Algebra and topology of right-angled Artin groups

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The braid group

Definition (E. Artin 1926/1947)

$$B_n = \left\langle \sigma_1, \dots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & 1 \le i \le n-2 \\ \sigma_i\sigma_j = \sigma_j\sigma_i & |i-j| > 1 \end{array} \right\rangle$$

Fits into exact sequence

$$1 \longrightarrow P_n \longrightarrow B_n \xrightarrow{\pi} S_n \longrightarrow 1 ,$$

where S_n is the symmetric group, π sends a braid to the corresponding permutation, and P_n is the group of “pure" braids.

Serre's realization problem

Problem

Which finitely presented groups G are (quasi-) projective*, i.e., can be realized as $G = \pi_1(M)$, with M a connected, smooth, complex (quasi-) projective variety?*

Theorem (J.-P. Serre 1958)

All finite groups are projective.

Theorem (Fox–Neuwirth 1962)

Both P_n and B_n are quasi-projective groups.

$$P_n = \pi_1(F(\mathbb{C}, n)),$$

where $F(\mathbb{C}, n) = \mathbb{C}^n \setminus \{z_i = z_j\}$ is the configuration space of n ordered points in $\mathbb{C}$, or, the complement of the braid arrangement.

$$B_n = \pi_1(C(\mathbb{C}, n)),$$

where $C(\mathbb{C}, n) = F(\mathbb{C}, n)/S_n = \mathbb{C}^n \setminus \{\Delta_n = 0\}$ is the configuration space of n unordered points in $\mathbb{C}$, or, the complement of the discriminant hypersurface.

Remark

As we shall see later, neither P_n nor B_n is a projective group.

Artin groups

Let $\Gamma = (V, E)$ be a finite, simple graph, and $\ell \colon E \to \mathbb{Z}_{\geq 2}$ an edge-labeling. The associated *Artin group*:

$$G_{\Gamma,\ell} = \langle v \in V \mid \underbrace{vwv\cdots}_{\ell(e)} = \underbrace{wvw\cdots}_{\ell(e)}, \text{ for } e = \{v, w\} \in E.\rangle$$

E.g.: (Γ, ℓ) Dynkin diagram of type $\mathrm{A}_{n-1} \Rightarrow G_{\Gamma,\ell} = B_n$

The corresponding *Coxeter group*,

$$W_{\Gamma,\ell} = G_{\Gamma,\ell} / \langle v^2 = 1 \mid v \in V \rangle$$

fits into exact sequence

$$1 \longrightarrow P_{\Gamma,\ell} \longrightarrow G_{\Gamma,\ell} \xrightarrow{\pi} W_{\Gamma,\ell} \longrightarrow 1 .$$

Serre's problem for Artin groups

Theorem (Brieskorn 1971)

If $W_{\Gamma,\ell}$ is finite, then $G_{\Gamma,\ell}$ is quasi-projective.

Idea: let

- $\mathcal{A}_{\Gamma,\ell}$ = reflection arrangement of type $W_{\Gamma,\ell}$ (over $\mathbb{C}$)
- $X_{\Gamma,\ell} = \mathbb{C}^n \setminus \bigcup_{H\in\mathcal{A}_{\Gamma,\ell}} H$, where $n = |\mathcal{A}_{\Gamma,\ell}|$
- $P_{\Gamma,\ell} = \pi_1(X_{\Gamma,\ell})$

then:

$$G_{\Gamma,\ell} = \pi_1(X_{\Gamma,\ell}/W_{\Gamma,\ell}) = \pi_1(\mathbb{C}^n \setminus \{\delta_{\Gamma,\ell} = 0\})$$

Theorem (Kapovich–Millson 1998)

There exist infinitely many (Γ,ℓ) such that $G_{\Gamma,\ell}$ is not *quasi-projective.*

Right-angled Artin groups

Important particular case: $\ell(e) = 2$, for all $e \in E$. Simply write:

$$G_\Gamma = \langle v \in \mathsf{V} \mid vw = wv \text{ if } \{v, w\} \in \mathsf{E}\rangle.$$

- $\Gamma = \overline{K}_n$ (discrete graph) $\Rightarrow G_\Gamma = F_n$
- $\Gamma = K_n$ (complete graph) $\Rightarrow G_\Gamma = \mathbb{Z}^n$
- $\Gamma = \Gamma' \coprod \Gamma'' \Rightarrow G_\Gamma = G_{\Gamma'} * G_{\Gamma''}$
- $\Gamma = \Gamma' * \Gamma'' \Rightarrow G_\Gamma = G_{\Gamma'} \times G_{\Gamma''}$

Theorem (Kim–Makar-Limanov–Neggers–Roush 80/Droms 87)

$$\Gamma \cong \Gamma' \Leftrightarrow G_\Gamma \cong G_{\Gamma'}$$

Serre's problem for right-angled Artin groups

Theorem (Dimca–Papadima–S. 2009)

The following are equivalent:

1. G_Γ *is a quasi-projective group*
2. $\Gamma = \overline{K}_{n_1} * \cdots * \overline{K}_{n_r}$
3. $G_\Gamma = F_{n_1} \times \cdots \times F_{n_r}$

1. G_Γ *is a projective group*
2. $\Gamma = K_{2r}$
3. $G_\Gamma = \mathbb{Z}^{2r}$

Partial product construction

Input:

- L, a simplicial complex on $[n] = \{1, \dots, n\}$.
- (X, A), a pair of topological spaces, $A \neq \emptyset$.

Output:

$$(X, A)^L = \bigcup_{\sigma \in L} (X, A)^\sigma \subset X^{\times n}$$

where $(X, A)^\sigma = \{x \in X^{\times n} \mid x_i \in A \text{ if } i \notin \sigma\}$.

Interpolates between $A^{\times n}$ and $X^{\times n}$.

Converts simplicial joins to direct products:

$$(X, A)^{K * L} \cong (X, A)^K \times (X, A)^L.$$

Toric complexes

Definition

The *toric complex* associated to L is the space $T_L = (S^1, *)^L$.

In other words:

- Circle $S^1 = e^0 \cup e^1$, with basepoint $* = e^0$.
- Torus $T^n = (S^1)^{\times n}$, with product cell structure:

$$(k-1)\text{-simplex } \sigma = \{i_1, \dots, i_k\} \quad \rightsquigarrow \quad k\text{-cell } e^\sigma = e^1_{i_1} \times \cdots \times e^1_{i_k}$$

- $T_L = \bigcup_{\sigma \in L} e^\sigma$.

Examples:

- $T_\emptyset = *$
- $T_{n \text{ points}} = \bigvee^n S^1$
- $T_{\partial\Delta^{n-1}} = (n-1)\text{-skeleton of } T^n$
- $T_{\Delta^{n-1}} = T^n$

- k-cells in $T_L \longleftrightarrow (k-1)$-simplices in L.
- $C_*^{\mathrm{CW}}(T_L)$ is a subcomplex of $C_*^{\mathrm{CW}}(T^n)$; thus, all $\partial_k = 0$, and

$$H_k(T_L, \mathbb{Z}) = C_{k-1}^{\text{simplicial}}(L, \mathbb{Z}) = \mathbb{Z}^{\#\,(k-1)\text{-simplices of } L}.$$

- $H^*(T_L, \Bbbk)$ is the *exterior Stanley-Reisner ring* $\Bbbk\langle L\rangle = E/J_L$, where
  - E = exterior algebra (over $\Bbbk$) on generators $v_1^*, \dots, v_n^*$ in degree 1;
  - J_L = ideal generated by all monomials $v_{i_1}^* \cdots v_{i_k}^*$ corresponding to simplices $\{i_1, \dots, i_k\} \notin L$.
- Clearly, $\pi_1(T_L) = G_\Gamma$, where $\Gamma = L^{(1)}$.
- T_L is formal, and so G_Γ is 1-formal. (Notbohm–Ray 2005)
- In fact, $G_{\Gamma,\ell}$ is 1-formal. (Kapovich–Millson)

- $K(G_\Gamma, 1) = T_{\Delta_\Gamma}$, where Δ_Γ is the *flag complex* of Γ.
  (Davis–Charney 1995, Meier–VanWyk 1995)

- Hence, $H^*(G_\Gamma, \Bbbk) = \bigwedge_{\Bbbk}(v_1^*, \dots, v_n^*)/J_\Gamma$, where

$$J_\Gamma = \text{ideal}(v_i^* v_j^* \mid \{v_i, v_j\} \notin \mathsf{E}(\Gamma))$$

- Since J_Γ is a quadratic monomial ideal, $A = E/J_\Gamma$ is a Koszul algebra (Fröberg 1975), i.e.,

$$\text{Tor}_i^A(\Bbbk, \Bbbk)_j = 0, \quad \text{for all } i \neq j.$$

Associated graded Lie algebra

Let G be a finitely-generated group. Define:

- *LCS series*: $G = G_1 \triangleright G_2 \triangleright \cdots \triangleright G_k \triangleright \cdots$, where $G_{k+1} = [G_k, G]$
- *LCS quotients*: $\operatorname{gr}_k G = G_k / G_{k+1}$ (f.g. abelian groups)
- *LCS ranks*: $\phi_k(G) = \operatorname{rank}(\operatorname{gr}_k G)$
- *Associated graded Lie algebra*: $\operatorname{gr}(G) = \bigoplus_{k \geq 1} \operatorname{gr}_k(G)$, with Lie bracket $[\,,\,]\colon \operatorname{gr}_k \times \operatorname{gr}_\ell \to \operatorname{gr}_{k+\ell}$ induced by group commutator.

Example (Witt, Magnus)

Let $G = F_n$ (free group of rank n).
Then $\operatorname{gr} G = \mathrm{Lie}_n$ (free Lie algebra of rank n), with LCS ranks given by

$$\prod_{k=1}^{\infty}(1-t^k)^{\phi_k} = 1 - nt.$$

Explicitly: $\phi_k(F_n) = \frac{1}{k}\sum_{d|k}\mu(d)n^{k/d}$, where μ is Möbius function.

Example (Kohno 1985)

Let $G = P_n$. Then:

$$\prod_{k=1}^{\infty}(1-t^k)^{\phi_k} = (1-t)\cdots(1-(n-1)t)$$

In other words, $\phi_k(P_n) = \phi_k(F_1 \times \cdots \times F_{n-1})$.

Holonomy Lie algebra

Definition (Chen 1977, Markl–Papadima 1992)

Let G be a finitely generated group, with $H_1 = H_1(G, \mathbb{Z})$ torsion-free. The *holonomy Lie algebra* of G is the quadratic, graded Lie algebra

$$\mathfrak{h}_G = \text{Lie}(H_1)/\text{ideal}(\text{im}(\nabla)),$$

where $\nabla \colon H_2(G, \mathbb{Z}) \to H_1 \wedge H_1 = \text{Lie}_2(H_1)$ is the comultiplication map.

Let $G = \pi_1(X)$ and $A = H^*(X, \mathbb{Q})$.

- (Löfwall 1986) $U(\mathfrak{h}_G \otimes \mathbb{Q}) \cong \bigoplus_{k\geq 1} \text{Ext}_A^k(\mathbb{Q}, \mathbb{Q})_k$.
- There is a canonical epimorphism $\mathfrak{h}_G \twoheadrightarrow \text{gr}(G)$.
- (Sullivan 1977) If G is 1-formal, then $\mathfrak{h}_G \otimes \mathbb{Q} \xrightarrow{\simeq} \text{gr}(G) \otimes \mathbb{Q}$.

Example

$G = F_n$, then clearly $\mathfrak{h}_G = \text{Lie}_n$, and so $\mathfrak{h}_G = \text{gr}(G)$.

Let $\Gamma = (\mathsf{V}, \mathsf{E})$ graph, and $P_\Gamma(t) = \sum_{k\geq 0} f_k(\Gamma) t^k$ its clique polynomial.

Theorem (Duchamp–Krob 1992, Papadima–S. 2006)

For $G = G_\Gamma$:

1. $\mathrm{gr}(G) \cong \mathfrak{h}_G$.
2. *Graded pieces are torsion-free, with ranks given by*

$$\prod_{k=1}^{\infty} (1 - t^k)^{\phi_k} = P_\Gamma(-t).$$

Idea of proof:

1. $A = \bigwedge_{\Bbbk} V^*/J_\Gamma \Rightarrow \mathfrak{h}_G \otimes \Bbbk = L_\Gamma := \mathrm{Lie}(\mathsf{V})/([v,w] = 0 \text{ if } \{v,w\} \in \mathsf{E})$.
2. Shelton–Yuzvinsky: $U(L_\Gamma) = A^!$ (Koszul dual).
3. Koszul duality: $\mathrm{Hilb}(A^!, t) \cdot \mathrm{Hilb}(A, -t) = 1$.
4. $\mathrm{Hilb}(\mathfrak{h}_G \otimes \Bbbk, t)$ independent of $\Bbbk \Rightarrow \mathfrak{h}_G$ torsion-free.
5. But $\mathfrak{h}_G \twoheadrightarrow \mathrm{gr}(G)$ is iso over $\mathbb{Q}$ (by 1-formality) $\Rightarrow$ iso over $\mathbb{Z}$.
6. LCS formula follows from (3) and PBW.

Chen Lie algebras

Definition (Chen 1951)

The *Chen Lie algebra* of a finitely generated group G is $\operatorname{gr}(G/G'')$, i.e., the assoc. graded Lie algebra of its maximal metabelian quotient.

Write $\theta_k(G) = \operatorname{rank}\operatorname{gr}_k(G/G'')$ for the Chen ranks. Facts:

- $\operatorname{gr}(G) \twoheadrightarrow \operatorname{gr}(G/G'')$, and so $\phi_k(G) \geq \theta_k(G)$, with equality for $k \leq 3$.
- The map $\mathfrak{h}_G \twoheadrightarrow \operatorname{gr}(G)$ induces epimorphism $\mathfrak{h}_G/\mathfrak{h}''_G \twoheadrightarrow \operatorname{gr}(G/G'')$.
- (P.–S. 2004) If G is 1-formal, then $\mathfrak{h}_G/\mathfrak{h}''_G \otimes \mathbb{Q} \xrightarrow{\simeq} \operatorname{gr}(G/G'') \otimes \mathbb{Q}$.

Example (Chen 1951)

$\operatorname{gr}(F_n/F_n'')$ is torsion-free, with ranks $\theta_1 = n$, and

$$\theta_k = (k-1)\cdot\binom{n+k-2}{k}, \text{ for } k \geq 2.$$

Example (Cohen–S. 1995)

$\operatorname{gr}(P_n/P_n'')$ is torsion-free, with ranks $\theta_1 = \binom{n}{2}$, $\theta_2 = \binom{n}{3}$, and

$$\theta_k = (k-1)\cdot\binom{n+1}{4}, \text{ for } k \geq 3.$$

In particular, $\theta_k(P_n) \neq \theta_k(F_1 \times \cdots \times F_{n-1})$, for $n \geq 4$ and $k \geq 4$.

The Chen Lie algebra of a RAAG

Theorem (P.–S. 2006)

For $G = G_\Gamma$:

1. $\mathrm{gr}(G/G'') \cong \mathfrak{h}_G/\mathfrak{h}''_G$.
2. *Graded pieces are torsion-free, with ranks given by*

$$\sum_{k=2}^{\infty} \theta_k t^k = Q_\Gamma\left(\frac{t}{1-t}\right),$$

where $Q_\Gamma(t) = \sum_{j\geq 2} c_j(\Gamma) t^j$ is the "cut polynomial" of Γ, with

$$c_j(\Gamma) = \sum_{\mathsf{W}\subset\mathsf{V}:\ |\mathsf{W}|=j} \tilde{b}_0(\Gamma_{\mathsf{W}}).$$

Idea of proof:

1. Write $A := H^*(G, \Bbbk) = E/J_\Gamma$, where $E = \bigwedge_\Bbbk(v_1^*, \dots, v_n^*)$.
2. Write $\mathfrak{h} = \mathfrak{h}_G \otimes \Bbbk$.
3. By Fröberg and Löfwall (2002)

$$\left(\mathfrak{h}'/\mathfrak{h}''\right)_k \cong \operatorname{Tor}^E_{k-1}(A, \Bbbk)_k, \quad \text{for } k \geq 2$$

4. By Aramova–Herzog–Hibi & Aramova–Avramov–Herzog (97-99):

$$\sum_{k\geq 2} \dim_\Bbbk \operatorname{Tor}^E_{k-1}(E/J_\Gamma, \Bbbk)_k = \sum_{i\geq 1} \dim_\Bbbk \operatorname{Tor}^S_i(S/I_\Gamma, \Bbbk)_{i+1} \cdot \left(\frac{t}{1-t}\right)^{i+1},$$

where $S = \Bbbk[x_1, \dots, x_n]$ and I_Γ = ideal $\langle x_i x_j \mid \{v_i, v_j\} \notin \mathsf{E}\rangle$.

5. By Hochster (1977):

$$\dim_\Bbbk \operatorname{Tor}^S_i(S/I_\Gamma, \Bbbk)_{i+1} = \sum_{\mathsf{W}\subset \mathsf{V}:\ |\mathsf{W}|=i+1} \dim_\Bbbk \widetilde{H}_0(\Gamma_{\mathsf{W}}, \Bbbk) = c_{i+1}(\Gamma).$$

6. The answer is independent of $\Bbbk \Rightarrow \mathfrak{h}_G/\mathfrak{h}''_G$ is torsion-free.
7. Using formality of G_Γ, together with $\mathfrak{h}_G/\mathfrak{h}''_G \otimes \mathbb{Q} \xrightarrow{\simeq} \operatorname{gr}(G/G'') \otimes \mathbb{Q}$ ends the proof.

Example

Let Γ be a pentagon, and Γ' a square with an edge attached to a vertex. Then:

- $P_\Gamma = P_{\Gamma'} = 1 - 5t + 5t^2$, and so

$$\phi_k(G_\Gamma) = \phi_k(G_{\Gamma'}), \quad \text{for all } k \geq 1.$$

- $Q_\Gamma = 5t^2 + 5t^3$ but $Q_{\Gamma'} = 5t^2 + 5t^3 + t^4$, and so

$$\theta_k(G_\Gamma) \neq \theta_k(G_{\Gamma'}), \quad \text{for } k \geq 4.$$

Resonance varieties

Let X be a connected CW-complex with finite k-skeleton ($k \geq 1$).
Let $\mathbb{k}$ be a field; if $\operatorname{char} \mathbb{k} = 2$, assume $H_1(X, \mathbb{Z})$ has no 2-torsion.
Let $A = H^*(X, \mathbb{k})$. Then: $a \in A^1 \Rightarrow a^2 = 0$. Thus, get cochain complex

$$(A, \cdot a)\colon \; A^0 \xrightarrow{a} A^1 \xrightarrow{a} A^2 \longrightarrow \cdots$$

Definition (Falk 1997, Matei–S. 2000)

The *resonance varieties* of X (over $\mathbb{k}$) are the algebraic sets

$$\mathcal{R}^i_d(X, \mathbb{k}) = \{a \in A^1 \mid \dim_{\mathbb{k}} H^i(A, a) \geq d\},$$

defined for all integers $0 \leq i \leq k$ and $d > 0$.

- $\mathcal{R}^i_d$ are homogeneous subvarieties of $A^1 = H^1(X, \mathbb{k})$
- $\mathcal{R}^i_1 \supseteq \mathcal{R}^i_2 \supseteq \cdots \supseteq \mathcal{R}^i_{b_i+1} = \emptyset$, where $b_i = b_i(X, \mathbb{k})$.
- $\mathcal{R}^1_d(X, \mathbb{k})$ depends only on $G = \pi_1(X)$, so denote it by $\mathcal{R}_d(G, \mathbb{k})$.

Resonance of toric complexes

Recall $A = H^*(T_L, \Bbbk)$ is the exterior Stanley-Reisner ring of L.
Identify $A^1 = \Bbbk^{\mathsf{V}}$ — the $\Bbbk$-vector space with basis $\{v \mid v \in \mathsf{V}\}$.

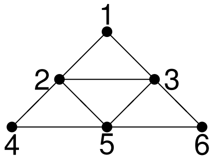
Theorem (Papadima–S. 2009)

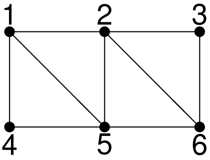
$$\mathcal{R}^i_d(T_L, \Bbbk) = \bigcup_{\substack{\mathsf{W} \subset \mathsf{V} \\ \sum_{\sigma \in L_{\mathsf{V}\setminus\mathsf{W}}} \dim_{\Bbbk} \widetilde{H}_{i-1-|\sigma|}(\mathrm{lk}_{L_{\mathsf{W}}}(\sigma), \Bbbk) \geq d}} \Bbbk^{\mathsf{W}},$$

where L_{W} is the subcomplex induced by L on W, and $\mathrm{lk}_K(\sigma)$ is the link of a simplex σ in a subcomplex $K \subseteq L$.

In particular:

$$\mathcal{R}_1(G_\Gamma, \Bbbk) = \bigcup_{\substack{\mathsf{W} \subseteq \mathsf{V} \\ \Gamma_{\mathsf{W}} \text{ disconnected}}} \Bbbk^{\mathsf{W}}.$$





Example

Let Γ and Γ' be the two graphs above. Both have

$$P(t) = 1 + 6t + 9t^2 + 4t^3, \quad \text{and} \quad Q(t) = t^2(6 + 8t + 3t^2).$$

Thus, G_Γ and $G_{\Gamma'}$ have the same LCS and Chen ranks.
Each resonance variety has 3 components, of codimension 2:

$$\mathcal{R}_1(G_\Gamma, \Bbbk) = \Bbbk^{\overline{23}} \cup \Bbbk^{\overline{25}} \cup \Bbbk^{\overline{35}}, \qquad \mathcal{R}_1(G_{\Gamma'}, \Bbbk) = \Bbbk^{\overline{15}} \cup \Bbbk^{\overline{25}} \cup \Bbbk^{\overline{26}}.$$

Yet the two varieties are not isomorphic, since

$$\dim(\Bbbk^{\overline{23}} \cap \Bbbk^{\overline{25}} \cap \Bbbk^{\overline{35}}) = 3, \quad \text{but} \quad \dim(\Bbbk^{\overline{15}} \cap \Bbbk^{\overline{25}} \cap \Bbbk^{\overline{26}}) = 2.$$

Projective varieties

Let M be a connected, smooth, complex projective variety. Then:

1. $H^*(M, \mathbb{Z})$ admits a Hodge structure
2. Hence, the odd Betti numbers of M are even
3. M is formal (Deligne–Griffiths–Morgan–Sullivan 1975)

This puts strong restrictions on $G = \pi_1(M)$:

1. $b_1(G)$ is even
2. G is 1-formal, i.e., its Malcev Lie algebra $\mathfrak{m}(G)$ is quadratic
3. G cannot split non-trivially as a free product (Gromov 1989)

In particular, B_n is not a projective group, since $b_1(B_n) = 1$.

Quasi-projective varieties

Let X be a connected, smooth, quasi-projective variety. Then:

- $H^*(X, \mathbb{Z})$ has a mixed Hodge structure
  (Deligne 1972–74)
- $X = \mathbb{CP}^n \setminus \{\text{hyperplane arrangement}\} \Rightarrow X$ is formal
  (Brieskorn 1973)
- $W_1(H^1(X, \mathbb{C})) = 0 \Rightarrow \pi_1(X)$ is 1-formal
  (Morgan 1978)
- $X = \mathbb{CP}^n \setminus \{\text{hypersurface}\} \Rightarrow \pi_1(X)$ is 1-formal
  (Kohno 1983)

A structure theorem

Theorem (D.–P.–S. 2009)

Let X be a smooth, quasi-projective variety, and $G = \pi_1(X)$. Let $\{L_\alpha\}_\alpha$ be the non-zero irred components of $\mathcal{R}_1(G)$. If G is 1-formal, then

1. *Each L_α is a p-isotropic linear subspace of $H^1(G, \mathbb{C})$, with* $\dim L_\alpha \geq 2p + 2$, *for some* $p = p(\alpha) \in \{0, 1\}$.
2. *If $\alpha \neq \beta$, then $L_\alpha \cap L_\beta = \{0\}$.*
3. $\mathcal{R}_d(G) = \{0\} \cup \bigcup_\alpha L_\alpha$, *where the union is over all α for which* $\dim L_\alpha > d + p(\alpha)$.

Furthermore,

4. *If X is compact, then G is 1-formal, and each L_α is 1-isotropic.*
5. *If $W_1(H^1(X, \mathbb{C})) = 0$, then G is 1-formal, and each L_α is 0-isotropic.*

Proof uses two basic ingredients:

- A structure theorem for $\mathcal{V}^i_d(X, \mathbb{C})$—the jump loci for cohomology with coefficients in rank 1 local systems on a quasi-projective variety X (Arapura 1997).
- A “tangent cone theorem", equating $TC_1(\mathcal{V}^1_d(G, \mathbb{C}))$ with $\mathcal{R}^1_d(G, \mathbb{C})$, for G a 1-formal group.

Serre's problem for RAAGs revisited

Using this theorem, together with the computation of $\mathcal{R}_1(G_\Gamma, \mathbb{C})$, the characterization of those graphs Γ for which G_Γ can be realized as a (quasi-) projective fundamental group follows.

Example

Let Γ be the graph above. The maximal disconnected subgraphs are $\Gamma_{\{134\}}$ and $\Gamma_{\{124\}}$. Thus:

$$\mathcal{R}_1(G_\Gamma, \mathbb{C}) = \mathbb{C}^{\{134\}} \cup \mathbb{C}^{\{124\}}.$$

But $\mathbb{C}^{\{134\}} \cap \mathbb{C}^{\{124\}} = \mathbb{C}^{\{14\}}$, which is a non-zero subspace.
But recall G_Γ is 1-formal. Thus, G_Γ is not a quasi-projective group.

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