milnor fiber* is $F(\mathcal{A}) := f^{-1}(1)$. The monodromy, $h\colon F \to F$, is given by $h(z) = e^{2\pi i/m}z$, where $m = |\mathcal{A}|$.
- F is a Stein manifold. It has the homotopy type of a finite CW-complex of dimension $d-1$ (connected if $d > 1$).
- MHS on F may not be pure; $\pi_1(F)$ may be non-1-formal [Zuber].
- $H_1(F, \mathbb{Z})$ may have torsion [Yoshinaga].

- F is the regular, $\mathbb{Z}_m$-cover of $U = \mathbb{P}(M)$, classified by the epimorphism $\pi_1(U) \twoheadrightarrow \mathbb{Z}_m$, $x_H \mapsto 1$.
- To study $\pi_1(F)$, we may assume w.l.o.g. that $d = 3$.
- Let $\iota\colon F \hookrightarrow M$ be the inclusion. Induced maps on π_1:

$$
\begin{tikzcd}
 & & 1 \arrow[d] & & \\
 & & \mathbb{Z} \arrow[d] & & \\
1 \arrow[r] & \pi_1(F) \arrow[r, "\iota_\sharp"] & \pi_1(M) \arrow[r, "f_\sharp"] \arrow[d, "p_\sharp"] & \mathbb{Z} \arrow[r] & 1 \\
 & & \pi_1(U) \arrow[d] & & \\
 & & 1 & &
\end{tikzcd}
$$

- $b_1(F) \geqslant m - 1$, and may be computed from $\mathcal{V}^1_k(U)$. Combinatorial formulas are known in some cases (e.g., if $\mathbb{P}(\mathcal{A})$ has only double or triple points [Papadima–S.]), but not in general.

Corollary

If $\iota_* \colon H_1(F,\mathbb{Z}) \to H_1(M,\mathbb{Z})$ *is injective, then*

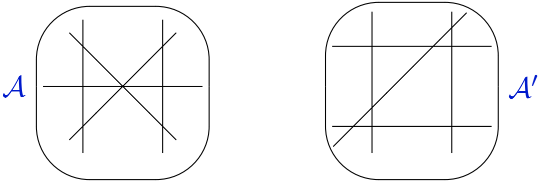
- $\iota^* \colon \mathbb{T}_M \twoheadrightarrow \mathbb{T}_F$ *restricts to maps* $\iota^* \colon \mathcal{V}_k(M) \to \mathcal{V}_k(F)$*; moreover,* $\iota^* \colon \mathcal{V}_1(M) \twoheadrightarrow \mathcal{V}_1(F)$ *is surjective.*
- $\operatorname{gr}_{\geqslant 2}(\pi_1(F)) \cong \operatorname{gr}_{\geqslant 2}(G)$ *and* $\operatorname{gr}_{\geqslant 2}(\pi_1(F)/\pi_1(F)'') \cong \operatorname{gr}_{\geqslant 2}(G/G'')$.

Corollary

If $\iota_* \colon H_1(F,\mathbb{Q}) \to H_1(M,\mathbb{Q})$ *is injective, then*

- $\iota^* \colon \mathbb{T}_M \twoheadrightarrow \mathbb{T}^0_F$ *restricts to maps* $\iota^* \colon \mathcal{V}_k(M) \to \mathcal{W}_k(F)$ *for all* $k \geqslant 1$*; moreover,* $\iota^* \colon \mathcal{V}_1(M) \twoheadrightarrow \mathcal{W}_1(F)$ *is surjective.*
- $\iota^* \colon H^1(M;\mathbb{C}) \twoheadrightarrow H^1(F;\mathbb{C})$ *restricts to maps* $\iota^* \colon \mathcal{R}_k(M) \to \mathcal{R}_k(F)$ *for all* $k \geqslant 1$*; moreover,* $\iota^* \colon \mathcal{R}_1(M) \twoheadrightarrow \mathcal{R}_1(F)$ *is surjective.*
- $\operatorname{gr}^{\mathbb{Q}}_{\geqslant 2}(\pi_1(F)) \cong \operatorname{gr}^{\mathbb{Q}}_{\geqslant 2}(G)$ *and* $\operatorname{gr}^{\mathbb{Q}}_{\geqslant 2}(\pi_1(F)/\pi_1(F)'') \cong \operatorname{gr}^{\mathbb{Q}}_{\geqslant 2}(G/G'')$.
- $\phi_n(\pi_1(F)) = \phi_n(\pi_1(U))$ *and* $\theta_n(\pi_1(F)) = \theta_n(\pi_1(U))$ *for all* $n \geqslant 1$.

Falk's pair of arrangements



- Both $\mathcal{A}$ and $\mathcal{A}'$ have 2 triple points and 9 double points, yet $L(\mathcal{A}) \not\cong L(\mathcal{A}')$. Nevertheless, $M(\mathcal{A}) \simeq M(\mathcal{A}')$.
- $\mathcal{V}_1(M)$ and $\mathcal{V}_1(M')$ consist of two 2-dimensional subtori of $(\mathbb{C}^*)^6$, corresponding to the triple points; $\mathcal{V}_2(M) = \mathcal{V}_2(M') = \{1\}$.
- Both Milnor fibrations have trivial $\mathbb{Z}$-monodromy.
- $\mathcal{V}_1(F)$ and $\mathcal{V}_1(F')$ consist of two 2-dimensional subtori of $(\mathbb{C}^*)^5$.
- On the other hand, $\mathcal{V}_2(F) \cong \mathbb{Z}_3$, yet $\mathcal{V}_2(F') = \{1\}$.
- Thus, $\pi_1(F) \not\cong \pi_1(F')$.

Yoshinaga's icosidodecahedral arrangement

- The icosidodecahedron is the convex hull of 30 vertices given by the even permutations of $(0, 0, \pm 1)$ and $\frac{1}{2}(\pm 1, \pm\phi, \pm\phi^2)$, where $\phi = (1 + \sqrt{5})/2$.
- It gives rise to an arrangement of 16 hyperplanes in $\mathbb{R}^3$, whose complexification is the icosidodecahedral arrangement $\mathcal{A}$ in $\mathbb{C}^3$.
- $M(\mathcal{A})$ is a $K(G, 1)$.
- $H_1(F, \mathbb{Z}) = \mathbb{Z}^{15} \oplus \mathbb{Z}_2$. Thus, the algebraic monodromy of the Milnor fibration is trivial over $\mathbb{Q}$ and $\mathbb{Z}_p$ ($p > 2$), but not over $\mathbb{Z}$.
- Hence, $\operatorname{gr}(\pi_1(F)) \cong \operatorname{gr}(\pi_1(U))$, away from the prime 2. Moreover,
    - $\operatorname{gr}_1(\pi_1(F)) = \mathbb{Z}^{15} \oplus \mathbb{Z}_2$
    - $\operatorname{gr}_2(\pi_1(F)) = \mathbb{Z}^{45} \oplus \mathbb{Z}_2^7$
    - $\operatorname{gr}_3(\pi_1(F)) = \mathbb{Z}^{250} \oplus \mathbb{Z}_2^{43}$
    - $\operatorname{gr}_4(\pi_1(F)) = \mathbb{Z}^{1405} \oplus \mathbb{Z}_2^{?}$

References

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