Fundamental groups and cohomology jumping loci

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The fundamental group

Definition (Poincaré 1904)

Given a topological space X, and a basepoint $x_0 \in X$, let

$$\pi_1(X, x_0) = \{\text{loops at } x_0\}/\simeq$$

This is a group, with multiplication = concatenation of loops, unit = constant loop, and inverse = reversal of loop.

- If X is path-connected, $\pi_1(X)$ does not depend on basepoint.
- If $f\colon X \to Y$ is a continuous map, then $f_\sharp\colon \pi_1(X, x_0) \to (Y, f(x_0))$, $f_\sharp([\alpha]) = [f \circ \alpha]$ is a group homomorphism.
- $f \simeq g \implies f_\sharp = g_\sharp$.
- $X \simeq Y \implies \pi_1(X) \cong \pi_1(Y)$.

Example

$\pi_1(S^1) = \mathbb{Z}$, and $\pi_1(S^n) = 0$, for $n > 1$.

Example (Poincaré)

Let $I < \mathrm{SO}(3)$ be the group of isometries of the icosahedron ($I \cong A_5$). Let $p\colon S^3 \to \mathrm{SO}(3)$ be the double cover. Then

$$I^* := p^{-1}(I) \cong \mathrm{SL}(2, \mathbb{F}_5)$$

is the binary icosahedral group (of order 120), and

$$\Sigma^3 := S^3/I^* = \Sigma(2, 3, 5)$$

is the Poincaré sphere (a smooth, compact, connected, orientable, 3-dimensional manifold). We have:

$$H_*(\Sigma^3, \mathbb{Z}) \cong H_*(S^3, \mathbb{Z}),$$

but

$$\pi_1(\Sigma^3) \ncong \pi_1(S^3).$$

Thus, $\Sigma^3 \not\approx S^3$.

Realizing finitely presented groups

- If M is a smooth, compact, connected [for short, closed] manifold, then $\pi_1(M)$ admits a finite presentation:

$$\pi_1(M) = \langle x_1, \dots x_p \mid r_1, \dots, r_q \rangle.$$

- Conversely, every finitely presented group G can be realized as $G = \pi_1(M)$ for a closed manifold M^n of dimension $n \geq 4$.
- M^n can be chosen to be orientable.
- M^n (n even) can be chosen to be symplectic (Gompf 1995).
- M^n (n even, $n \geq 6$) can be chosen to be complex (Taubes 1992).
- Requiring $n = 3$ puts severe restrictions on G, e.g.:

$$G \text{ abelian 3-manifold group} \iff G \in \{\mathbb{Z}/m\mathbb{Z}\ (m \geq 1),\ \mathbb{Z},\ \mathbb{Z} \oplus \mathbb{Z}_2,\ \mathbb{Z}^3\}.$$

Kähler manifolds and Kähler groups

Definition

A compact, connected, complex manifold M is called a *Kähler manifold* if M admits a Hermitian metric h for which the imaginary part $\omega = \Im(h)$ is a closed 2-form.

Examples: Riemann surfaces, $\mathbb{CP}^n$, and, more generally, smooth, complex projective varieties.

Definition

A group G is a *Kähler group* if $G = \pi_1(M)$, for some compact Kähler manifold M.

G is *projective* if M is actually a smooth projective variety.

- G finite $\Rightarrow$ G is a projective group (Serre 1958).
- G_1, G_2 Kähler groups $\Rightarrow$ $G_1 \times G_2$ is a Kähler group
- G Kähler group, $H < G$ finite-index subgroup $\Rightarrow$ H is a Kähler gp

Problem (Serre 1958)

Which finitely presented groups are Kähler (or projective) groups?

The Kähler condition puts strong restrictions on M:

1. $H^*(M, \mathbb{Z})$ admits a Hodge structure
2. Hence, the odd Betti numbers of M are even
3. M is formal, i.e., $(\Omega(M), d) \simeq (H^*(M, \mathbb{R}), 0)$ (Deligne–Griffiths–Morgan–Sullivan 1975)

The Kähler condition also puts strong restrictions on $G = \pi_1(M)$:

1. $b_1(G)$ is even
2. G is 1-formal, i.e., its Malcev Lie algebra $\mathfrak{m}(G)$ is quadratic
3. G cannot split non-trivially as a free product (Gromov 1989)

Kähler 3-manifold groups

Question (Donaldson–Goldman 1989, Reznikov 1993)

Which 3-manifold groups are Kähler groups?

Clearly, $\{$abelian, 3-manifold, Kähler groups$\} = \{$finite cyclic groups$\}$.
Partial answers (much harder):

Theorem (Reznikov 2002)

Let M be an irreducible, atoroidal 3-manifold. Suppose there is a homomorphism $\rho\colon \pi_1(M) \to \mathrm{SL}(2,\mathbb{C})$ with Zariski dense image. Then $G = \pi_1(M)$ is not a Kähler group.

Theorem (Hernández-Lamoneda 2001)

Let M be a geometrizable 3-manifold, with all pieces hyperbolic. Then $G = \pi_1(M)$ is not a Kähler group.

Theorem (Dimca–S., JEMS 2009)

Let G be the fundamental group of a closed 3-manifold. If G is a Kähler group, then G is finite.

By (Perelman 2003):

$$\pi_1(M^3) \text{ finite} \iff M^3 \text{ has a metric of constant positive curvature}$$

Hence,

$$\{\text{Kähler groups}\} \cap \{\text{3-manifold groups}\}$$
$$= \{\text{finite subgroups of O(4), acting freely on } S^3\}$$

By (Hopf 1925) and (Milnor 1957), these groups are:

$$1,\ D^*_{4m},\ O^*,\ I^*,\ D_{2^k(2m+1)},\ P'_{8\cdot 3^k},$$

and products of one of these with a cyclic group of relatively prime order.

Quasi-Kähler manifolds

Definition

A manifold X is called *quasi-Kähler* if $X = \overline{X} \setminus D$, where $\overline{X}$ is a compact Kähler manifold and D is a divisor with normal crossings.

Similar definition for X quasi-projective.
The notions of quasi-Kähler group and quasi-projective group are defined as above.

- X quasi-projective $\Rightarrow H^*(X, \mathbb{Z})$ has a mixed Hodge structure (Deligne 1972–74)
- $X = \mathbb{CP}^n \setminus \{\text{hyperplane arrangement}\} \Rightarrow X$ is formal (Brieskorn 1973)
- X quasi-projective, $W_1(H^1(X, \mathbb{C})) = 0 \Rightarrow \pi_1(X)$ is 1-formal (Morgan 1978)
- $X = \mathbb{CP}^n \setminus \{\text{hypersurface}\} \Rightarrow \pi_1(X)$ is 1-formal (Kohno 1983)

Question

Which 3-manifold groups are quasi-Kähler groups?

Theorem (Dimca–Papadima–S., arXiv:0810.2158)

Let G be the fundamental group of a closed, orientable 3-manifold. Assume G is 1-formal. Then the following are equivalent:

1. $\mathfrak{m}(G) \cong \mathfrak{m}(\pi_1(X))$, *for some quasi-Kähler manifold X.*
2. $\mathfrak{m}(G) \cong \mathfrak{m}(\pi_1(M))$, *where M is either* S^3, $\#^n S^1 \times S^2$, *or* $S^1 \times \Sigma_g$.

Remark

There are many 3-manifold groups which are quasi-Kähler, yet are not 1-formal. We only have partial results in this case.

Proposition

Let $(X, 0)$ be an isolated surface singularity with $\mathbb{C}^*$-action, and let M be its singularity link. Then $G = \pi_1(M)$ is a quasi-projective, 3-manifold group, yet G is not 1-formal, provided $b_1(M) > 0$.

Example

The Heisenberg nilmanifold, $M = H_{\mathbb{R}}/H_{\mathbb{Z}}$, where $H = \{3 \times 3$ unipotent matrices$\}$, occurs as the Brieskorn manifold $\Sigma(2, 3, 6)$.

We also construct examples of irreducible, smooth *affine* surfaces U, with $\pi_1(U)$ not 1-formal. (Necessarily, $W_1 H^1(U, \mathbb{C}) \neq 0$, by Morgan.)

Example

Take $U = X \setminus V(f)$, where X is the surface in $\mathbb{C}^3$ given by $x^d + y^d + z^d = 0$, with $d \geq 3$, and $f = x + y^2 + z^3$.

Toric complexes and right-angled Artin groups

Definition

Let L be simplicial complex on n vertices. The associated *toric complex*, T_L, is the subcomplex of the n-torus obtained by deleting the cells corresponding to the missing simplices of L.

- Special case of "generalized moment angle complex".
- $\pi_1(T_L)$ is the *right-angled Artin group* associated to graph $\Gamma = L^{(1)}$:

$$G_\Gamma = \langle v \in V(\Gamma) \mid vw = wv \text{ if } \{v, w\} \in E(\Gamma)\rangle.$$

- $K(G_\Gamma, 1) = T_{\Delta_\Gamma}$, where Δ_Γ is the *flag complex* of Γ.
  (Davis–Charney 1995, Meier–VanWyk 1995)
- $H^*(T_L, \Bbbk)$ is the *exterior Stanley-Reisner ring* of L, with generators the duals v^*, and relations the monomials corresponding to the missing simplices of L.
- T_L is formal, and so G_Γ is 1-formal. (Notbohm–Ray 2005)

Example

- $\Gamma = \overline{K}_n \Rightarrow G_\Gamma = F_n$
- $\Gamma = K_n \Rightarrow G_\Gamma = \mathbb{Z}^n$
- $\Gamma = \Gamma' \coprod \Gamma'' \Rightarrow G_\Gamma = G_{\Gamma'} * G_{\Gamma''}$
- $\Gamma = \Gamma' * \Gamma'' \Rightarrow G_\Gamma = G_{\Gamma'} \times G_{\Gamma''}$

Theorem (Dimca–Papadima–S., Duke 2009)

The following are equivalent:

1. G_Γ *is a quasi-Kähler group*
2. $\Gamma = K_{n_1,\dots,n_r} := \overline{K}_{n_1} * \cdots * \overline{K}_{n_r}$
3. $G_\Gamma = F_{n_1} \times \cdots \times F_{n_r}$

1. G_Γ *is a Kähler group*
2. $\Gamma = K_{2r}$
3. $G_\Gamma = \mathbb{Z}^{2r}$

Bestvina–Brady groups: $N_\Gamma = \ker(\nu\colon G_\Gamma \twoheadrightarrow \mathbb{Z})$, where $\nu(v) = 1$

Theorem (D.–P.–S., JAG 2008)

The following are equivalent:

1. N_Γ *is a quasi-Kähler group*
2. Γ *is either a tree, or* $\Gamma = K_{n_1,\dots,n_r}$*, with some* $n_i = 1$*, or all* $n_i \geq 2$ *and* $r \geq 3$*.*

The following are equivalent:

1. N_Γ *is a Kähler group*
2. $\Gamma = K_{2r+1}$
3. $N_\Gamma = \mathbb{Z}^{2r}$

Characteristic varieties

- X connected CW-complex with finite k-skeleton ($k \geq 1$)
- $G = \pi_1(X, x_0)$: a finitely generated group
- $\Bbbk$ field; Hom$(G, \Bbbk^\times)$ character variety

Definition (Green–Lazarsfeld 1987, Beauville 1988, Simpson 1992, ...)

The *characteristic varieties* of X (over $\Bbbk$):

$$\mathcal{V}^i_d(X, \Bbbk) = \{\rho \in \mathrm{Hom}(G, \Bbbk^\times) \mid \dim_{\Bbbk} H_i(X, \Bbbk_\rho) \geq d\},$$

for $0 \leq i \leq k$ and $d > 0$.

- For each i, get stratification $\mathrm{Hom}(G, \Bbbk^\times) \supseteq \mathcal{V}^i_1 \supseteq \mathcal{V}^i_2 \supseteq \cdots$
- If $\Bbbk \subseteq \mathbb{K}$ extension: $\mathcal{V}^i_d(X, \Bbbk) = \mathcal{V}^i_d(X, \mathbb{K}) \cap \mathrm{Hom}(G, \Bbbk^\times)$
- For G of type F_k, set: $\mathcal{V}^i_d(G, \Bbbk) := \mathcal{V}^i_d(K(G, 1), \Bbbk)$
- Note: $\mathcal{V}_d(X, \Bbbk) := \mathcal{V}^1_d(X, \Bbbk) = \mathcal{V}^1_d(\pi_1(X), \Bbbk)$

Tangent cones and exponential tangent cones

The homomorphism $\mathbb{C} \to \mathbb{C}^{\times}$, $z \mapsto e^z$ induces

$$\exp\colon \operatorname{Hom}(G,\mathbb{C}) \to \operatorname{Hom}(G,\mathbb{C}^{\times}), \quad \exp(0) = 1$$

Let $W = V(I)$ be a Zariski closed subset in $\operatorname{Hom}(G,\mathbb{C}^{\times})$.

Definition

- The *tangent cone* at 1 to W:

$$TC_1(W) = V(\operatorname{in}(I))$$

- The *exponential tangent cone* at 1 to W:

$$\tau_1(W) = \{z \in \operatorname{Hom}(G,\mathbb{C}) \mid \exp(tz) \in W,\ \forall t \in \mathbb{C}\}$$

Both types of tangent cones

- are homogeneous subvarieties of $\mathrm{Hom}(G, \mathbb{C})$
- are non-empty iff $1 \in W$
- depend only on the analytic germ of W at 1
- commute with finite unions and arbitrary intersections

Moreover,

- $\tau_1(W) \subseteq TC_1(W)$
  - $=$ if all irred components of W are subtori
  - $\neq$ in general
- $\tau_1(W)$ is a finite union of rationally defined subspaces

Resonance varieties

Let $A = H^*(X, \Bbbk)$. If char $\Bbbk = 2$, assume $H_1(X, \mathbb{Z})$ has no 2-torsion. Then: $a \in A^1 \Rightarrow a^2 = 0$. Get cochain complex ("Aomoto complex")

$$(A, \cdot a)\colon\ A^0 \xrightarrow{a} A^1 \xrightarrow{a} A^2 \longrightarrow \cdots$$

Definition (Falk 1997, Matei–S. 2000)

The *resonance varieties* of X (over $\Bbbk$):

$$\mathcal{R}^i_d(X, \Bbbk) = \{a \in A^1 \mid \dim_{\Bbbk} H^i(A, \cdot a) \geq d\}$$

Homogeneous subvarieties of $A^1 = H^1(X, \Bbbk)$: $\mathcal{R}^i_1 \supseteq \mathcal{R}^i_2 \supseteq \cdots$

Theorem (Libgober 2002)

$$TC_1(\mathcal{V}^i_d(X, \mathbb{C})) \subseteq \mathcal{R}^i_d(X, \mathbb{C})$$

Equality does not hold in general (Matei–S. 2002)

Example: resonance of toric complexes

Recall $A = H^*(T_L, \Bbbk)$ is the exterior Stanley-Reisner ring of L.

Theorem (Papadima–S., Adv. Math. 2009)

$$\mathcal{R}^i_d(T_L, \Bbbk) = \bigcup_{\substack{\mathsf{W} \subset \mathsf{V} \\ \sum_{\sigma \in L_{\mathsf{V}\setminus\mathsf{W}}} \dim_{\Bbbk} \widetilde{H}_{i-1-|\sigma|}(\operatorname{lk}_{L_{\mathsf{W}}}(\sigma), \Bbbk) \geq d}} \Bbbk^{\mathsf{W}},$$

where L_{W} is the subcomplex induced by L on W, and $\operatorname{lk}_K(\sigma)$ is the link of a simplex σ in a subcomplex $K \subseteq L$.

Similar formula holds for $\mathcal{V}^i_d(T_L, \Bbbk)$, with $\Bbbk^{\mathsf{W}}$ replaced by $(\Bbbk^{\times})^{\mathsf{W}}$. In particular,

$$TC_1(\mathcal{R}^i_d(T_L, \mathbb{C})) = \mathcal{V}^i_d(T_L, \mathbb{C}).$$

Tangent cone theorem

Theorem (D.–P.–S., Duke 2009)

If G is 1-formal, then $\exp\colon (\mathcal{R}^1_d(G,\mathbb{C}),0) \xrightarrow{\simeq} (\mathcal{V}^1_d(G,\mathbb{C}),1)$. *Hence*

$$\tau_1(\mathcal{V}^1_d(G,\mathbb{C})) = TC_1(\mathcal{V}^1_d(G,\mathbb{C})) = \mathcal{R}^1_d(G,\mathbb{C})$$

In particular, $\mathcal{R}^1_d(G,\mathbb{C})$ is a union of rationally defined subspaces in $H^1(G,\mathbb{C}) = \mathrm{Hom}(G,\mathbb{C})$.

Example

Let $G = \langle x_1, x_2, x_3, x_4 \mid [x_1, x_2], [x_1, x_4][x_2^{-2}, x_3], [x_1^{-1}, x_3][x_2, x_4]\rangle$. Then

$$\mathcal{R}^1_1(G,\mathbb{C}) = \{x \in \mathbb{C}^4 \mid x_1^2 - 2x_2^2 = 0\}$$

splits into subspaces over $\mathbb{R}$ but not over $\mathbb{Q}$. Thus, G is *not* 1-formal.

Example

- $X = F(\Sigma_g, n)$: the configuration space of n labeled points of a Riemann surface of genus g (a smooth, quasi-projective variety).
- $\pi_1(X) = P_{g,n}$: the pure braid group on n strings on Σ_g.

Using computation of $H^*(F(\Sigma_g, n), \mathbb{C})$ by Totaro (1996), get

$$\mathcal{R}^1_1(P_{1,n}, \mathbb{C}) = \left\{ (x, y) \in \mathbb{C}^n \times \mathbb{C}^n \;\middle|\; \begin{array}{l} \sum_{i=1}^n x_i = \sum_{i=1}^n y_i = 0, \\ x_i y_j - x_j y_i = 0, \text{ for } 1 \le i < j < n \end{array} \right\}$$

For $n \geq 3$, this is an irreducible, non-linear variety (a rational normal scroll). Hence, $P_{1,n}$ is not 1-formal.

Σ-invariants

G finitely generated group $\rightsquigarrow \mathcal{C}(G)$ Cayley graph.
$\chi\colon G \to \mathbb{R}$ homomorphism $\rightsquigarrow \mathcal{C}_\chi(G)$ induced subgraph on vertex set $G_\chi = \{g \in G \mid \chi(g) \geq 0\}$.

Definition

$\Sigma^1(G) = \{\chi \in \operatorname{Hom}(G,\mathbb{R}) \setminus \{0\} \mid \mathcal{C}_\chi(G) \text{ is connected}\}$

An open, conical subset of $\operatorname{Hom}(G,\mathbb{R}) = H^1(G,\mathbb{R})$, independent of choice of generating set for G.

Definition

$\Sigma^k(G,\mathbb{Z}) = \{\chi \in \operatorname{Hom}(G,\mathbb{R}) \setminus \{0\} \mid \text{the monoid } G_\chi \text{ is of type } \mathrm{FP}_k\}$

Here, G is of type FP_k if there is a projective $\mathbb{Z}G$-resolution $P_\bullet \to \mathbb{Z}$, with P_i finitely generated for all $i \leq k$.

- The BNSR invariants $\Sigma^q(G,\mathbb{Z})$ form a descending chain of *open* subsets of $\mathrm{Hom}(G,\mathbb{R})\setminus\{0\}$.
- $\Sigma^k(G,\mathbb{Z})\neq\emptyset \implies G$ is of type FP_k.
- $\Sigma^1(G,\mathbb{Z})=\Sigma^1(G)$.
- The Σ-invariants control the finiteness properties of normal subgroups $N\triangleleft G$ with G/N is abelian:

$$N \text{ is of type } \mathrm{FP}_k \iff S(G,N)\subseteq\Sigma^k(G,\mathbb{Z})$$

  where $S(G,N)=\{\chi\in\mathrm{Hom}(G,\mathbb{R})\setminus\{0\}\mid\chi(N)=0\}$.
- In particular:

$$\ker(\chi\colon G\twoheadrightarrow\mathbb{Z}) \text{ is f.g.} \iff \{\pm\chi\}\subseteq\Sigma^1(G)$$

Let X be a connected CW-complex with finite 1-skeleton, $G = \pi_1(X)$.

Definition

The *Novikov-Sikorav completion* of $\mathbb{Z}G$:

$$\widehat{\mathbb{Z}G}_\chi = \left\{ \lambda \in \mathbb{Z}^G \mid \{g \in \operatorname{supp}\lambda \mid \chi(g) < c\} \text{ is finite, } \forall c \in \mathbb{R} \right\}$$

$\widehat{\mathbb{Z}G}_\chi$ is a ring, contains $\mathbb{Z}G$ as a subring $\implies$ $\widehat{\mathbb{Z}G}_\chi$ is a $\mathbb{Z}G$-module.

Definition

$$\Sigma^q(X, \mathbb{Z}) = \{\chi \in \operatorname{Hom}(G, \mathbb{R}) \setminus \{0\} \mid H_i(X, \widehat{\mathbb{Z}G}_{-\chi}) = 0, \ \forall\, i \leq q\}$$

Bieri: G of type $\mathrm{FP}_k \implies \Sigma^q(G, \mathbb{Z}) = \Sigma^q(K(G,1), \mathbb{Z}), \forall q \leq k$.

Exponential tangent cone upper bound

Theorem (Papadima–S., PLMS)

If X has finite k-skeleton, then, for every $q \leq k$,

$$\Sigma^q(X,\mathbb{Z}) \subseteq \Big(\tau_1^{\mathbb{R}}\big(\bigcup_{i\leq q} \mathcal{V}_1^i(X,\mathbb{C})\big)\Big)^{\complement}. \qquad (*)$$

Thus: Each Σ-invariant is contained in the complement of a union of rationally defined subspaces. Bound is sharp:

Example

Let G be a finitely generated nilpotent group. Then

$$\Sigma^q(G,\mathbb{Z}) = \mathrm{Hom}(G,\mathbb{R}) \setminus \{0\}, \quad V_1^q(G,\mathbb{C}) = \{1\}, \quad \forall q$$

and so (*) holds as an equality.

Resonance upper bound

Corollary

Suppose $\exp\colon (\mathcal{R}^i_1(X,\mathbb{C}),0) \xrightarrow{\simeq} (\mathcal{V}^i_1(X,\mathbb{C}),1)$, *for* $i \le q$. *Then:*

$$\Sigma^q(X,\mathbb{Z}) \subseteq \Big(\bigcup_{i\le q} \mathcal{R}^i_1(X,\mathbb{R})\Big)^{\complement}.$$

Corollary

Suppose G *is a* 1*-formal group. Then* $\Sigma^1(G) \subseteq \mathcal{R}^1_1(G,\mathbb{R})^{\complement}$.
In particular, if $\mathcal{R}^1_1(G,\mathbb{R}) = H^1(G,\mathbb{R})$, *then* $\Sigma^1(G) = \emptyset$.

Example

The above inclusion may be strict: Let $G = \langle a,b \mid aba^{-1} = b^2\rangle$.
Then G is 1-formal, $\Sigma^1(G) = (-\infty,0)$, yet $\mathcal{R}^1_1(G,\mathbb{R}) = \{0\}$.

Characteristic varieties of quasi-Kähler manifolds

Theorem (Arapura 1997)

Let $X = \overline{X} \setminus D$ be a quasi-Kähler manifold. Then:

1. *Each component of $\mathcal{V}^1_1(X)$ is either an isolated unitary character, or of the form $\rho \cdot f^*(H^1(C, \mathbb{C}^\times))$, for some torsion character ρ and some admissible map $f\colon X \to C$.*
2. *If either $X = \overline{X}$ or $b_1(\overline{X}) = 0$, then each component of $\mathcal{V}^i_d(X)$ is of the form $\rho \cdot f^*(H^1(T, \mathbb{C}^\times))$, for some unitary character ρ and some holomorphic map $f\colon X \to T$ to a complex torus.*

Here, $f\colon X \to C$ is *admissible* (or, a *pencil*) if f is a holomorphic, surjective map to a connected, smooth complex curve C, and there is a holomorphic, surjective extension $\overline{f}\colon \overline{X} \to \overline{C}$ with connected fibers.

Resonance varieties of quasi-Kähler manifolds

Theorem (D.–P.–S., Duke 2009)

Let X be a quasi-Kähler manifold, $G = \pi_1(X)$. Let $\{\mathcal{V}^\alpha\}_\alpha$ be the irred components of $\mathcal{V}_1^1(G)$ containing 1. Set $\mathcal{T}^\alpha = TC_1(\mathcal{V}^\alpha)$. Then:

1. *Each $\mathcal{T}^\alpha$ is a p-isotropic subspace of $H^1(G, \mathbb{C})$, of $\dim \geq 2p+2$, for some $p = p(\alpha) \in \{0, 1\}$.*
2. *If $\alpha \neq \beta$, then $\mathcal{T}^\alpha \bigcap \mathcal{T}^\beta = \{0\}$.*

Assume further that G is 1-formal. Let $\{\mathcal{R}^\alpha\}_\alpha$ be the irred components of $\mathcal{R}_1^1(G)$. Then:

3. $\{\mathcal{T}^\alpha\}_\alpha = \{\mathcal{R}^\alpha\}_\alpha$.
4. $\mathcal{R}_d^1(G) = \{0\} \cup \bigcup_{\alpha:\ \dim \mathcal{R}^\alpha > d + p(\alpha)} \mathcal{R}^\alpha$.

Here we used the following

Definition

A non-zero subspace $U \subseteq H^1(G,\mathbb{C})$ is *p-isotropic* with respect to

$$\cup_G \colon H^1(G,\mathbb{C}) \wedge H^1(G,\mathbb{C}) \to H^2(G,\mathbb{C})$$

if the restriction of $\cup_G$ to $U \wedge U$ has rank p.

Example

Let C be a smooth complex curve with $\chi(C) < 0$. Then

$$\mathcal{R}^1_1(\pi_1(C),\mathbb{C}) = H^1(C,\mathbb{C})$$

and this space is either 1- or 0-isotropic, according to whether C is compact or not.

Σ-invariants of quasi-Kähler groups

Theorem (P.–S., PLMS)

Let X be a quasi-Kähler manifold, and $G = \pi_1(X)$. Then:

1. $\Sigma^1(G) \subseteq TC_1^{\mathbb{R}}(\mathcal{V}_1^1(G, \mathbb{C}))^{\complement}$.
2. *If X is Kähler, or $W_1(H^1(X, \mathbb{C})) = 0$, then $\mathcal{R}_1^1(G, \mathbb{R})$ is a finite union of rationally defined linear subspaces, and $\Sigma^1(G) \subseteq \mathcal{R}_1^1(G, \mathbb{R})^{\complement}$.*

Example

Assumption from (2) is necessary. E.g., let X be the complex Heisenberg manifold: bundle $\mathbb{C}^\times \to X \to (\mathbb{C}^\times)^2$ with $e = 1$. Then:

1. X is a smooth quasi-projective variety;
2. $G = \pi_1(X)$ is nilpotent (and not 1-formal);
3. $\Sigma^1(G) = \mathbb{R}^2 \setminus \{0\}$ and $\mathcal{R}_1^1(G, \mathbb{R}) = \mathbb{R}^2$.

Thus, $\Sigma^1(G) \not\subseteq \mathcal{R}_1^1(G, \mathbb{R})^{\complement}$.

For Kähler manifolds, we can say precisely when the resonance upper bound for Σ^1 is attained.

Theorem (P.–S., PLMS)

Let M be a compact Kähler manifold with $b_1(M) > 0$*, and* $G = \pi_1(M)$*. The following are equivalent:*

1. $\Sigma^1(G) = \mathcal{R}^1_1(G, \mathbb{R})^{\complement}$.
2. *If* $f\colon M \to C$ *is an elliptic pencil, then f has no multiple fibers.*

Proof uses results of Arapura, DPS, and Delzant.

Hyperplane arrangements

Let $\mathcal{A}$ be an arrangement of hyperplanes in $\mathbb{C}^\ell$, with complement $X = \mathbb{C}^\ell \setminus \bigcup_{H \in \mathcal{A}} H$, and group $G = G(\mathcal{A}) = \pi_1(X)$.

- X is a smooth, quasi-projective variety, and so G is a quasi-projective group.
- X is formal, and so $G = \pi_1(X)$ is 1-formal.
- $A = H^*(X, \mathbb{Z})$ is the Orlik-Solomon algebra, determined by the intersection lattice, $L(\mathcal{A})$.
- Resonance varieties $\mathcal{R}^1_d(X, \mathbb{C})$ are very well understood.
- Tangent cone formula holds. In particular, components of $\mathcal{V}^1_d(X, \mathbb{C})$ passing through 1 are combinatorially determined.
- $\mathcal{V}^1_1(X, \mathbb{C})$ may contain translated subtori.
- $\Sigma^q(X, \mathbb{Z}) \subseteq \mathcal{R}^q_1(X, \mathbb{R})^\complement$

Let $\mathcal{A}$ be an arrangement of lines in $\mathbb{C}^2$, with group $G = G(\mathcal{A})$.

Theorem (S. 2009)

The following are equivalent:

1. *G is a Kähler group.*
2. *G is a free abelian group of even rank.*
3. *$\mathcal{A}$ consists of an even number of lines in general position.*

Also equivalent:

1. *G is a right-angled Artin group.*
2. *G is a finite direct product of finitely generated free groups.*
3. *The multiplicity graph $\Gamma(\mathcal{A})$ is a forest.*

Donaldson–Goldman problem revisited

Proposition

Let M be a closed, orientable 3-manifold. Then:

1. $H^1(M,\mathbb{C})$ *is not 1-isotropic.*
2. *If* $b_1(M)$ *is even, then* $\mathcal{R}_1(M,\mathbb{C}) = H^1(M,\mathbb{C})$*.*

Proposition

Let M be a compact Kähler manifold with $b_1(M) \neq 0$*. If* $\mathcal{R}_1(M,\mathbb{C}) = H^1(M,\mathbb{C})$*, then* $H^1(M,\mathbb{C})$ *is 1-isotropic.*

But $G = \pi_1(M)$, with M Kähler $\Rightarrow b_1(G)$ even.
Thus, if G is both a 3-mfd group and a Kähler group $\Rightarrow b_1(G) = 0$.
Using work of Fujiwara (1999) and Reznikov (2002) on Kazhdan's property (T) $\Rightarrow G$ finite.

Boundary manifolds of line arrangements

Let $\mathcal{A} = \{\ell_0, \dots, \ell_n\}$ be an arrangement of lines in $\mathbb{CP}^2$. The *boundary manifold* of $\mathcal{A}$ is the closed, orientable 3-manifold $M = M(\mathcal{A})$ obtained by taking the boundary of a regular neighborhood of $\bigcup_{i=0}^n \ell_i$ in $\mathbb{CP}^2$.

Theorem (Cohen–S., GTM 08, Dimca–Papadima–S., IMRN 08)

Let $\mathcal{A} = \{\ell_0, \dots, \ell_n\}$ be an arrangement of lines in $\mathbb{CP}^2$, and let M be the corresponding boundary manifold. The following are equivalent:

1. *The manifold M is formal.*
2. *The group $G = \pi_1(M)$ is 1-formal.*
3. $TC_1(V_1(G, \mathbb{C})) = \mathcal{R}_1(G, \mathbb{C})$.
4. *The group G is quasi-projective.*
5. *$\mathcal{A}$ is either a pencil (and so $M = \sharp^n S^1 \times S^2$), or $\mathcal{A}$ is a near-pencil (and so $M = S^1 \times \Sigma_{n-1}$).*

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