Residual finiteness properties of fundamental groups

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FUNDAMENTAL GROUPS OF MANIFOLDS

- Every finitely presented group π can be realized as $\pi = \pi_1(M)$, for some smooth, compact, connected manifold M^n of dim $n \geqslant 4$.
- M^n can be chosen to be orientable.
- If n even, $n \geqslant 4$, then M^n can be chosen to be symplectic (Gompf).
- If n even, $n \geqslant 6$, then M^n can be chosen to be complex (Taubes).
- Requiring that $n = 3$ puts severe restrictions on the (closed) 3-manifold group $\pi = \pi_1(M^3)$.

PROJECTIVE GROUPS

- A group π is said to be a *projective group* if $\pi = \pi_1(M)$, for some smooth, projective variety M.
- The class of projective groups is closed under finite direct products and passing to finite-index subgroups.
- Every finite group is a projective group. [Serre $\sim$1955]
- The projectivity condition puts strong restrictions on π, e.g.:
  - π is finitely presented.
  - $b_1(\pi)$ is even. [by Hodge theory]
  - π is 1-formal [Deligne–Griffiths–Morgan–Sullivan 1975]
  - π cannot split non-trivially as a free product. [Gromov 1989]
  - $\pi = \pi_1(N)$ for some closed 3-manifold N iff π is a finite subgroup of $\mathrm{O}(4)$. [Dimca–S. 2009]

QUASI-PROJECTIVE GROUPS

- A group π is said to be a *quasi-projective group* if $\pi = \pi_1(M \backslash D)$, where M is a smooth, projective variety and D is a divisor.
- Qp groups are finitely presented. The class of qp groups is closed under direct products and passing to finite-index subgroups.
- For a qp group π,
  - $b_1(\pi)$ can be arbitrary (e.g., the free groups F_n).
  - π may be non-1-formal (e.g., the Heisenberg group).
  - π can split as a non-trivial free product (e.g., $F_2 = \mathbb{Z} * \mathbb{Z}$).

COMPLEMENTS OF HYPERSURFACES

- A subclass of quasi-projective groups consists of fundamental groups of complements of hypersurfaces in $\mathbb{CP}^n$.
- By the Lefschetz hyperplane sections theorem, this class coincides the class of fundamental groups of complements of plane algebraic curves.
- All such groups are 1-formal.
- Even more special are the *arrangement groups*, i.e., the fundamental groups of complements of complex hyperplane arrangements (or, equivalently, complex line arrangements).

PLANE ALGEBRAIC CURVES

- Let $\mathcal{C} \subset \mathbb{CP}^2$ be a plane algebraic curve, defined by a homogeneous polynomial $f \in \mathbb{C}[z_1, z_2, z_3]$.
- Zariski commissioned Van Kampen to find a presentation for the fundamental group of the complement, $U(\mathcal{C}) = \mathbb{CP}^2 \backslash \mathcal{C}$.
- Using the Alexander polynomial, Zariski showed that $\pi = \pi_1(U)$ is *not* fully determined by the combinatorics of $\mathcal{C}$, but depends on the position of its singularities.

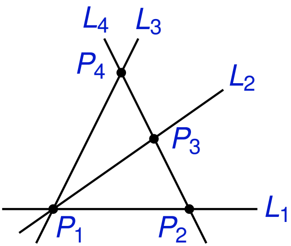
PROBLEM (ZARISKI)

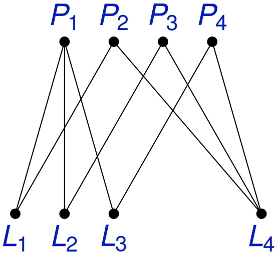
Is π residually finite?

- That is, given $g \in \pi$, $g \neq 1$, is there is a homomorphism $\varphi\colon \pi \to G$ onto some finite group G such that $\varphi(g) \neq 1$.
- Equivalently, is the canonical morphism to the profinite completion $\pi \to \pi^{\text{alg}} := \varprojlim_{N \triangleleft \pi : [\pi:N] < \infty} \pi/N$, injective?

LINE ARRANGEMENTS

- Let $\mathcal{A}$ be an *arrangement of lines* in $\mathbb{CP}^2$, defined by a polynomial $f = \prod_{L\in\mathcal{A}} f_L$, with f_L linear forms so that $L = \mathbb{P}(\ker(f_L))$.
- The combinatorics of $\mathcal{A}$ is encoded in the *intersection poset*, $\mathcal{L}(\mathcal{A})$, with $\mathcal{L}_1(\mathcal{A}) = \{\text{lines}\}$ and $\mathcal{L}_2(\mathcal{A}) = \{\text{intersection points}\}$.





- The group $\pi = \pi_1(U(\mathcal{A}))$ has a finite presentation with
  - Meridional generators $x_1, \dots, x_n$, where $n = |\mathcal{A}|$, and $\prod x_i = 1$.
  - Commutator relators $x_i \alpha_j(x_i)^{-1}$, where $\alpha_1, \dots \alpha_s \in P_n \subset \mathrm{Aut}(F_n)$, and $s = |\mathcal{L}_2(\mathcal{A})|$.
- Let $\gamma_1(\pi) = \pi, \ \ \gamma_2(\pi) = \pi' = [\pi, \pi], \ \ \gamma_k(\pi) = [\gamma_{k-1}(\pi), \pi]$, be the LCS of π. Then:
  - $\pi_{\mathrm{ab}} = \pi/\gamma_2$ equals $\mathbb{Z}^{n-1}$.
  - π/γ_3 is determined by $L(\mathcal{A})$.
  - π/γ_4 (and thus, π) is *not* determined by $L(\mathcal{A})$ (G. Rybnikov).

PROBLEM (ORLIK)

Is π torsion-free?

- Answer is yes if $U(\mathcal{A})$ is a $K(\pi, 1)$. This happens if the cone on $\mathcal{A}$ is a simplicial arrangement (Deligne), or supersolvable (Terao).

THE RFRp PROPERTY

Let G be a finitely generated group and let p be a prime.

We say that G is *residually finite rationally* p if there exists a sequence of subgroups $G = G_0 > \cdots > G_i > G_{i+1} > \cdots$ such that

1. $G_{i+1} \lhd G_i$.
2. $\bigcap_{i\geqslant 0} G_i = \{1\}$.
3. G_i / G_{i+1} is an elementary abelian p-group.
4. $\ker(G_i \to H_1(G_i, \mathbb{Q})) < G_{i+1}$.

Remarks:

- We may assume that each $G_i \lhd G$.
- G is RFRp if and only if $\mathrm{rad}_p(G) := \bigcap_i G_i$ is trivial.
- For each prime p, there exists a finitely presented group G_p which is RFRp, but not RFRq for any prime $q \neq p$.

- G RFRp $\Rightarrow$ residually p $\Rightarrow$ residually finite and residually nilpotent.
- G RFRp $\Rightarrow$ G torsion-free.
- G finitely presented and RFRp $\Rightarrow$ G has solvable word problem.
- The class of RFRp groups is closed under these operations:
  - Taking subgroups.
  - Finite direct products.
  - Finite free products.
- The following groups are RFRp, for all p:
  - Finitely generated free groups.
  - Closed, orientable surface groups.
  - Right-angled Artin groups.
- The following groups are *not* RFRp, for any p:
  - Finite groups
  - Non-abelian nilpotent groups

- Let G be a finitely-generated group, and let $\widehat{G} = \operatorname{Hom}(G, \mathbb{C}^*)$.
- The (degree 1) *characteristic varieties* of G are the closed algebraic subsets

$$V_i(G) = \{\chi \in \widehat{G} \mid \dim H^1(G, \mathbb{C}_\chi) \geqslant i\}.$$

LEMMA

Let $G'' = [G', G']$. The projection map $\pi\colon G \to G/G''$ induces an isomorphism $\hat{\pi}\colon \widehat{G/G''} \to \widehat{G}$ which restricts to isomorphisms $V_i(G/G'') \to V_i(G)$ for all $i \geqslant 1$.

A group G is *large* if G virtually surjects onto a non-abelian free group.

LEMMA (KOBERDA 2014)

An f.p. group G is large if and only if there exists a finite-index subgroup $H < G$ such that $V_1(H)$ has infinitely many torsion points.

THEOREM

Let G be a non-abelian, finitely generated group which is RFRp for infinitely many primes. Then:

- *G/G'' is not finitely presented.*
- *G' is not finitely generated.*
- *$V_1(G)$ contains infinitely many torsion points.*

As a consequence, we obtain the following 'Tits alternative' for RFRp groups.

COROLLARY

Let G be a finitely presented group which is RFRp for infinitely many primes. Then either:

1. *G is abelian.*
2. *G is large.*

BNS INVARIANTS

- The Bieri–Neumann–Strebel invariant of a f.g. group G is the set

$$\Sigma^1(G) = \{\chi \in S(G) \mid \mathrm{Cay}_\chi(G) \text{ is connected}\},$$

where

  - $S(G)$ is the unit sphere in $H^1(G, \mathbb{R})$.
  - For each non-zero homomorphism $\chi\colon G \to \mathbb{R}$, we let $\mathrm{Cay}_\chi(G)$ be the induced subgraph on vertices $g \in G$ such that $\chi(g) \geqslant 0$.
  - Although $\mathrm{Cay}_\chi(G)$ depends on the choice of a (finite, symmetric) generating set for G, its connectivity is independent of such choice.

THEOREM (PAPADIMA–S. (2010))

$\Sigma^1(G) \subset \left(\tau_1^{\mathbb{R}}(V_1(G))\right)^{\mathrm{c}}$.

Here, if $V \subset (\mathbb{C}^*)^n$, then $\tau_1(V) = \{z \in \mathbb{C}^n \mid \exp(\lambda z) \in V, \ \forall \lambda \in \mathbb{C}\}$.

For $N \lhd G$, write $S(G, N) = \{\chi \in S(G) \mid \chi|_N = 0\}$.

THEOREM (BNS 1988)

Let G be a finitely generated group, and let G/N be an infinite abelian quotient. Then the group N is finitely generated if and only if $S(G, N) \subset \Sigma^1(G)$. In particular, G' is finitely generated if and only if $\Sigma^1(G) = S(G)$.

By analogy with a result of Beauville on the structure of Kähler groups, we have:

THEOREM

Let G be a finitely generated group which is RFRp for infinitely many primes p. If $\Sigma^1(G) = S(G)$, then G is abelian.

THE RFRp TOPOLOGY

- Let G be a finitely generated group, and fix a prime p.
- The RFRp topology on G has basis the cosets of the standard RFRp filtration $\{G_i\}$ of G.
- G is RFRp iff this topology is Hausdorff.
- Let $\phi_i \colon G \to G/G_i$ be the canonical projection. A subgroup $H < G$ is *closed* iff for each $g \in G \setminus H$, there is an i such that $\phi_i(g) \notin \phi_i(H)$.

PROPOSITION

Let $r \colon G \to H$ be a retraction to a subgroup $H < G$. Then

1. *The RFRp topology on G induces the RFRp topology on H.*
2. *Moreover, if G is RFRp, then H is a closed subgroup of G.*

A COMBINATION THEOREM

THEOREM

Fix a prime p. Let $G = G_\Gamma$ be a finite graph of finitely generated groups with vertex groups $\{G_v\}_{v\in V(\Gamma)}$ and edge groups $\{G_e\}_{e\in E(\Gamma)}$ satisfying the following conditions:

1. *For each $v \in V(\Gamma)$, the group G_v is RFRp.*
2. *For each $v \in V(\Gamma)$, the RFRp topology on G induces the RFRp topology on G_v.*
3. *For each $e \in E(\Gamma)$ and each $v \in e$, the image of G_e in G_v is closed in the RFRp topology on G_v.*

Then G is RFRp.

3-MANIFOLDS

- Let M be a compact, connected, orientable 3-manifold M.
- We will assume that $\chi(M) = 0$ and M is prime, i.e., it cannot be decomposed as a nontrivial connected sum.
- M is said to be *geometric* if it admits a finite volume complete metric modeled on one of the eight Thurston geometries, $S^3, S^2 \times \mathbb{R}, \mathbb{R}^3, \mathrm{Nil}, \mathrm{Sol}, \mathbb{H}^2 \times \mathbb{R}, \widetilde{\mathrm{PSL}_2(\mathbb{R})}$, or $\mathbb{H}^3$.
- Perelman's Geometrization Theorem: every prime 3-manifold can be cut up along a canonical collection of incompressible tori into finitely many pieces, each one of which is geometric.

THEOREM

Let $G = \pi_1(M)$ be a geometric 3-manifold group. Then there is a finite index subgroup $G_0 < G$ which is RFRp for every prime p if and only if M admits one of the following geometries: $S^3, S^2 \times \mathbb{R}, \mathbb{R}^3, \mathbb{H}^2 \times \mathbb{R}, \mathbb{H}^3$. Otherwise, no finite index subgroup of G is RFRp for any prime p.

- A *graph manifold* is a prime 3-manifold which can be cut up along incompressible tori into pieces, each of which is Seifert fibered.
- Let $\mathcal{X}$ be the class of graph manifolds M satisfying:

1. The underlying graph Γ is finite, connected, and bipartite with colors $\mathcal{P}$ and $\mathcal{L}$, and each vertex in $\mathcal{P}$ has degree at least two.
2. Each vertex manifold M_v is homeomorphic to a trivial circle bundle over an orientable surface with boundary.
3. If M_v is colored by $\mathcal{L}$ then at least one boundary component of M_v is a boundary component of M, and $e(M_v) = 0$.
4. If M_v is colored by $\mathcal{P}$ then no boundary component of M_v is a boundary component of M, and $e(M_v) \neq 0$.
5. The gluing maps are given by flips.

THEOREM

Suppose M is a graph manifold satisfying the above conditions. Then, for each prime p, the group $\pi_1(M)$ is RFRp.

BOUNDARY MANIFOLDS OF CURVES

- Let $\mathcal{C}$ be a (reduced) algebraic curve in $\mathbb{CP}^2$, and let T be a regular neighborhood of $\mathcal{C}$.
- The *boundary manifold* of $\mathcal{C}$ is defined as $M_{\mathcal{C}} = \partial T$. This is a compact, orientable, smooth manifold of dimension 3.
- The homeomorphism type of $M = M_{\mathbb{C}}$ is independent of the choices made in constructing T, and depends only on $\mathcal{C}$ (Durfee).

EXAMPLE

Let $\mathcal{A}$ be a pencil of n lines in $\mathbb{CP}^2$, defined by $f = z_1^n - z_2^n$.
If $n = 1$, then $M = S^3$. If $n > 1$, then $M = \sharp^{n-1} S^1 \times S^2$.

EXAMPLE

Let $\mathcal{A}$ be a near-pencil of n lines in $\mathbb{CP}^2$, defined by $f = z_1(z_2^{n-1} - z_3^{n-1})$. Then $M = S^1 \times \Sigma_{n-2}$, where $\Sigma_g = \sharp^g S^1 \times S^1$.

In both examples, $\pi_1(M)$ is RFRp for all primes p.

EXAMPLE

Suppose $\mathcal{C}$ has a single irreducible component C, which we assume to be smooth. Then C is homeomorphic to an orientable surface Σ_g of genus $g = \binom{d-1}{2}$, where d is the degree of C, and $C \cdot C = d^2$. Thus, M is a circle bundle over Σ_g with Euler number $e = d^2$.

In this example, $\pi_1(M)$ is not RFRp, for any prime p, provided $d \geqslant 2$.

EXAMPLE

Suppose $\mathcal{C} = C \cup L$ consists of a smooth conic and a transverse line. The graph Γ is a square, the vertex manifolds are thickened tori $S^1 \times S^1 \times I$, and $M_{\mathcal{C}}$ is the Heisenberg nilmanifold.

In this example, $\pi_1(M)$ is not RFRp, for any prime p.

QUESTION

For which plane algebraic curves $\mathcal{C}$ is the fundamental group of the boundary manifold $M_{\mathcal{C}}$ an RFRp group (for some p or all primes p)?

- The boundary manifold of an affine plane curve is defined as $M = \partial T \cap D^4$, for some sufficiently large 4-ball D^4.

THEOREM

Let $\mathcal{C}$ be a plane algebraic curve such that

- *Each irreducible component of $\mathcal{C}$ is smooth and transverse to the line at infinity.*
- *Each singular point of $\mathcal{C}$ is a type A singularity.*

Then the boundary manifold $M_{\mathcal{C}}$ lies in $\mathcal{X}$.

- More precisely, $\mathcal{L}$ is the set of irreducible components of $\mathcal{C}$, while $\mathcal{P}$ is the set of multiple points of $\mathcal{C}$.
- The graph Γ has vertex set $V(\Gamma) = \mathcal{L} \cup \mathcal{P}$ and edge set $E(\Gamma) = \{(L, P) \mid P \in L\}$.
- For each $v \in V(\Gamma)$, there is a vertex manifold $M_v = S^1 \times S_v$, with $S_v = S^2 \setminus \bigcup_{\{v,w\} \in E(\Gamma)} D^2_{v,w}$.

THEOREM

Let $\mathcal{C}$ be an algebraic curve in $\mathbb{C}^2$. Suppose each irreducible component of $\mathcal{C}$ is smooth and transverse to the line at infinity, and all singularities of $\mathcal{C}$ are of type A. Then $\pi_1(M_{\mathcal{C}})$ is RFRp, for all primes p.

COROLLARY

If M is the boundary manifold of a line arrangement in $\mathbb{C}^2$, then $\pi_1(M)$ is RFRp, for all primes p.

CONJECTURE

Arrangement groups are RFRp, for all primes p.