LOWER CENTRAL SERIES AND SPLIT EXTENSIONS

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ABSTRACT. Following Lazard, we study the *N*-series of a group *G* and their associated graded Lie algebras. The main examples we consider are the lower central series and Stallings' rational and mod-*p* versions of this series. Building on the work of Massuyeau and Guaschi–Pereiro, we describe these *N*-series and Lie algebras in the case when *G* splits as a semidirect product, in terms of the relevant data for the factors and the monodromy action. As applications, we recover a well-known theorem of Falk–Randell regarding split extensions with trivial monodromy on abelianization and its mod-*p* version due to Bellingeri–Gervais and prove an analogous result for the rational lower central series of split extensions with trivial monodromy on torsion-free abelianization.

1. INTRODUCTION

1.1. *N*-series. The main goal of this paper is to analyze the behavior of lower central series and associated graded Lie algebras under split extensions of groups. We do this for both the usual lower central series, and its rational and mod-*p* analogues, working for the most part in the unified context of *N*-series.

The study of the lower central series and the associated graded Lie algebra of a group goes back to the work of Hall [19] and Magnus [26, 27] from the 1930s. In his thesis, [25], Lazard greatly developed these concepts, and introduced the notion of an *N*-series, which serves as a powerful abstraction of the lower central series. An *N*-series for a group *G* is a descending filtration by subgroups $\{K_n\}_{n\geq 1}$, starting at $K_1 = G$ and such that $[K_m, K_n] \subseteq K_{m+n}$, for all $m, n \geq 1$. Clearly, this is a central series (and thus, a normal series), and so the quotient groups, K_n/K_{n+1} , are abelian. The direct sum of these quotients, $\operatorname{gr}^K(G) = \bigoplus_{n\geq 1} K_n/K_{n+1}$, acquires the structure of a graded Lie algebra, with Lie bracket induced from the group commutator.

1.2. Lower central series. The prototypical example of an N-series is the lower central series, $\gamma(G) = \{\gamma_n(G)\}_{n \ge 1}$, that starts at $\gamma_1(G) = G$ and is defined recursively by

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 $\gamma_{n+1}(G) = [G, \gamma_n(G)]$. In a seminal paper from 1965, [38], Stallings established an important relationship between the low-dimensional homology of a group and its lower central series (LCS). In the process, he also considered the rational and mod-*p* versions of this series, which we denote by $\gamma^{\mathbb{Q}}(G)$ and $\gamma^p(G)$, respectively. Both these series (which were later investigated in depth by Cochran and Harvey [8, 9]), start at *G* and are defined recursively by

(1)
$$\gamma_{n+1}^{\mathbb{Q}}(G) = \sqrt{[G,\gamma_n^{\mathbb{Q}}(G)]} \text{ and } \gamma_{n+1}^{p}(G) = (\gamma_n^{p}(G))^{p}[G,\gamma_n^{p}(G)].$$

Here, for a subset $S \subseteq G$, we let \sqrt{S} be the isolator of S in G (i.e., the set of elements of G which have a positive power that belongs to S), and we let S^p denote the subgroup generated by all the p-th powers of elements in S. It is known that the rational and mod-plower central series are also N-series; moreover, the graded quotients of $\gamma^{\mathbb{Q}}(G)$ are torsionfree abelian groups, while those of $\gamma^p(G)$ are elementary abelian p-groups; see [35, 3, 29, 33]. The graded Lie algebras associated to $\gamma(G)$, $\gamma^{\mathbb{Q}}(G)$, and $\gamma^p(G)$ are denoted by $\operatorname{gr}(G)$, $\operatorname{gr}^{\mathbb{Q}}(G)$, and $\operatorname{gr}^p(G)$, respectively.

1.3. **Split extensions.** For most of the paper we focus our attention on groups that arise as split extensions of the form $B = A \rtimes_{\varphi} C$, where A is a normal subgroup of B with quotient group C. The projection map $B \to C$ admits a splitting $\sigma: C \to B$, which determines (and is determined by) the monodromy of the extension, $\varphi: C \to Aut(A)$; in particular, we may view C as a subgroup of B.

In order to describe the lower central series $\gamma(B)$ and the associated graded Lie algebra gr(*B*), we consider a series $L = \{L_n\}_{n \ge 1}$ of normal subgroups of *A*, which is defined inductively by setting $L_1 = A$ and letting L_{n+1} be the subgroup generated by the commutators $[A, L_n], [A, \gamma_n(C)], \text{ and } [L_n, C]$. This series was recently introduced by Guaschi and Pereiro, who showed in [17, Theorem 1.1] that the monodromy of the extension restricts to maps $\varphi: \gamma_n(C) \to \operatorname{Aut}(L_n)$, and that

(2)
$$\gamma_n(B) = L_n \rtimes_{\varphi} \gamma_n(C)$$

for all $n \ge 1$. We prove in Theorem 4.6 that the Guaschi–Pereiro series is, in fact, an *N*-series. This additional information allows us to give in Theorem 4.8 a roughly similar, yet more streamlined and transparent proof of their result. Furthermore, we show in Theorem 5.1 that the associated graded Lie algebra of *B* splits as a semidirect product of graded Lie algebras,

(3)
$$\operatorname{gr}(B) = \operatorname{gr}^{L}(A) \rtimes_{\bar{\varphi}} \operatorname{gr}(C),$$

where $\operatorname{gr}^{L}(A)$ is the associated graded Lie algebra of A with respect to the L-filtration, and where the monodromy $\overline{\varphi}$: $\operatorname{gr}(C) \to \operatorname{Der}(\operatorname{gr}^{L}(A))$ is induced by φ . As we show by means of several examples, it is not possible, in general, to replace $\operatorname{gr}^{L}(A)$ by $\operatorname{gr}(A)$ in (3).

Nevertheless, there is an important class of split extensions where this is possible. Suppose $B = A \rtimes_{\varphi} C$ is an almost direct product of groups; that is, *C* acts trivially on the abelianization $A_{ab} = gr_1(A)$. In this case, we show in Theorem 6.3 that $L_n = \gamma_n(A)$ for all

 $n \ge 1$. When combined with formulas (2) and (3), this theorem recovers two well-known results of Falk and Randell from [14], namely, the semidirect product decompositions, $\gamma_n(B) = \gamma_n(A) \rtimes_{\varphi} \gamma_n(C)$ and $\operatorname{gr}(B) = \operatorname{gr}(A) \rtimes_{\bar{\varphi}} \operatorname{gr}(C)$.

1.4. **Rational LCS.** Building on the approach outlined above, we establish analogous results for the rational and mod-*p* lower central series. In the first instance, we start by showing in Proposition 7.2 that $\gamma_n^{\mathbb{Q}}(G) = \sqrt{\gamma_n(G)}$, for all groups *G*. When $B = A \rtimes_{\varphi} C$, work of Massuyeau [29] implies that $\gamma^{\mathbb{Q}}(B)$ and \sqrt{L} are *N*-series for *B* and *A*, respectively. We then prove in Theorem 8.4 and Corollary 8.5 the following result.

Theorem 1.1. Let $B = A \rtimes_{\varphi} C$ be a semidirect product of groups. Then,

(1)
$$\sqrt[B]{\gamma_n(B)} = \sqrt[A]{L_n} \rtimes_{\varphi} \sqrt[C]{\gamma_n(C)} \text{ for all } n \ge 1$$

(2) $\operatorname{gr}^{\mathbb{Q}}(B) = \operatorname{gr}^{\sqrt{L}}(A) \rtimes_{\bar{\varphi}} \operatorname{gr}^{\mathbb{Q}}(C).$

The arguments we give parallel those in the integral case, though they do require an additional careful analysis (based on lemmas from §2.1 and §3.2) of the way commutators and powers interact in split extensions.

Noteworthy is the case of rational almost direct products, i.e., split extensions of the form $B = A \rtimes_{\varphi} C$, with *C* acting trivially on the torsion-free abelianization $A_{abf} = gr_1^{\mathbb{Q}}(A)$. For such extensions, we show in Theorem 9.4 that $\sqrt{L_n} = \gamma_n^{\mathbb{Q}}(A)$ for all $n \ge 1$. Using the above theorem, we obtain rational analogues of the results of Falk and Randell; namely,

(4)
$$\gamma_n^{\mathbb{Q}}(B) = \gamma_n^{\mathbb{Q}}(A) \rtimes_{\varphi} \gamma_n^{\mathbb{Q}}(C)$$

for all $n \ge 1$, from which we conclude in Theorem 9.5 that the rational associated graded Lie algebra of *B* decomposes as $gr^{\mathbb{Q}}(B) = gr^{\mathbb{Q}}(A) \rtimes_{\bar{\varphi}} gr^{\mathbb{Q}}(C)$.

1.5. **Mod**-*p* **LCS.** Turning now to the mod-*p* lower central series, we define by analogy with the integral and rational cases a sequence L^p of subgroups of *A*, by setting $L_1^p = A$ and letting L_{n+1}^p be the subgroup generated by $(L_n^p)^p$, $[A, L_n^p]$, $[A, \gamma_n^p(C)]$, and $[L_n^p, C]$. We then prove in Theorem 11.4 that the sequence $L^p = \{L_n^p\}_{n\geq 1}$ forms a *p*-torsion *N*-series for *A*. Furthermore, we estanlish in Theorem 11.6 and Corollary 11.7 the following result.

Theorem 1.2. Let $B = A \rtimes_{\varphi} C$ be a split extension and let p be a prime. Then,

(1) $\gamma_n^p(B) = L_n^p \rtimes_{\varphi} \gamma_n^p(C).$ (2) $\operatorname{gr}^p(B) = \operatorname{gr}^{L^p}(A) \rtimes_{\bar{\varphi}} \operatorname{gr}^p(C).$

More can be said when the split extensions $B = A \rtimes_{\varphi} C$ is a mod-*p* almost direct product, that is, when *C* acts trivially on $A_{ab} \otimes \mathbb{Z}_p = \operatorname{gr}_1^p(A)$. In this case, we show in Theorem 12.2 that $L_n^p = \gamma_n^p(A)$ for all $n \ge 1$. In conjunction with the above theorem, this allows us to recover in Theorem 12.3 a result first proved by Bellingeri and Gervais in [5], to wit,

(5)
$$\gamma_n^p(B) = \gamma_n^p(A) \rtimes_{\varphi} \gamma_n^p(C),$$

for all $n \ge 1$. As an application, we show that $\operatorname{gr}^p(B) = \operatorname{gr}^p(A) \rtimes_{\bar{\varphi}} \operatorname{gr}^p(C)$.

1.6. **Residual properties and further directions.** One of the many reasons for studying the lower central series and its rational and modular variants comes from the fact that these series control the corresponding residual properties of a group *G*. Namely, *G* is residually nilpotent, respectively, residually torsion-free nilpotent, or residually *p* if and only if the intersection of the terms of the series $\gamma(G)$, respectively $\gamma^{\mathbb{Q}}(G)$, or $\gamma^{p}(G)$ is trivial.

Now suppose that $B = A \rtimes C$ is a split extension. As noted in a special case by Falk and Randell [15], if the factors A and C are residually nilpotent groups, and C acts trivially on A_{ab} , then, as a consequence of (2), the group B is also a residually nilpotent. Applying formula (4) we show in Theorem 9.7 that an analogous result holds for a rational almost direct product of residually torsion-free nilpotent (RTFN) groups.

Theorem 1.3. Let $B = A \rtimes C$ be a split extension of RTFN groups. If C acts trivially on A_{abf} , then B is also RTFN.

When the group A is finitely generated, the hypothesis of the theorem may relaxed by only assuming that C acts trivially on $H_1(A; \mathbb{Q})$. Finally, as noted by Bellingeri and Gervais in [5], formula (5) implies that split extensions of residually p groups are residually p, provided C acts trivially on $H_1(A; \mathbb{Z}_p)$.

We pursue this investigation in [39], where we study the derived series, $G \triangleright G' \triangleright G'' \triangleright \cdots$, and the Alexander invariant, B(G) = G'/G'', as well as the rational and *p*-versions of these objects, together with the cohomology jump loci of a finitely generated group *G*. The framework developed here, and the results herein, are used in an essential way in [39], as well as in a planned sequel, [40], where we apply these techniques to the study of Milnor fibrations of hyperplane arrangements.

2. N-SERIES AND ASSOCIATED GRADED LIE ALGEBRAS

2.1. **Commutators.** Let *G* be a group. Given elements $x, y, z \in G$, we will write ${}^{x}y = xyx^{-1}$, $y^{x} = x^{-1}yx$, and $[x, y] = xyx^{-1}y^{-1}$. The following "Hall–Witt" identities then hold for all $z \in G$:

(6)
$$[x, yz] = [x, y] \cdot {}^{y}[x, z] = [x, y][x, z][[z, x], y],$$

(7)
$$[[x, y], {}^{y}z][[y, z], {}^{z}x][[z, x], {}^{x}y] = 1.$$

Given subgroups H_1 and H_2 of G, define their commutator, $[H_1, H_2]$, to be the subgroup of G generated by all elements of the form $[x_1, x_2]$ with $x_1 \in H_1$ and $x_2 \in H_2$. In particular, G' = [G, G] is the derived (or, commutator) subgroup of G.

It is readily seen that $[H_1, H_2] = [H_2, H_1]$, and that $[H_1, H_2]$ is contained in $\langle H_1, H_2 \rangle$, the subgroup generated by H_1 and H_2 . In general, $[H_1, H_2]$ is not contained in H_1 ; nevertheless, if H_1 is a normal subgroup, then $[H_1, H_2] \subseteq H_1$. Furthermore, $[H_1, H_2]$ need not be a normal subgroup, but it is normalized by both H_1 and H_2 ; if both H_1 and H_2 are normal subgroups, then $[H_1, H_2]$ is also a normal subgroup. Note also that $\langle H_1, H_2 \rangle = H_1 H_2$ precisely when H_1 and H_2 are permuting subgroups, i.e., $H_1 H_2 = H_2 H_1$.

Very useful in this context is the following "Three Subgroup Lemma" of P. Hall [19].

Lemma 2.1. Let H_1, H_2, H_3 be three subgroups of *G*, and let $N \triangleleft G$ be a normal subgroup.

(1) If $[[H_1, H_2], H_3] \subseteq N$ and $[[H_2, H_3], H_1] \subseteq N$, then $[[H_3, H_1], H_2] \subseteq N$. (2) If $H_i \lhd G$, then $[[H_1, H_2], H_3] \subseteq [[H_2, H_3], H_1] \cdot [[H_3, H_1], H_2]$.

Proof. Both assertions follow from (7); see [35, Lemma 11.1.6] and [28, Theorem 5.2], respectively. \Box

The following lemma will also be of much use later on.

Lemma 2.2. Let H_1, H_2 be subgroups of a group G. Then $[x, y^k] \equiv [x, y]^k$ modulo $[[H_1, H_2], H_2]$, for all $x \in H_1$, $y \in H_2$, and $k \ge 1$.

Proof. The proof is by induction on k, with the base case k = 1 tautologically true. By (6), we have that [x, yz] = [x, y][x, z][[z, x], y] for every $z \in H_2$. Therefore, $[x, yz] \equiv [x, y][x, z] \mod [[H_1, H_2], H_2]$. Taking $z = y^{k-1}$ and using the induction hypothesis gives $[x, y^k] \equiv [x, y][x, y^{k-1}] \equiv [x, y]^k \mod [[H_1, H_2], H_2]$, and we are done.

2.2. *N*-series. The following notion is due to Lazard [25]. An *N*-series (or, strongly central series) for a group G is a descending filtration

(8)
$$G = K_1 \ge K_2 \ge \cdots \ge K_n \ge \cdots$$

by subgroups of *G* satisfying

(9)
$$[K_m, K_n] \subseteq K_{m+n} \text{ for all } m, n \ge 1.$$

In particular, $K = \{K_n\}_{n \ge 1}$ is a *central series*, i.e., $[G, K_n] \subseteq K_{n+1}$ for all $n \ge 1$, and thus, a *normal series*, that is, $K_n \lhd G$ for all $n \ge 1$. Consequently, each quotient K_n/K_{n+1} lies in the center of G/K_{n+1} , and thus is an abelian group; furthermore, G/K_{n+1} is a nilpotent group. If, moreover, all the quotients K_n/K_{n+1} (or, equivalently, all the quotients G/K_{n+1}) are torsion-free, K is called an N_0 -series.

2.3. The lower central series. The quintessential example of an *N*-series is the *lower central series*. For a group *G*, this is the series $\gamma(G) = {\gamma_n(G)}_{n \ge 1}$, defined inductively by $\gamma_1(G) = G$ and

(10)
$$\gamma_{n+1}(G) = [G, \gamma_n(G)].$$

The fact that $\gamma(G)$ is an *N*-series was first established by P. Hall [19], using induction and Lemma 2.1; see also [28, 35, 36]. As the next, well-known lemma shows, $\gamma(G)$ exhibits the fastest descent among all central series of *G*, and thus, it is the fastest descending *N*-series of *G*.

Lemma 2.3. If $K = \{K_n\}_{n \ge 1}$ is a central series for G, then $\gamma_n(G) \le K_n$ for all $n \ge 1$.

Proof. Induction on *n*, with the base case n = 1 being obvious: assuming $\gamma_n(G) \subseteq K_n$, we have that $\gamma_{n+1}(G) = [G, \gamma_n(G)] \subseteq [G, K_n] \subseteq K_{n+1}$, and we are done.

The successive quotients of the series, $\gamma_n(G)/\gamma_{n+1}(G)$, are abelian groups. The first such quotient, $G/\gamma_2(G)$, coincides with the abelianization $G_{ab} = H_1(G; \mathbb{Z})$. If $\varphi: G \to H$ is a group homomorphism, an easy induction argument shows that $\varphi(\gamma_n(G)) = \gamma_n(\varphi(G))$, for all $n \ge 1$. Thus, $\varphi(\gamma_n(G)) \subseteq \gamma_n(H)$, with equality if φ is surjective.

By definition, *G* is *nilpotent* if $\gamma(G)$ terminates in finitely many steps. For each $n \ge 1$, the quotient group $\Gamma_n = G/\gamma_n(G)$ is a nilpotent group, to wit, the maximal (n-1)-step nilpotent quotient of *G*. Since this group is nilpotent, its torsion elements, $\text{Tors}(\Gamma_n)$, form a (characteristic) subgroup; the quotient group, $\Gamma_n/\text{Tors}(\Gamma_n)$, is the maximal (n-1)-step torsion-free nilpotent quotient of *G*.

2.4. **Residual properties.** The lower central series and other related series control certain residual properties of groups. A group *G* is said to be *residually* \mathscr{P} , where \mathscr{P} is a class of groups, if for any $g \in G$, $g \neq 1$, there exists a group $Q \in \mathscr{P}$ and an epimorphism $\psi: G \twoheadrightarrow Q$ such that $\psi(g) \neq 1$.

Lemma 2.4. A group G is residually nilpotent if and only if the intersection of its lower central series, $\gamma_{\omega}(G) := \bigcap_{n \ge 1} \gamma_n(G)$, is the trivial subgroup.

This well-known lemma is a consequence of the following observation: If $\psi: G \twoheadrightarrow Q$ is an epimorphism to a nilpotent group Q, then ψ factors through the projection $G \twoheadrightarrow G/\gamma_n(G)$, for some $n \ge 1$. Finitely generated, residually nilpotent groups are residually finite, and therefore Hopfian (i.e., not isomorphic to a proper factor group).

Example 2.5. For each $0 < m \le |n|$, let G = BS(m, n) be the Baumslag–Solitar group, with presentation $G = \langle t, a | ta^m t^{-1} = a^n \rangle$. Then G is residually finite if and only if m = 1 or m = |n|, and it is residually nilpotent if and only if m = 1 and $n \ne 2$, or $m = |n| = p^r$ for some prime p and some r > 0; see [2, 23] and references therein. In particular, BS(1, 2) is residually finite but not residually nilpotent, whereas the Klein bottle group BS(1, -1) is residually nilpotent.

2.5. The Lie algebra associated to an *N*-series. Let $K = \{K_n\}_{n \ge 1}$ be an *N*-series for a group *G*. Since this is a central series, the successive quotients, $gr_n^K(G) := K_n/K_{n+1}$, are abelian groups. Following Lazard [25], set

(11)
$$\operatorname{gr}^{K}(G) = \bigoplus_{n \ge 1} \operatorname{gr}^{K}_{n}(G).$$

Using the Witt-Hall identities (6)–(7) and the assumption that $[K_m, K_n] \subseteq K_{m+n}$ for all $m, n \ge 1$, it is readily verified that $\operatorname{gr}^K(G)$ has the structure of a graded Lie algebra over \mathbb{Z} . The addition is induced from the group multiplication, while the Lie bracket, which is induced by the group commutator, restricts to bilinear maps [,]: $\operatorname{gr}^K_m(G) \times \operatorname{gr}^K_n(G) \to \operatorname{gr}^K_{m+n}(G)$. These operations then satisfy the identities [x, x] = 0, [x, y] + [y, x] = 0, and [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 for all homogeneous elements in $\operatorname{gr}^K(G)$.

Now let *G* and *H* be two groups endowed with *N*-series $K = \{K_n\}_{n \ge 1}$ and $L = \{L_n\}_{n \ge 1}$, respectively, and let $\varphi \colon G \to H$ be a group homomorphism which is compatible with these

filtrations, i.e., $\varphi(K_n) \subseteq L_n$ for all $n \ge 1$. Then φ induces a map, $\operatorname{gr}^{K,L}(\varphi)$: $\operatorname{gr}^K(G) \to \operatorname{gr}^L(H)$. It is readily seen that this map preserves degrees and Lie brackets and respects compositions; that is, $\operatorname{gr}^{K,L}(\varphi)$ is a morphism in the category of graded Lie algebras.

2.6. The associated graded Lie algebra of a group. Returning to the setup from §2.5, consider the case when the *N*-series *K* is $\gamma(G)$, the lower central series of *G*. The resulting Lie algebra, $\operatorname{gr}^{K}(G)$, denoted simply by $\operatorname{gr}(G)$, is called the *associated graded Lie algebra* of *G* (over the ring \mathbb{Z}).

Since $\gamma_{n+1}(G) = [G, \gamma_n(G)]$, the Lie bracket map $[,]: gr_1(G) \otimes gr_n(G) \rightarrow gr_{n+1}(G)$ is surjective. It follows by induction on *n* that gr(G) is generated as a Lie algebra by its degree 1 piece, $gr_1(G) = G_{ab}$. An inductive argument now shows the following: if G_{ab} is finitely generated, then the groups $gr_n(G)$ are also finitely generated, for all $n \ge 1$.

Example 2.6. Let F_X be the free group on a set X, and let Lie(X) be the free Lie algebra on this set. Work of P. Hall [19], W. Magnus [26, 27], and E. Witt from the 1930s (see [28, 36]) shows that the canonical map $X \to \text{gr}_1(F_X)$ induces an isomorphism of graded Lie algebras, $\text{Lie}(X) \xrightarrow{\simeq} \text{gr}(F_X)$. Consequently, the groups $\text{gr}_n(F_X)$ are torsion-free; moreover, in the case when X is finite, those groups have rank equal to $\frac{1}{n} \sum_{d|n} \mu(d) |X|^{n/d}$, where $\mu: \mathbb{N} \to \{0, \pm 1\}$ is the Möbius function.

Now let $\varphi: G \to H$ be a group homomorphism. As noted in §2.3, we have $\varphi(\gamma_n(G)) = \gamma_n(\varphi(G))$ for all $n \ge 1$; it follows that φ induces a morphism of graded Lie algebras, $\operatorname{gr}(\varphi): \operatorname{gr}(G) \to \operatorname{gr}(H)$. It is readily seen that the assignment $\varphi \rightsquigarrow \operatorname{gr}(\varphi)$ is functorial; moreover, if φ is surjective, then $\operatorname{gr}(\varphi)$ is also surjective.

Given any *N*-series $K = \{K_n\}_{n \ge 1}$, recall that we have inclusions $\gamma_n(G) \subseteq K_n$ for all $n \ge 1$. Consequently, we have an induced morphism of graded Lie algebras, $gr(G) \rightarrow gr^K(G)$. In degree 1, this map is surjective, but in higher degrees it may fail to be either injective or surjective.

3. Split exact sequences and commutators

3.1. **Semidirect products.** We now switch our focus, from arbitrary groups to those that arise as a semidirect product of two other groups. To get started, consider a split exact sequence of groups,

(12)
$$1 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 1,$$

where $\beta \circ \sigma = \mathrm{id}_C$. Let $\mathrm{Aut}(A)$ be the group of automorphisms of A, with group operation $\psi_1 \cdot \psi_2 = \psi_1 \circ \psi_2$. The splitting homomorphism σ defines an action of C on A via the homomorphism $\varphi \colon C \to \mathrm{Aut}(A)$ given by

(13)
$$\alpha(\varphi(c)(a)) = \sigma(c)\alpha(a)\sigma(c)^{-1}.$$

This procedure realizes the group B as a split extension, $B = A \rtimes_{\varphi} C$; that is, the set $A \times C$, endowed with the group operation given by

(14)
$$(a_1, c_1) \cdot (a_2, c_2) = (a_1 \varphi(c_1)(a_2), c_1 c_2).$$

Conversely, every split extension $B = A \rtimes_{\varphi} C$ gives rise to a split exact sequence of the form (12) by defining $\alpha(a) = (a, 1), \beta(a, c) = c$, and $\sigma(c) = (1, c)$.

In what follows, we will identify the group *C* with its image under the splitting σ , and thus view *C* as a subgroup of $B = A \rtimes_{\varphi} C$. Likewise, we will identify *A* with its image under the inclusion α and view it as a normal subgroup of *B*. With these identifications, the action of *C* on *A* becomes the restriction of the conjugation action in *B*, that is,

(15)
$$\varphi(c)(a) = cac^{-1}.$$

Furthermore, every element $b \in B$ can be written in a unique way as a product, b = ac, for some $a \in A$ and $c \in C$.

3.2. Commutators and powers. Observe that $\varphi(c)(a) \cdot a^{-1} = [c, a]$ and $a \cdot \varphi(c)(a^{-1}) = [a, c] = [c, a]^{-1}$. Let us also note the way *C* acts on two types of commutators: if $g, c \in C$ and $a, a' \in A$, then

(16)
$$\varphi(g)([a,a']) = [\varphi(g)(a), \varphi(g)(a')],$$

(17)
$$\varphi(g)([c,a]) = [gc,a] \cdot {}^a[g,a^{-1}].$$

Moreover, conjugation by an element $x \in A$ acts on a commutator [c, a] as follows:

(18)
$$x[c,a]x^{-1} = [c,x^ca][c,x^c]^{-1}$$

(19)
$$= [c, x^{c}a][x^{c}, a][a, x].$$

We conclude this section with two lemmas that will be needed later on. In both lemmas, $B = A \rtimes C$ is a semidirect product of groups.

Lemma 3.1. Let $A_1 \leq A$ and $C_1 \leq C$, and assume $[A_1, C_1] \leq A_1$. Then $[A_1, C_1]$ is a normal subgroup of A_1 . In particular, $[A, C_1] \lhd A$ and $[A, C] \lhd A$.

Proof. By our assumption, if $c \in C_1$ and $x \in A_1$, then $x^c = [c^{-1}, x]x \in A_1$. Thus, if $a \in A_1$, then $x[c, a]x^{-1}$ belongs to $[A_1, C_1]$, by formula (18). The claim and its consequences follow at once.

Lemma 3.2. Let $A_1 \leq A$ and $C_1 \leq C$, and assume $[A_1, C_1] \leq A_1$. Then $(ac)^k \equiv a^k c^k \mod [A_1, C_1]$, for all $a \in A_1$, $c \in C_1$, and $k \ge 1$.

Proof. We prove the claim by induction on k, with the base case k = 1 being obvious. Suppose $(ac)^{k-1} \equiv a^{k-1}c^{k-1}$ modulo $[A_1, C_1]$, that is, $(ac)^{k-1} = a^{k-1}a'c^{k-1}$, for some $a' \in [A_1, C_1]$. Then

$$(ac)^{k} = a^{k-1}a'c^{k-1}ac = a^{k} \cdot [a^{-1}a', c^{k-1}][c^{k-1}, a^{-1}a'a] \cdot a^{-1}a'a \cdot c^{k}.$$

But $a^{-1}a'a \in [A_1, C_1]$ by Lemma 3.1, and we are done.

4. The lower central series of a split extension

4.1. The Guaschi–Pereiro series. Our goal in this section is to analyze the lower central series $\gamma(B) = \{\gamma_n(B)\}_{n \ge 1}$ of a split extension of groups, $B = A \rtimes_{\varphi} C$, and describe it in terms of the corresponding lower central series for the factors, $\gamma(A)$ and $\gamma(C)$. In [17], Guaschi and Pereiro associate to such a split extension a sequence of subgroups of A which plays a central role in this analysis.

This sequence, $L = \{L_n\}_{n \ge 1}$, is defined inductively by setting $L_1 = A$ and letting L_{n+1} be the subgroup of A generated by the commutators $[A, L_n]$, $[A, \gamma_n(C)]$, and $[L_n, C]$; that is,

(20)
$$L_{n+1} = \langle [A, L_n], [A, \gamma_n(C)], [L_n, C] \rangle.$$

For instance, $L_2 = \langle A', [A, C] \rangle$ and $L_3 = \langle \gamma_3(A), [A, C'], [A', C], [[A, C], A], [[A, C], C] \rangle$. Let us emphasize that this sequence depends in an essential way on the monodromy $\varphi \colon C \to \operatorname{Aut}(A)$ of the split extension $B = A \rtimes_{\varphi} C$.

Remark 4.1. If *A* is abelian, then $[A, L_n] = \{1\}$ and so the recursion simplifies to $L_{n+1} = \langle [A, \gamma_n(C)], [L_n, C] \rangle$. On the other hand, if *C* is abelian, then $\gamma_n(C) = \{1\}$ for $n \ge 2$ and so $L_{n+1} = \langle [A, L_n], [L_n, C] \rangle$ for $n \ge 2$. Finally, if both *A* and *C* are abelian, then $L_{n+1} = [L_n, C]$ for $n \ge 2$.

Example 4.2. Let $G = F_2/\gamma_3(F_2)$ be the 2-step free nilpotent group of rank 2. This group can be realized as a split extension of the form $\mathbb{Z}^2 \rtimes_{\varphi} \mathbb{Z}$, with monodromy given by the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. It is readily seen that $L_2 = \mathbb{Z}$ and $L_3 = 0$.

The next two results are proved in [17, Lemma 3.3]. For completeness, and since we will need the basic outline of the arguments later on, we provide self-contained proofs.

Lemma 4.3 ([17]). The groups L_n are normal subgroups of A.

Proof. We prove the claim by induction on *n*, with the base case $L_1 = A$ tautologically true. So assume $L_n \lhd A$. As noted in §2.1, the commutator of two normal subgroups is again a normal subgroup; thus, $[A, L_n] \lhd A$. Furthermore, by Lemma 3.1, $[A, \gamma_n(C)] \lhd A$. Finally, if $x \in A$, then $x[L_n, C]x^{-1}$ is contained in $[L_n, C] \cdot [A, L_n]$, by formula (19), and thus $x[L_n, C]x^{-1} \subseteq L_{n+1}$. Therefore, in view of definition (20), we have that $L_{n+1} \lhd A$, and the induction is complete.

Lemma 4.4 ([17]). The subgroups $\{L_n\}_{n\geq 1}$ form a descending series for A.

Proof. We need to show that, for each $n \ge 1$, we have an inclusion $L_{n+1} \subseteq L_n$, or, equivalently, that $[A, L_n]$, $[A, \gamma_n(C)]$, and $[L_n, C]$ are all included in L_n . We consider the three cases in turn.

- (a) $[A, L_n] \subseteq L_n$: follows at once from Lemma 4.3.
- (b) $[A, \gamma_n(C)] \subseteq L_n$: we prove this by induction on *n*, with the base case, $[A, C] \subseteq A$, being obvious. We have $[[A, C], \gamma_n(C)] \subseteq [A, \gamma_n(C)] \subseteq L_{n+1}$, by (20) and

 $[[A, \gamma_n(C)], C] \subseteq [L_n, C] \subseteq L_{n+1}$, by the induction hypothesis and (20). Thus, $[A, \gamma_{n+1}(C)] = [A, [C, \gamma_n(C)]]$ is contained in L_{n+1} , by Lemma 2.1, and the induction is complete.

- (c) $[L_n, C] \subseteq L_n$: we prove this again by induction on *n*, with the base case n = 1 being obvious. For the induction step, there are three sub-cases to consider.
 - (i) Clearly, $[[A, C], L_n] \subseteq [A, L_n]$, while $[A, [L_n, C]] \subseteq [A, L_n]$ by the induction hypothesis. Hence, $[[A, L_n], C] \subseteq [A, L_n] \subseteq L_{n+1}$, by Lemma 2.1 and (20).
 - (ii) We have $[[A, C], \gamma_n(C)] \subseteq [A, \gamma_n(C)] \subseteq L_{n+1}$, by (20), and $[A, [C, \gamma_n(C)]] = [A, \gamma_{n+1}(C)] \subseteq L_{n+1}$, by case (b). Thus, $[[A, \gamma_n(C)], C] \subseteq L_{n+1}$, by Lemma 2.1.

(iii) Finally, $[[L_n, C], C] \subseteq [L_n, C] \subseteq L_{n+1}$, by the induction hypothesis and (20). This shows that $[L_{n+1}, C] \subseteq L_{n+1}$, thereby completing the induction for case (c).

This ends the proof.

4.2. The *L* series is an *N*-series. Our next objective is to show that the Guaschi–Pereiro series is, in fact, an *N*-series. We start with a preparatory lemma, which will be of much use later on.

Lemma 4.5. $[L_n, \gamma_m(C)] \subseteq L_{n+m}$, for all $n, m \ge 1$.

Proof. We prove the claim by induction on *m*. The base case m = 1, which amounts to $[L_n, C] \subseteq L_{n+1}$ for all $n \ge 1$, follows directly from (20). Assume now that $[L_n, \gamma_m(C)] \subseteq L_{n+m}$, for all $n \ge 1$. Then $[[L_n, C], \gamma_m(C)] \subseteq [L_{n+1}, \gamma_m(C)] \subseteq L_{n+m+1}$, and likewise $[[L_n, \gamma_m(C)], C] \subseteq [L_{n+m}, C] \subseteq L_{n+m+1}$. Therefore, by Lemma 2.1, $[L_n, \gamma_{m+1}(C)] = [L_n, [C, \gamma_m(C)]]$ is contained in L_{n+m+1} , and so the induction step is complete.

Theorem 4.6. The subgroups $\{L_n\}_{n\geq 1}$ form an N-series for A.

Proof. From Lemma 4.4, we know that $L = \{L_n\}_{n \ge 1}$ is a descending series of (normal) subgroups of A. Thus, we only need to show that $[L_n, L_m] \subseteq L_{n+m}$ for all $n, m \ge 1$.

We prove this claim by induction on *m*. The base case m = 1, which amounts to $[L_n, A] \subseteq L_{n+1}$ for all $n \ge 1$, is built in definition (20). Assume now that $[L_n, L_m] \subseteq L_{n+m}$ for all $n \ge 1$. For the induction step, we use repeatedly Lemmas 2.1 and 4.5, as well as (20). There are three cases to consider.

- (a) $[[L_n, A], L_m] \subseteq [L_{n+1}, L_m] \subseteq L_{n+1+m}$ and $[[L_n, L_m], A] \subseteq [L_{n+m}, A] \subseteq L_{n+1+m}$. Therefore, $[L_n, [A, L_m]] \subseteq L_{n+1+m}$.
- (b) $[[L_n, A], \gamma_m(C)] \subseteq [L_{n+1}, \gamma_m(C)] \subseteq L_{n+1+m}$ and $[[L_n, \gamma_m(C)], A] \subseteq [L_{n+m}, A] \subseteq L_{n+1+m}$. Therefore, $[L_n, [A, \gamma_m(C)]] \subseteq L_{n+1+m}$.
- (c) $[[L_n, C], L_m] \subseteq [L_{n+1}, L_m] \subseteq L_{n+1+m}$ and $[[L_n, L_m], C] \subseteq [L_{n+m}, C] \subseteq L_{n+1+m}$. Therefore, $[L_n, [L_m, C]] \subseteq L_{n+1+m}$.

This shows that $[L_n, L_{m+1}] \subseteq L_{n+m+1}$ for all $n \ge 1$, thereby completing the induction. \Box

We conclude this subsection with one more lemma.

Lemma 4.7. The inclusions $\gamma_n(A) \subseteq L_n \subseteq \gamma_n(B)$ hold for all $n \ge 1$.

Proof. The first claim follows at once from Lemma 2.3 and Theorem 4.6.

The second claim is proved again by induction on *n*, with the base case n = 1 being obvious. Assuming $L_n \subseteq \gamma_n(B)$, we have that all three subgroups generating L_{n+1} in (20) are included in $[B, \gamma_n(B)]$, whence $L_{n+1} \subseteq \gamma_{n+1}(B)$.

4.3. The lower central series of a split extension. A recent result of Guaschi and Pereiro ([17, Theorem 1.1]) expresses the lower central series of a semidirect product in terms of the lower central series of the factors and the extension data. For completeness, and since we will use this approach in both the rational and modular contexts, we provide a self-contained proof of this theorem. The proof given here is by and large modeled on the original one, yet it is more condensed.

Theorem 4.8 ([17]). Let $B = A \rtimes_{\varphi} C$ be a split extension of groups. For each $n \ge 1$, the following hold.

- (1) The homomorphism $\varphi \colon C \to \operatorname{Aut}(A)$ restricts to a homomorphism $\varphi \colon \gamma_n(C) \to \operatorname{Aut}(L_n)$.
- (2) $\gamma_n(B) = L_n \rtimes_{\varphi} \gamma_n(C)$, where φ is the monodromy action from part (1).

Proof. We prove claim (1) by induction on *n*, the case n = 1 being tautological. So assume the map $\varphi: C \to \operatorname{Aut}(A)$ restricts to a map $\varphi: \gamma_{n-1}(C) \to \operatorname{Aut}(L_{n-1})$. Let *g* be in $\gamma_n(C)$, and thus in $\gamma_{n-1}(C)$, too. There are three cases to consider.

- (a) Since $\varphi(g)$ leaves the subgroup L_{n-1} invariant, formula (16) implies that $\varphi(g)$ also leaves $[A, L_{n-1}]$ invariant.
- (b) Since $gc \in \gamma_{n-1}(C)$ for $c \in \gamma_{n-1}(C)$, formula (17) and Lemma 3.1 imply that $\varphi(g)$ leaves $[A, \gamma_{n-1}(C)]$ invariant.
- (c) By Lemma 3.1, $[L_{n-1}, C]$ is a normal subgroup of *A*. Formula (17) now implies that $\varphi(g)$ leaves $[L_{n-1}, C]$ invariant.

Given the way the subgroup L_n was defined in (20), it follows that $\varphi(g)$ leaves it invariant. Since the map $\varphi(g): A \to A$ is injective, its restriction to L_n is also injective. To show that the map $\varphi(g): L_n \to L_n$ is surjective, we go through the above three cases one more time. In case (a), the surjectivity of the maps $\varphi(g): A \to A$ and $\varphi(g): L_{n-1} \to L_{n-1}$ yields the claim. In cases (b) and (c) the claim follows from (17), by observing that $[c, a] = \varphi(g)(x)$, where $x = [g^{-1}c, a] \cdot {}^a[g^{-1}, a^{-1}]$.

Claim (2) is also proved by induction on *n*, with the case n = 1 being tautological. Assume that $\gamma_{n-1}(B) = L_{n-1} \rtimes_{\varphi} \gamma_{n-1}(C)$. To show that $L_n \rtimes_{\varphi} \gamma_n(C) \subseteq \gamma_n(B)$, observe that $\gamma_n(C) \subseteq \gamma_n(B)$, and also $L_n \subseteq \gamma_n(B)$, by Lemma 4.7. For the reverse inclusion, the induction hypothesis and definition (20) show that

(21)
$$\gamma_n(B) = [B, \gamma_{n-1}(B)] = [A \rtimes_{\varphi} C, L_{n-1} \rtimes_{\varphi} \gamma_{n-1}(C)] \subseteq L_n \rtimes_{\varphi} \gamma_n(C).$$

This completes the proof.

5. The associated graded Lie algebra of a split extension

5.1. Split extensions of Lie algebras. Before proceeding, let us review some notions from the realm of Lie algebras. Given a graded Lie algebra g, define the Lie algebra of degree-0 derivations, Der(g), as the additive group of degree-preserving endomorphisms $\delta: g \to g$ with the property that $\delta[x, y] = [\delta x, y] + [x, \delta y]$ for all $x, y \in g$, and with Lie bracket given by $[\delta, \delta'] = \delta \circ \delta' - \delta' \circ \delta$. Now let

(22)
$$0 \longrightarrow \mathfrak{n} \xrightarrow{\iota} \mathfrak{g} \xrightarrow{\tau} \mathfrak{h} \longrightarrow 0$$

be a split exact sequence of graded Lie algebras. Then g can be identified with the semidirect product $\mathfrak{n} \rtimes_{\theta} \mathfrak{h}$, where the monodromy $\theta \colon \mathfrak{h} \to \text{Der}(\mathfrak{n})$ is the Lie algebra map defined by

(23)
$$\iota(\theta(y)(x)) = [\tau(y), \iota(x)]$$

for $x \in \mathfrak{n}$ and $y \in \mathfrak{h}$. That is, $\mathfrak{g} = \mathfrak{n} \times \mathfrak{h}$ as an abelian group, with Lie bracket given by $[(x_1, y_1), (x_2, y_2)] = ([x_1, x_2] + \theta(y_1)(x_2) - \theta(y_2)(x_1), [y_1, y_2]).$

5.2. The associated graded Lie algebra of a split extension. Returning to our previous setup, consider a split extension of groups, $B = A \rtimes_{\varphi} C$, and recall that such an extension defines an *N*-series, $L = \{L_n\}_{n \ge 1}$, for *A*. Let $\operatorname{gr}^L(A)$ be the graded Lie algebra associated to this *N*-series according to formula (11). As noted in Lemma 4.7, the inclusion map $\alpha : A \to B$ sends L_n to $\gamma_n(B)$. We thus have an induced morphism of associated graded Lie algebras, $\operatorname{gr}^L(\alpha) : \operatorname{gr}^L(A) \to \operatorname{gr}(B)$.

Theorem 5.1. Let $1 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 1$ be a split exact sequence of groups, with monodromy $\varphi \colon C \to \operatorname{Aut}(A)$, and let L be the N-series for A defined in (20). There is then an induced split exact sequence of graded Lie algebras,

(24)
$$0 \longrightarrow \operatorname{gr}^{L}(A) \xrightarrow{\operatorname{gr}^{L}(\alpha)} \operatorname{gr}(B) \xrightarrow{\operatorname{gr}(\beta)} \operatorname{gr}(C) \longrightarrow 0$$

Consequently, $\operatorname{gr}(B) \cong \operatorname{gr}^{L}(A) \rtimes_{\bar{\varphi}} \operatorname{gr}(C)$, where the monodromy $\bar{\varphi} \colon \operatorname{gr}(C) \to \operatorname{Der}(\operatorname{gr}^{L}(A))$ is the map of Lie algebras induced by φ .

Proof. By Theorem 4.8, for each $n \ge 1$, there is a split exact sequence,

(25)
$$1 \longrightarrow L_n \xrightarrow{\alpha} \gamma_n(B) \xrightarrow{\beta} \gamma_n(C) \longrightarrow 1.$$

For each $n \ge 1$, the Snake Lemma yields split exact sequences of abelian groups,

(26)
$$0 \to L_n/L_{n+1} \xrightarrow{\operatorname{gr}_n^L(\alpha)} \gamma_n(B)/\gamma_{n+1}(B) \xrightarrow{\operatorname{gr}_n(\beta)} \gamma_n(C)/\gamma_{n+1}(C) \to 0.$$

In turn, these sequences assemble in the exact sequence (24) of graded Lie algebras.

If $\sigma: C \to B$ is a splitting for the map β , the above construction induces a splitting $gr(\sigma): gr(C) \to gr(B)$ for $gr(\beta)$. The homomorphism $\varphi: C \to Aut(A)$ defined by (13) in

terms of the splitting σ and conjugation in *B* induces a morphism of graded Lie algebras, $\bar{\varphi}$: gr(*C*) \rightarrow Der(gr^{*L*}(*A*)), defined by (23) in terms of the splitting gr(σ) and the Lie bracket in gr(*B*). This realizes the associated graded Lie algebra of *B* as a split extension of gr(*C*) with gr^{*L*}(*A*) with monodromy $\bar{\varphi}$, as claimed.

5.3. **Discussion and examples.** In general, we cannot replace $gr^{L}(A)$ by gr(A) in Theorem 5.1. Indeed, the natural map of graded Lie algebras, $gr(A) \rightarrow gr^{L}(A)$, need not be an isomorphism, due to the fact that the inclusions $\gamma_n(A) \hookrightarrow L_n$ need not be surjective. We illustrate this phenomenon with a few examples.

Example 5.2. Let $K = BS(1, -1) = \langle a, t | tat^{-1} = a^{-1} \rangle$ be the fundamental group of the Klein bottle. Then *K* is a split extension of the form $A \rtimes_{\varphi} C$, where $C = \langle t \rangle$ acts by inversion on $A = \langle a \rangle$. It is readily verified that $L_n = \langle a^{2^{n-1}} \rangle$ for $n \ge 1$ (see [17, p. 19]), and thus $gr_n^L(A) = \mathbb{Z}_2$ for $n \ge 1$, although of course $\gamma_n(A) = \{1\}$ and $gr_n(A) = 0$ for n > 1. It also follows from Theorem 4.8 that $\gamma_n(K) = L_n$ for n > 1, and thus $\gamma_{\omega}(K) = \{1\}$, verifying the aforementioned known fact that K = BS(1, -1) is residually nilpotent.

Example 5.3. Let $G = \pi_1(X)$ be the fundamental group of a knot complement, $X = S^3 \setminus K$. For simplicity, we assume the knot is fibered and non-trivial, so that X is the mapping torus of a homeomorphism $h: S \to S$, where S is a punctured orientable surface of genus g > 0. Then $G = F \rtimes_{\varphi} \mathbb{Z}$, where $F = \pi_1(S)$ is a free group of rank 2g and the monodromy $\varphi: \mathbb{Z} \to \operatorname{Aut}(F)$ sends 1 to the automorphism $h_*: \pi_1(S) \to \pi_1(S)$. The projection map $\beta: G \to \mathbb{Z}$ induces an isomorphism $\beta_{ab}: G_{ab} \xrightarrow{\simeq} \mathbb{Z}$, which extends to an isomorphism $gr(\beta): gr(G) \xrightarrow{\simeq} gr(\mathbb{Z})$, concentrated in degree 1. Clearly, $gr(F) = \operatorname{Lie}(\mathbb{Z}^{2g})$ yet $\operatorname{gr}^L(F) = 0$, showing how far apart these two Lie algebras are in this case.

Example 5.4. The Formanek–Procesi "poison group" is a split extension of the form $G = A \rtimes_{\varphi} C$, with factors $A = F_3 = \langle a_1, a_2, a_3 \rangle$ and $C = F_2 = \langle c_1, c_2 \rangle$, and with monodromy action given by $\varphi(c_i)(a_j) = a_j$ and $\varphi(c_i)(a_3) = a_3a_i$, for $1 \leq i, j \leq 2$. Its notoriety comes from the fact that G (and thus Aut (F_n) for $n \geq 3$, or any other group in which this poison group embeds) does not admit a finite-dimensional linear representation. In [10, Proposition 7.1], Cohen, Cohen, and Prassidis analyzed the associated graded Lie algebra of G, and concluded that the kernel of the projection map $gr(G) \rightarrow gr(C)$ is $\text{Lie}(\mathbb{Z}) = \mathbb{Z}$, generated by a_3 . By Theorem 5.1, though, this kernel is the Lie algebra $gr^L(A)$, which indeed has degree 1 piece equal to $\mathbb{Z} = \langle a_3 \rangle$, but also has degree 2 piece equal to $\mathbb{Z}^2 = \langle [a_1, a_3], [a_2, a_3] \rangle$, and so on. In fact, it follows from [37, 30] that $gr_n^L(A) \cong \text{Lie}_{n-1}(\mathbb{Z}^2)$ for $n \geq 2$, although of course $gr(A) = \text{Lie}(\mathbb{Z}^3)$.

Nevertheless, as we shall see in the next section, there is a noteworthy class of split extensions of the form $A \rtimes C$ for which all the maps $\gamma_n(A) \hookrightarrow L_n$ are isomorphisms, and thus $\operatorname{gr}(A) \cong \operatorname{gr}^L(A)$.

6. TRIVIAL ACTION ON ABELIANIZATION

6.1. *N*-series for almost direct products. Once again, let $B = A \rtimes_{\varphi} C$ be a split extension of groups, with monodromy $\varphi \colon C \to \operatorname{Aut}(A)$. Following [14], we say that *B* is an *almost direct product* of *C* and *A* if *C* acts trivially on the abelianization $A_{ab} = H_1(A; \mathbb{Z})$. In other words, the monodromy of the extension factors through a map $\varphi \colon C \to \mathcal{T}(A)$, where $\mathcal{T}(A) \coloneqq \operatorname{ker} (\operatorname{Aut}(A) \to \operatorname{Aut}(A_{ab}))$ is the Torelli group of *A*.

On elements, the above condition says that $\varphi(c)(a) \cdot a^{-1} \in A'$, for all $c \in C$ and $a \in A$. If we view *C* as a subgroup of *G* via the splitting $\sigma \colon C \to B$, so that $\varphi(c)(a) \cdot a^{-1} = [c, a]$, the condition reads more simply as

$$[A,C] \subseteq \gamma_2(A)$$

As noted in [6, Proposition 6.3], the property of a split extension $B = A \rtimes_{\varphi} C$ being an almost direct product does not depend on the choice of splitting. That is, if $\sigma' \colon C \to B$ is another splitting of the projection $\beta \colon B \to C$, and if $\varphi' \colon C \to \operatorname{Aut}(A)$ is the corresponding monodromy action, then the split extension $B = A \rtimes_{\varphi'} C$ is again an almost direct product.

Example 6.1. Let P_n be the Artin pure braid group on *n* strands. We then have a split exact sequence, $1 \rightarrow F_{n-1} \rightarrow P_n \rightarrow P_{n-1} \rightarrow 1$, with monodromy given by the Artin embedding, $P_{n-1} \rightarrow \operatorname{Aut}(F_{n-1})$. Since pure braids act trivially on $H_1(F_{n-1};\mathbb{Z})$, the extension is an almost direct product. More generally, fundamental groups of complements of fiber-type arrangements can be realized as iterated almost direct products of finitely generated free groups, see [14].

Example 6.2. Let $G_{\Gamma} = \langle v \in V | [v, w] = 1$ if $\{v, w\} \in E \rangle$ be the right-angled Artin group associated to a finite simplicial graph $\Gamma = (V, E)$. The corresponding Bestvina–Brady group, N_{Γ} , is the kernel of the homomorphism $G_{\Gamma} \twoheadrightarrow \mathbb{Z}$ that sends each generator $v \in V$ to $1 \in \mathbb{Z}$, see [7]. As shown in [32], if Γ is connected, then the group \mathbb{Z} acts trivially on $H_1(N_{\Gamma}; \mathbb{Z})$, and so the split extension $G_{\Gamma} = N_{\Gamma} \rtimes \mathbb{Z}$ is an almost direct product.

6.2. The *L*-series of an almost direct product. Here is the key result of this section.

Theorem 6.3. Let $B = A \rtimes_{\varphi} C$ be an almost direct product of groups, and let $L = \{L_n\}_{n \ge 1}$ be the corresponding N-series for A. Then $L_n = \gamma_n(A)$ for each $n \ge 1$.

Proof. By Lemma 4.7, we have that $\gamma_n(A) \subseteq L_n$, for all $n \ge 1$. We prove the reverse inclusion by induction on n, with the base case n = 1 being obvious. So assume $L_r \subseteq \gamma_r(A)$ for $r \le n$. In view of (20), we then have

(28)
$$[\gamma_{n-1}(A), C] \subseteq \gamma_n(A), \quad [A, \gamma_{n-1}(C)] \subseteq \gamma_n(A),$$

and so $[[A, \gamma_{n-1}(C)], A]$ is included in $[\gamma_n(A), A] = \gamma_{n+1}(A)$. Thus, by Lemma 2.1, $[[A, A], \gamma_{n-1}(C)]$ is also included in $\gamma_{n+1}(A)$; that is,

(29)
$$[\gamma_2(A), \gamma_{n-1}(C)] \subseteq \gamma_{n+1}(A).$$

For the induction step, it is enough to show that all three subgroups from (20) that generate L_{n+1} are included in $\gamma_{n+1}(A)$; we do this next, keeping in mind that, by the induction hypothesis, $L_n = \gamma_n(A)$.

- (a) $[A, \gamma_n(A)] \subseteq \gamma_{n+1}(A)$. This is part of the definition of $\gamma(A)$.
- (b) $[\gamma_n(A), C] \subseteq \gamma_{n+1}(A)$. Since $\gamma(A)$ is an *N*-series, we have

$$\begin{split} & \left[\left[A, C \right], \gamma_{n-1}(A) \right] \underset{(27)}{\subseteq} \left[\gamma_2(A), \gamma_{n-1}(A) \right] = \gamma_{n+1}(A), \\ & \left[\left[\gamma_{n-1}(A), C \right], A \right] \underset{(28)}{\subseteq} \left[\gamma_n(A), A \right] = \gamma_{n+1}(A). \end{split}$$

Hence, by Lemma 2.1, $[\gamma_n(A), C] = [[\gamma_{n-1}(A), A], C]$ is included in $\gamma_{n+1}(A)$.

(c) $[A, \gamma_n(C)] \subseteq \gamma_{n+1}(A)$. We have

$$\begin{split} & [[A,C],\gamma_{n-1}(C)] \underset{(27)}{\subseteq} [\gamma_2(A),\gamma_{n-1}(C)] \underset{(29)}{\subseteq} \gamma_{n+1}(A) \\ & [[A,\gamma_{n-1}(C)],C] \underset{(28)}{\subseteq} [\gamma_n(A),C] \underset{(b)}{\subseteq} \gamma_{n+1}(A). \end{split}$$

Hence, by Lemma 2.1, $[A, \gamma_n(C)] = [A, [\gamma_{n-1}(C), C]]$ is included in $\gamma_{n+1}(A)$. This verifies the induction step, and thus completes the proof.

6.3. **Applications.** We now derive several applications of Theorem 6.3, based on the results from the previous section. We start by recovering a technical result of Falk and Randell ([14, Lemma 3.4]; the case m = 1 is [14, Lemma 3.3]).

Corollary 6.4 ([14]). *Let* $B = A \rtimes C$ *be an almost direct product. Then* $[\gamma_n(A), \gamma_m(C)] \subseteq \gamma_{n+m}(A)$, for all $n, m \ge 1$.

Proof. By Theorem 6.3, $\gamma_n(A) = L_n$. The claim now follows from Lemma 4.5.

Next, we recover a well-known result of Falk and Randell [14, p. 85].

Corollary 6.5 ([14]). Let $B = A \rtimes_{\varphi} C$ be an almost direct product. Then

- (1) $\gamma_n(B) = \gamma_n(A) \rtimes_{\varphi} \gamma_n(C)$ for all $n \ge 1$.
- (2) The corresponding split exact sequence, $1 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 1$, restricts to split exact sequences $1 \rightarrow \gamma_n(A) \xrightarrow{\alpha} \gamma_n(B) \xrightarrow{\beta} \gamma_n(C) \rightarrow 1$ for all $n \ge 1$.

Proof. Follows at once from Theorems 4.8 and 6.3.

The following corollary extends another result of Falk and Randell ([15, Theorem 2.6]) to a more general setting.

Corollary 6.6 ([15]). Suppose $B = A \rtimes_{\varphi} C$ is an almost direct product of two residually nilpotent groups. Then B is also residually nilpotent.

Proof. Let $1 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 1$, be the corresponding split exact sequence. By Lemma 2.4, we have that $\gamma_{\omega}(A) = \{1\}$ and $\gamma_{\omega}(C) = \{1\}$. Let $g \in \gamma_{\omega}(B)$. Then $\beta(g) \in \gamma_{\omega}(C)$, and so $\beta(g) = 1$. Thus, there is an element $a \in A$ such that $\alpha(a) = g$.

Now fix an index $n \ge 1$. Then $g \in \gamma_n(B)$, and since $\beta(g) = 1$, Corollary 6.5 ensures the existence of an element $a_n \in \gamma_n(A)$ such that $\alpha(a_n) = g$. But the map $\alpha \colon A \to B$ is injective, and so we must have $a = a_n$, whence $a \in \gamma_n(A)$. It follows that $a \in \gamma_{\omega}(A)$, and thus a = 1. Therefore, g = 1, and we are done.

Next, we recover another well-known result of Falk and Randell ([14, Theorem 3.1]), in a slightly stronger form.

Corollary 6.7 ([14]). Let $1 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 1$ be a split exact sequence of groups. Suppose the resulting action, $\varphi: C \rightarrow \text{Aut}(A_{ab})$, is trivial. Then the sequence of induced maps,

(30)
$$0 \longrightarrow \operatorname{gr}(A) \xrightarrow{\operatorname{gr}(\alpha)} \operatorname{gr}(B) \xrightarrow{\operatorname{gr}(\beta)} \operatorname{gr}(C) \longrightarrow 0,$$

is a split exact sequence of graded Lie algebras. Thus, $gr(B) \cong gr(A) \rtimes_{\bar{\varphi}} gr(C)$, where the monodromy $\bar{\varphi}$: $gr(C) \rightarrow Der(gr(A))$ is the map of Lie algebras induced by φ .

Proof. By Theorem 6.3, we have that $L_n = \gamma_n(A)$ for all $n \ge 1$, and so $\operatorname{gr}^L(A) = \operatorname{gr}(A)$. Both claims now follow from Theorem 5.1.

In [1, Proposition 2], Bardakov and Bellingeri obtain as an application of Corollary 6.5 the following result.

Corollary 6.8 ([1]). Suppose $B = A \rtimes_{\varphi} C$ is an almost direct product. Then $\sqrt[B]{\gamma_n(B)} = \sqrt[A]{\gamma_n(A)} \rtimes_{\varphi} \sqrt[C]{\gamma_n(C)}$.

We will strengthen this result in Theorem 9.4, where we will show that the hypothesis that *C* acts trivially on A_{ab} can be replaced by the much weaker hypothesis that *C* acts trivially on A_{abf} .

7. RATIONAL LOWER CENTRAL SERIES

7.1. **Isolators.** For a subset $S \subseteq G$, we let

(31)
$$\sqrt{S} \coloneqq \sqrt[6]{S} = \{g \in G \mid g^m \in S \text{ for some } m \in \mathbb{N}\}$$

be the *isolator* (or, *root set*) of *S* in *G*. Clearly, $S \subseteq \sqrt{S}$ and $\sqrt{\sqrt{S}} = \sqrt{S}$. Moreover, if $\varphi: G \to H$ is a homomorphism, and $\varphi(S) \subseteq T$, then $\varphi(\sqrt[c]{S}) \subseteq \sqrt[H]{T}$.

The isolator of a subgroup of G need not be a subgroup; for instance, $\sqrt[G]{\{1\}}$ is equal to Tors(G), the set of torsion elements in G, which is not a subgroup in general. Nevertheless, if G is nilpotent, then the isolator of any subgroup of G is again a subgroup. A subgroup

 $H \leq G$ is said to be *isolated* if $\sqrt{H} = H$; clearly, \sqrt{H} is the intersection of all isolated subgroups of G containg H.

Now suppose $N \lhd G$ is a normal subgroup. Then

(32)
$$\sqrt[G]{N} = \pi^{-1}(\operatorname{Tors}(G/N)),$$

where $\pi: G \to G/N$ is the canonical projection, and so $\sqrt[G]{N}/N \cong \text{Tors}(G/N)$. In particular, *N* is isolated if and only if G/N is torsion-free. When $K = \{K_n\}_{n \ge 1}$ is an *N*-series for *G*, each quotient G/K_n is nilpotent, and so the set of torsion elements forms a normal subgroup of G/K_n ; hence, $\sqrt{K_n}$ is a normal subgroup of *G*. The following result of Massuyeau ([29, Lemma 4.4]), builds on this observation.

Proposition 7.1 ([29]). Suppose $K = \{K_n\}_{n \ge 1}$ is an N-series for G. Then each isolator $\sqrt{K_n}$ is a normal subgroup of G; moreover, $\sqrt{K} := \{\sqrt{K_n}\}_{n \ge 1}$ is an N_0 -series for G.

In fact, \sqrt{K} is the fastest descending N_0 -series containing K: if L is any other N_0 -series, and $K_n \leq L_n$ for all $n \geq 1$, then $\sqrt{K_n} \leq L_n$ for all $n \geq 1$.

7.2. The rational lower central series. The rational version of the lower central series was introduced by Stallings in [38] (see also [8, 31]). For a group *G*, this is the series $\gamma^{\mathbb{Q}}(G) = \{\gamma_n^{\mathbb{Q}}(G)\}_{n \ge 1}$, defined inductively by $\gamma_1^{\mathbb{Q}}(G) = G$ and

(33)
$$\gamma_{n+1}^{\mathbb{Q}}(G) = \sqrt{[G,\gamma_n^{\mathbb{Q}}(G)]}.$$

As observed in [38], $\gamma^{\mathbb{Q}}(G)$ is the most rapidly descending central series whose successive quotients are torsion-free abelian groups; in particular, it is the fastest descending N_0 -series for G. Clearly, the terms of this series are fully invariant subgroups of G, i.e., $\varphi(\gamma_n^{\mathbb{Q}}(G)) \subseteq \gamma_n^{\mathbb{Q}}(H)$ for every homomorphism $\varphi: G \to H$. The next proposition gives another, very useful description of this series.

Proposition 7.2. $\gamma_n^{\mathbb{Q}}(G) = \sqrt{\gamma_n(G)}$, for all $n \ge 1$.

Proof. First we show that $\sqrt{\gamma_n(G)} \subseteq \gamma_n^{\mathbb{Q}}(G)$ for all *n*. We use induction on *n*, the base case n = 1 being clear. For the induction step, we have

$$\sqrt{\gamma_{n+1}(G)} = \sqrt{[G,\gamma_n(G)]} \subseteq \sqrt{\left[G,\sqrt{\gamma_n(G)}\right]} \subseteq \sqrt{\left[G,\gamma_n^{\mathbb{Q}}(G)\right]} = \gamma_{n+1}^{\mathbb{Q}}(G).$$

For the reverse inclusion, $\gamma_n^{\mathbb{Q}}(G) \subseteq \sqrt{\gamma_n(G)}$, we use again induction on *n*. For the induction step, we have

$$\gamma_{n+1}^{\mathbb{Q}}(G) = \sqrt{\left[G, \gamma_n^{\mathbb{Q}}(G)
ight]} \subseteq \sqrt{\left[G, \sqrt{\gamma_n(G)}
ight]} \subseteq \sqrt{\sqrt{\gamma_{n+1}(G)}} = \sqrt{\gamma_{n+1}(G)},$$

where at the third step we used that, as a consequence of Proposition 7.1, $\sqrt{\gamma(G)}$ is a normal series. This completes the proof.

It follows from (33) and the above proposition that, for all $n \ge 1$.

(34)
$$\sqrt{\gamma_{n+1}(G)} = \sqrt{\left[G, \sqrt{\gamma_n(G)}\right]}.$$

Corollary 7.3. The series $\gamma^{\mathbb{Q}}(G)$ is an N_0 -series.

Proof. By [34, Lemma 1.3] and [35, Lemma 11.1.8], or as a corollary to Proposition 7.1, we have that $\sqrt{\gamma(G)}$ is an N_0 -series. The claim now follows from Proposition 7.2.

Note that $G/\gamma_2^{\mathbb{Q}}(G) = G_{abf}$, where $G_{abf} = G_{ab}/\operatorname{Tors}(G_{ab})$ is the maximal torsion-free abelian quotient of G. More generally, as pointed out in [16, Appendix A],

(35)
$$G/\gamma_n^{\mathbb{Q}}(G) = \Gamma_n/\operatorname{Tors}(\Gamma_n),$$

where $\Gamma_n = G/\gamma_n(G)$, and this property defines the series $\gamma^{\mathbb{Q}}(G)$. Consequently, the quotients $G/\gamma_n^{\mathbb{Q}}(G)$ are torsion-free nilpotent groups, and the series $\gamma^{\mathbb{Q}}(G)$ is the fastest descending normal series with this property. Furthermore, it follows from formula (32) that $\gamma_n^{\mathbb{Q}}(G)/\gamma_n(G) \cong \text{Tors}(\Gamma_n)$.

7.3. Alternative definitions. We now describe several other ways in which one can define series which turn out to coincide the rational lower central series. The first one is the series $\gamma^{\mathbb{Z}}(G) = \{\gamma_n^{\mathbb{Z}}(G)\}_{n \ge 1}$, defined inductively by Bass and Lubotzky in [3, Definition 7.1] by means of the following three conditions:

(1) $\gamma_1^{\mathbb{Z}}(G) = G;$ (2) $[G, \gamma_n^{\mathbb{Z}}(G)] \leq \gamma_{n+1}^{\mathbb{Z}}(G);$ (3) $\gamma_{n+1}^{\mathbb{Z}}(G)/[G, \gamma_n^{\mathbb{Z}}(G)] = \operatorname{Tors}(\gamma_n^{\mathbb{Z}}(G)/[G, \gamma_n^{\mathbb{Z}}(G)]).$

The second condition implies that $\gamma_n^{\mathbb{Z}}(G) \triangleleft G$, while the third one is equivalent to $\gamma_n^{\mathbb{Z}}(G)/\gamma_{n+1}^{\mathbb{Z}}(G) = (\Gamma_n^{\mathbb{Z}})_{abf}$, where $\Gamma_n^{\mathbb{Z}} = \gamma_n^{\mathbb{Z}}(G)/[G, \gamma_n^{\mathbb{Z}}(G)]$. As shown in [3, Proposition 7.2(b)], for each $n \ge 1$ there is an exact sequence

(36)
$$1 \longrightarrow \gamma_n^{\mathbb{Z}}(G)/\gamma_n(G) \longrightarrow \Gamma_n \longrightarrow G/\gamma_n^{\mathbb{Z}}(G) \longrightarrow 1$$
,

where $\gamma_n^{\mathbb{Z}}(G)/\gamma_n(G) = \text{Tors}(\Gamma_n)$. It follows that the successive quotients of the series $\gamma^{\mathbb{Z}}(G)$ are torsion-free, and the natural maps $\gamma_n(G)/\gamma_{n+1}(G) \rightarrow \gamma_n^{\mathbb{Z}}(G)/\gamma_{n+1}^{\mathbb{Z}}(G)$ have finite kernel and cokernel. For n = 1, we have that $\gamma_1^{\mathbb{Z}}(G)/\gamma_2^{\mathbb{Z}}(G) = G_{abf}$, and the previous map is simply the projection $G_{ab} \rightarrow G_{abf}$. Finally, as noted in [3], $\gamma^{\mathbb{Z}}(G)$ is the fastest descending central series whose successive quotients are torsion-free. Therefore, $\gamma_n^{\mathbb{Z}}(G) = \gamma_n^{\mathbb{Q}}(G)$, for all $n \ge 1$.

Closely related is the descending series $\gamma^T(G) = \{\gamma_n^T(G)\}_{n \ge 1}$, defined by Koberda [22] by setting $\gamma_1^T(G) = G$ and $\gamma_n^T(G) = \ker (G \twoheadrightarrow \Gamma_n \twoheadrightarrow \Gamma_n / \operatorname{Tors}(\Gamma_n))$. It follows from either (35) or (36) that $\gamma_n^T(G) = \gamma_n^{\mathbb{Q}}(G) = \gamma_n^{\mathbb{Z}}(G)$ for all *n*.

Finally, another description is given in [21, §10.4], where Hillman defines inductively a sequence of subgroups, $\{G_n^{\mathbb{Q}}\}_{n\geq 1}$, as follows: $G_1^{\mathbb{Q}} = G$ and $G_{n+1}^{\mathbb{Q}}$ is the preimage in G of $\text{Tors}(G/[G, G_n^{\mathbb{Q}}])$. Then all the quotients $G/G_n^{\mathbb{Q}}$ are torsion-free nilpotent groups, and the

series $\{G_n^{\mathbb{Q}}\}_{n \ge 1}$ is the fastest descending normal series with this property. Consequently, $G_n^{\mathbb{Q}} = \gamma_n^{\mathbb{Q}}(G)$ for all *n*.

7.4. **RTFN groups.** A group G is said to be *residually torsion-free nilpotent* (RTFN) if every non-trivial element can be detected in a torsion-free nilpotent quotient. Clearly, such groups are residually nilpotent, but the converse does not hold. For instance, every finite nilpotent group is residually nilpotent, but not RTFN.

Lemma 7.4. A group G is RTFN if and only if the intersection of its rational lower central series, $\gamma_{\omega}^{\mathbb{Q}}(G) \coloneqq \bigcap_{n \ge 1} \gamma_n^{\mathbb{Q}}(G)$, is the trivial subgroup.

A proof of this well-known result is given in [3, Proposition 7.2(e)]. By [34, Ch. VI, Theorem 2.26], the group G is RTFN precisely when the group-algebra $\mathbb{Q}[G]$ is residually nilpotent, that is, $\bigcap_{n\geq 1} I^n = \{0\}$, where I is the augmentation ideal. When G is finitely generated, the RTFN condition is equivalent to the injectivity of the canonical map $G \rightarrow \mathfrak{M}(G)$ to its prounipotent (or, Malcev) completion, where $\mathfrak{M}(G)$ is the set of group-like elements in the Hopf algebra obtained by completing $\mathbb{Q}[G]$ with respect to the *I*-adic filtration (see for instance [41] and references therein).

Finitely generated RTFN groups are torsion-free and bi-orderable. If *G* is residually nilpotent and $gr_n(G)$ is torsion-free for $n \ge 1$, then *G* is residually torsion-free nilpotent. Examples of RTFN groups include free groups [26], orientable surface groups [4], right-angled Artin groups [13], and the pure braid groups [15].

7.5. The rational associated graded Lie algebra. Recall that the successive quotients of the rational lower central series, $gr_n^{\mathbb{Q}}(G) = \gamma_n^{\mathbb{Q}}(G)/\gamma_{n+1}^{\mathbb{Q}}(G)$, are torsion-free abelian groups. The direct sum of these quotients,

(37)
$$\operatorname{gr}^{\mathbb{Q}}(G) \coloneqq \bigoplus_{n \ge 1} \gamma_n^{\mathbb{Q}}(G) / \gamma_{n+1}^{\mathbb{Q}}(G),$$

with Lie bracket induced from the group commutator, is the *rational associated graded Lie* algebra of G. Since the terms of $\gamma^{\mathbb{Q}}(G)$ are fully invariant subgroups of G, this construction is again functorial.

Since $\gamma_n(G) \leq \gamma_n^{\mathbb{Q}}(G)$ for all *n*, we have an induced map between associated graded Lie algebras, $\operatorname{gr}(G) \to \operatorname{gr}^{\mathbb{Q}}(G)$, which is natural with respect to group homomorphisms. In [3, Proposition 7.2], Bass and Lubotzky show that, for each $n \geq 1$, there is an exact sequence,

$$(38) \quad 1 \longrightarrow \frac{\gamma_n(G) \cap \gamma_{n+1}^{\mathbb{Q}}(G)}{\gamma_{n+1}(G)} \longrightarrow \operatorname{gr}_n(G) \longrightarrow \operatorname{gr}_n^{\mathbb{Q}}(G) \longrightarrow \frac{\gamma_n^{\mathbb{Q}}(G)}{\gamma_n(G)\gamma_{n+1}^{\mathbb{Q}}(G)} \longrightarrow 1.$$

Since the groups on the left and the right are both finite, the next result follows.

Proposition 7.5 ([3]). *For a group G, the following hold.*

- (1) The map $gr(G) \to gr^{\mathbb{Q}}(G)$ has torsion kernel and cokernel in each degree.
- (2) The map $gr(G) \otimes \mathbb{Q} \to gr^{\mathbb{Q}}(G) \otimes \mathbb{Q}$ is an isomorphism.

As a consequence, the rational Lie algebra $\operatorname{gr}^{\mathbb{Q}}(G) \otimes \mathbb{Q}$ is generated in degree 1 by the rational vector space $G_{\operatorname{ab}} \otimes \mathbb{Q} = H_1(G; \mathbb{Q})$.

8. The rational lower central series of a split extension

8.1. A rational Guaschi–Pereiro series. In this section we study the rational lower series of a split extension of groups, $B = A \rtimes_{\varphi} C$. Let $L = \{L_n\}_{n \ge 1}$ be the sequence of subgroups of *A* defined by the recursion formula (20). We will focus on the sequence $\{\sqrt{L_n}\}_{n \ge 1}$, where $\sqrt{L_n} = \sqrt[A]{L_n}$ is the isolator of L_n in *A*. By definition,

(39)
$$\sqrt{L_{n+1}} = \sqrt{\langle [A, L_n], [A, \gamma_n(C)], [L_n, C] \rangle}.$$

Thus, every element in $\sqrt{L_{n+1}}$ admits a non-trivial power which can be expressed as a word in elements from the subgroups $[A, L_n]$, $[A, \gamma_n(C)]$, and $[L_n, C]$. We start with a crucial fact about this series.

Lemma 8.1. The sequence $\sqrt{L} = {\sqrt{L_n}}_{n \ge 1}$ is an N_0 -series for A.

Proof. Follows at once from Proposition 7.1 and Theorem 4.6.

From Lemma 4.5, we know that $[L_n, \gamma_m(C)] \subseteq L_{n+m}$ for all $n, m \ge 1$. The next two lemmas constitute a rational analogue of this result.

Lemma 8.2. $\left[\sqrt[A]{L_n}, \gamma_m(C)\right] \subseteq \sqrt[A]{L_{m+n}}$ for all $m, n \ge 1$.

Proof. The proof is by induction on *n*. Since $\sqrt{L_1} = A$, the base case n = 1 follows at once from Lemma 4.5. So assume that $[\sqrt{L_n}, \gamma_m(C)] \subseteq \sqrt{L_{m+n}}$ for all $m \ge 1$. Let $a \in \sqrt{L_{n+1}}$, so that $a^k \in L_{n+1}$, for some k > 0, and let $c \in \gamma_m(C)$. Since *a* belongs to $\sqrt{L_n}$, too, we may apply Lemma 2.2 to infer that

 $[c,a]^k \equiv [c,a^k] \mod \left[\sqrt{L_n}, \left[\sqrt{L_n}, \gamma_m(C)\right]\right].$

By the induction hypothesis, the subgroup on the right is contained in $[\sqrt{L_n}, \sqrt{L_{m+n}}]$, which in turn is a subgroup of $\sqrt{L_{m+2n}} \subseteq \sqrt{L_{m+n+1}}$ by Lemma 8.1 and the fact that $n \ge 1$. On the other hand, $[c, a^k]$ belongs to $[L_{n+1}, \gamma_m(C)]$, which is a subgroup of L_{m+n+1} , by Lemma 4.5. Therefore, $[c, a]^k \in \sqrt{L_{m+n+1}}$, and so $[c, a] \in \sqrt{L_{m+n+1}}$. This shows that $[\sqrt{L_{n+1}}, \gamma_m(C)] \subseteq \sqrt{L_{m+n+1}}$, thereby completing the induction.

Lemma 8.3. $\left[\sqrt[A]{L_n}, \sqrt[C]{\gamma_m(C)}\right] \subseteq \sqrt[A]{L_{m+n}}$ for all $m, n \ge 1$.

Proof. The proof is by induction on *m*. Since $\sqrt[c]{\gamma_1(C)} = C$, the base case m = 1 follows at once from Lemma 8.2. So assume that $\left[\sqrt[A]{L_n}, \sqrt[c]{\gamma_m(C)}\right] \subseteq \sqrt[A]{L_{m+n}}$ for all $n \ge 1$. Let $c \in \sqrt[c]{\gamma_{m+1}(C)}$, so that $c^k \in \gamma_{m+1}(C)$, for some k > 0, and let $a \in \sqrt[A]{L_n}$. Since *c* belongs to $\sqrt[c]{\gamma_m(C)}$, too, we may apply Lemma 2.2 to deduce that

$$[a,c]^k \equiv [a,c^k] \mod \left[\left[\sqrt[A]{L_n}, \sqrt[C]{\gamma_m(C)} \right], \sqrt[C]{\gamma_m(C)} \right].$$

Employing the induction hypothesis twice, we see that the subgroup on the right is contained in $[[\sqrt[4]{L_{m+n}}, \sqrt[6]{\gamma_m(C)}]] \subseteq \sqrt[4]{L_{2m+n}}$, which in turn is a subgroup of $\sqrt[4]{L_{m+n+1}}$, since $m \ge 1$. On the other hand, $[a, c^k] \in [\sqrt[4]{L_n}, \gamma_{m+1}(C)]$, which is a subgroup of $\sqrt{L_{m+n+1}}$, by Lemma 8.2. Hence, $[a, c]^k \in \sqrt{L_{m+n+1}}$, and so $[a, c] \in \sqrt{L_{m+n+1}}$. This shows that $[\sqrt[4]{L_n}, \sqrt[6]{\gamma_{m+1}(C)}] \subseteq \sqrt[4]{L_{m+n+1}}$, and the induction is complete.

8.2. The rational lower central series of a semidirect product. We are now ready to state and prove the main result of this section—an analogue of the Guaschi–Pereiro theorem for the rational lower central series.

Theorem 8.4. Let $B = A \rtimes_{\varphi} C$ be a semidirect product of groups. For each $n \ge 1$,

(1) The homomorphism $\varphi \colon C \to \operatorname{Aut}(A)$ restricts to homomorphisms $\varphi \colon \sqrt[C]{\gamma_n(C)} \to \operatorname{Aut}\left(\sqrt[A]{L_n}\right).$ (2) $\sqrt[B]{\gamma_n(B)} = \sqrt[A]{L_n} \rtimes_{\varphi} \sqrt[C]{\gamma_n(C)}.$

Proof. (1) From Theorem 4.8, we know that φ restricts to maps $\varphi: \gamma_n(C) \to \operatorname{Aut}(L_n)$, for all $n \ge 1$. We prove the first claim by induction on *n*, the case n = 1 being tautological. So assume the map $\varphi: C \to \operatorname{Aut}(A)$ restricts to a map $\varphi: \sqrt[C]{\gamma_{n-1}(C)} \to \operatorname{Aut}(\sqrt[A]{L_{n-1}})$.

Let g be in $\sqrt[c]{\gamma_n(C)}$, and thus also in $\sqrt[c]{\gamma_{n-1}(C)}$. Let $x \in \sqrt[A]{L_n}$, so that $x^{\ell} \in L_n$, for some $\ell > 0$. In view of (20), there are three cases to consider.

- (a) $x^{\ell} \in [A, L_{n-1}]$. Writing x^{ℓ} as a product of elements of the form $[a, a']^{\pm 1}$ with $a \in A$ and $a' \in L_{n-1}$, we have by (16) that $\varphi(g)(x^{\ell}) \in [A, L_{n-1}] \subseteq L_n$.
- (b) $x^{\ell} \in [A, \gamma_{n-1}(C)]$. Arguing as above, we may assume that $x^{\ell} = [c, a]$ with $a \in A$ and $c \in \gamma_{n-1}(C)$. We then have by (17) that $\varphi(g)(x^{\ell}) = [gc, a] \cdot {}^{a}[g, a^{-1}]$. By Lemma 3.1, this element belongs to $[A, \sqrt[c]{\gamma_{n-1}(C)}]$; thus, by Lemma 8.3, $\varphi(g)(x^{\ell}) \in \sqrt[A]{L_n}$.
- (c) $x^{\ell} \in [L_{n-1}, C]$. Again, we may assume that $x^{\ell} = [c, a]$ with $a \in L_{n-1}$ and $c \in C$, in which case we have by (17) that $\varphi(g)(x^{\ell}) \in [L_{n-1}, C] \subseteq L_n$.

Consequently, $\varphi(g)(x) \in \sqrt[A]{L_n}$, thus showing that $\varphi(g)$ leaves $\sqrt[A]{L_n}$ invariant. Since the map $\varphi(g): A \to A$ is injective, its restriction to $\sqrt[A]{L_n}$ is also injective. To show that the map $\varphi(g): \sqrt[A]{L_n} \to \sqrt[A]{L_n}$ is surjective, we go again through the above three cases. In case (a), the surjectivity of the maps $\varphi(g): A \to A$ and $\varphi(g): L_{n-1} \to L_{n-1}$ yields the claim. In cases (b) and (c), the claim follows from the observation that $\varphi(g)([g^{-1}c, a] \cdot {}^a[g^{-1}, a^{-1}]) = [c, a]$.

(2) Clearly, $\sqrt[C]{\gamma_n(C)} \subseteq \sqrt[B]{\gamma_n(B)}$; moreover, it follows from Lemma 4.7 that $\sqrt[A]{L_n} \subseteq \sqrt[B]{\gamma_n(B)}$. Therefore, $\sqrt[A]{L_n} \rtimes_{\varphi} \sqrt[C]{\gamma_n(C)} \subseteq \sqrt[B]{\gamma_n(B)}$.

To prove the reverse inclusion, let $b \in \sqrt[B]{\gamma_n(B)}$. Since $b \in B = A \rtimes C$, we may write in a unique way b = ac, with $a \in A$ and $c \in C$. From Theorem 4.8, we know that $\gamma_n(B) = L_n \rtimes_{\varphi} \gamma_n(C)$. Hence, there is an integer k > 0 such that $b^k = a'c'$, for some (uniquely defined) elements $a' \in L_n$ and $c' \in \gamma_n(C)$. Applying Lemma 3.2 with $A_1 = A$ and

 $C_1 = C$, we have that $(ac)^k \equiv a^k c^k$ modulo the commutator [A, C], which is a subgroup of L_2 . Therefore, $a^k c^k = a' a'' c'$, for some $a'' \in L_2 \leq A$. Hence, $c^k = c' \in \gamma_n(C)$, and so $c \in \sqrt[C]{\gamma_n(C)}$.

Applying once again Lemma 3.2, this time with $A_1 = A$ and $C_1 = \gamma_n(C)$, we find that $(ac)^k \equiv a^k c^k \mod [A, \sqrt[C]{\gamma_n(C)}]$, which by Lemma 8.3 is a subgroup of $\sqrt[A]{L_{n+1}}$. Hence, $a^k \equiv a' \mod \sqrt[A]{L_{n+1}}$; since $a' \in L_n$, we conclude that $a \in \sqrt[A]{L_n}$. Therefore, $\sqrt[B]{\gamma_n(B)}$ is included in $\sqrt[A]{L_n} \rtimes_{\varphi} \sqrt[C]{\gamma_n(C)}$, and the proof is complete.

8.3. **Rational associated graded Lie algebras.** Recall that $gr^{\mathbb{Q}}(G)$ denotes the graded Lie algebra associated to the *N*-series $\gamma^{\mathbb{Q}}(G) = \sqrt{\gamma(G)}$ for *G*. Since $\gamma^{\mathbb{Q}}(G)$ is, in fact, an N_0 -series, each graded piece is a torsion-free abelian group. Now let $B = A \rtimes_{\varphi} C$, and let $\sqrt{L} = \{\sqrt{L_n}\}_{n \ge 1}$ be the sequence of subgroups of *A* defined by (20) and (39). By Lemma 8.1, this is an N_0 -series for *A*; thus, the graded Lie algebra associated to this filtration, $gr^{\sqrt{L}}(A)$, also has the property that all its graded pieces are torsion-free.

Theorem 8.4 has the following corollary, whose proof is completely analogous to that of Theorem 5.1.

Corollary 8.5. Let $1 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 1$ be a split exact sequence of groups. There is then an induced split exact sequence of graded Lie algebras,

(40)
$$0 \longrightarrow \operatorname{gr}^{\sqrt{L}}(A) \xrightarrow{\operatorname{gr}^{\sqrt{L}}(\alpha)} \operatorname{gr}^{\mathbb{Q}}(B) \xrightarrow{\operatorname{gr}^{\mathbb{Q}}(\beta)} \operatorname{gr}^{\mathbb{Q}}(C) \longrightarrow 0$$

Consequently, $\operatorname{gr}^{\mathbb{Q}}(B) \cong \operatorname{gr}^{\sqrt{L}}(A) \rtimes_{\bar{\varphi}} \operatorname{gr}^{\mathbb{Q}}(C)$, where $\bar{\varphi} \colon \operatorname{gr}^{\mathbb{Q}}(C) \to \operatorname{Der}\left(\operatorname{gr}^{\sqrt{L}}(A)\right)$ is the map of Lie algebras induced by the monodromy $\varphi \colon C \to \operatorname{Aut}(A)$ of the split extension.

We illustrate the way this corollary works with a simple example.

Example 8.6. Let $K = A \rtimes C = \langle a, t | tat^{-1} = a^{-1} \rangle$ be the Klein bottle group from Example 5.2, with $A = C = \mathbb{Z}$. Then $gr^{\mathbb{Q}}(K) = gr^{\mathbb{Q}}(C) = \mathbb{Z}$ while $gr^{\sqrt{L}}(A) = 0$.

9. TRIVIAL ACTION ON TORSION-FREE ABELIANIZATION

9.1. **Rational almost-direct products.** In this section we analyze the rational lower central series of groups that arise as *rational almost-direct products*. By definition, these groups are split extensions of the form $B = A \rtimes_{\varphi} C$, where *C* acts trivially on the torsion-free abelianization $A_{abf} = H_1(A; \mathbb{Z})/$ Tors.

This condition is equivalent to $\varphi(c)(a) \cdot a^{-1} \in \sqrt{A'}$, for all $c \in C$ and $a \in A$. If we view *C* as a subgroup of *G* via the splitting $\sigma \colon C \to B$, so that $\varphi(c)(a) \cdot a^{-1} = [c, a]$, the condition reads as

(41)
$$[A,C] \subseteq \sqrt{\gamma_2(A)}.$$

Arguing as in the proofs of [6, Proposition 6.3] and [5, Proposition 3.2], it is readily seen that the property of a split extension being a rational almost direct product does not depend on the choice of splitting.

Clearly, if *C* acts trivially on A_{abf} , then it also acts trivially on $A_{abf} \otimes \mathbb{Q}$. The next result (proved in [39, Proposition 7.4]) provides a partial converse.

Lemma 9.1 ([39]). Let $B = A \rtimes_{\varphi} C$ be a split extension. If A_{abf} is finitely generated, then the extension is a rational almost-direct product if and only if C acts trivially on $H_1(A; \mathbb{Q})$.

In other words, if A_{abf} is finitely generated, then $B = A \rtimes_{\varphi} C$ is a rational almost-direct product if and only if the monodromy of the extension factors through a map $\varphi \colon C \to \mathscr{T}_{\mathbb{Q}}(A)$, where $\mathscr{T}_{\mathbb{Q}}(A) \coloneqq \ker (\operatorname{Aut}(A) \to \operatorname{Aut}(H_1(A; \mathbb{Q})))$ is the rational Torelli group of *A*. As the next example shows, though, the finite generation hypothesis on A_{abf} cannot be dropped.

Example 9.2. Let $G = BS(1, n) = \langle t, a \mid tat^{-1} = a^n \rangle$ be one of the groups from Example 2.5. In the extension $1 \to G' \to G \to G_{ab} \to 1$, the abelianization is isomorphic to \mathbb{Z} , generated by the image of *t*, while the derived subgroup is isomorphic to $\mathbb{Z}[1/n]$, normally generated by *a*. Thus, the extension is split exact, with monodromy given by $a \mapsto a^n$. Clearly, \mathbb{Z} acts trivially on $\mathbb{Z}[1/n] \otimes \mathbb{Q} = \mathbb{Q}$. But, if $n \ge 2$, then \mathbb{Z} acts non-trivially on the torsion-free, yet non-finitely generated abelian group $(G')_{abf} = G' = \mathbb{Z}[1/n]$.

9.2. The *L*-series of a rational almost direct product. The next lemma is the key technical result of this section.

Lemma 9.3. Assume that $[A, C] \subseteq \sqrt{\gamma_2(A)}$. Then, for all $m, n \ge 1$,

(42)
$$\left[\sqrt{\gamma_n(A)}, \gamma_m(C)\right] \subseteq \sqrt{\gamma_{m+n}(A)}.$$

Proof. We prove the claim by induction on *m*. The case m = 1, which amounts to $[\sqrt{\gamma_n(A)}, C] \subseteq \sqrt{\gamma_{n+1}(A)}$ for all $n \ge 1$, is proved by induction on *n*. The base case, $[A, C] \subseteq \sqrt{\gamma_2(A)}$, is our assumption. So assume $[\sqrt{\gamma_n(A)}, C] \subseteq \sqrt{\gamma_{n+1}(A)}$. Let $a \in \sqrt{\gamma_{n+1}(A)}$, so that $a^k \in \gamma_{n+1}(A)$, for some k > 0, and let $c \in C$. Since *a* belongs to $\sqrt{\gamma_n(A)}$, too, we may apply Lemma 2.2 to deduce that

$$[c,a]^k \equiv [c,a^k] \mod \left[\sqrt{\gamma_n(A)}, \left[\sqrt{\gamma_n(A)}, C\right]\right],$$

a subgroup contained in $[\sqrt{\gamma_n(A)}, \sqrt{\gamma_{n+1}(A)}]$, by the induction hypothesis. In turn, this is a subgroup of $\sqrt{\gamma_{2n+1}(A)} \subseteq \sqrt{\gamma_{n+2}(A)}$, since \sqrt{A} is an *N*-series and $n \ge 1$. On the other hand, $[c, a^k] \in [\gamma_{n+1}(A), C]$. Using once again our assumption and the induction hypothesis, we have

$$\begin{split} & [\gamma_n(A), [A, C]] \subseteq \left[\gamma_n(A), \sqrt{\gamma_2(A)}\right] \subseteq \sqrt{\gamma_{n+2}(A)}, \\ & [A, [\gamma_n(A), C]] \subseteq \left[A, \left[\sqrt{\gamma_n(A)}, C\right]\right] \subseteq \left[A, \sqrt{\gamma_{n+1}(A)}\right] \subseteq \sqrt{\gamma_{n+2}(A)} \end{split}$$

Thus, by Lemma 2.1, $[\gamma_{n+1}(A), C] = [[A, \gamma_n(A)], C]$ is a subgroup of $\sqrt{\gamma_{n+2}(A)}$. Hence, $[c, a]^k \in \sqrt{\gamma_{n+2}(A)}$, and so $[c, a] \in \sqrt{\gamma_{n+2}(A)}$. This shows that $[\sqrt{\gamma_{n+1}(A)}, C] \subseteq \sqrt{\gamma_{n+2}(A)}$, and the induction on *n* is complete.

Assume now that
$$\left[\sqrt{\gamma_n(A)}, \gamma_m(C)\right] \subseteq \sqrt{\gamma_{m+n}(A)}$$
 for all $n \ge 1$. Then

$$\begin{bmatrix} \left[\sqrt{\gamma_n(A)}, \gamma_m(C)\right], C \end{bmatrix} \subseteq \begin{bmatrix} \sqrt{\gamma_{m+n}(A)}, C \end{bmatrix} \subseteq \sqrt{\gamma_{m+n+1}(A)},$$

$$\begin{bmatrix} \left[\sqrt{\gamma_n(A)}, C\right], \gamma_m(C) \end{bmatrix} \subseteq \begin{bmatrix} \sqrt{\gamma_{n+1}(A)}, \gamma_m(C) \end{bmatrix} \subseteq \sqrt{\gamma_{m+n+1}(A)}$$

where we used both the induction hypothesis and the base case of the induction on *m*. Since $\gamma_{m+1}(C) = [C, \gamma_m(C)]$, Lemma 2.1 shows that $[\sqrt{\gamma_n(A)}, \gamma_{m+1}(C)] \subseteq \sqrt{\gamma_{m+n+1}(A)}$ for all $n \ge 1$, and the induction on *m* is complete.

We are now ready to state and prove the main result of this section.

Theorem 9.4. Let $B = A \rtimes_{\varphi} C$ be a split extension of groups, and let $L = \{L_n\}_{n \ge 1}$ be the corresponding N-series for A. Suppose C acts trivially on A_{abf} . Then, for each $n \ge 1$,

(1) $\sqrt[A]{L_n} = \sqrt[A]{\gamma_n(A)}.$ (2) $\sqrt[B]{\gamma_n(B)} = \sqrt[A]{\gamma_n(A)} \rtimes_{\varphi} \sqrt[C]{\gamma_n(C)}.$

Proof. By Lemma 4.7, $\gamma_n(A) \subseteq L_n$, and thus $\sqrt{\gamma_n(A)} \subseteq \sqrt{L_n}$. We prove the reverse inclusion by induction on *n*. Since $L_1 = A$, the base case n = 1 is immediate. So assume that $\sqrt{L_n} \subseteq \sqrt{\gamma_n(A)}$. Recall from (39) that every element in $\sqrt{L_{n+1}}$ admits a non-trivial power which can be expressed as a word in elements from the subgroups $[A, L_n]$, $[A, \gamma_n(C)]$, and $[L_n, C]$. It is enough, then, to show that each of these subgroups is contained in $\sqrt{\gamma_{n+1}(A)}$. We consider the three cases in turn.

- (a) By the induction hypothesis, we have that $[A, L_n] \subseteq [A, \sqrt{L_n}] \subseteq [A, \sqrt{\gamma_n(A)}]$, which in turn is equal to $\sqrt{\gamma_{n+1}(A)}$, by (34).
- (b) By the induction hypothesis and Lemma 9.3, we have that $[L_n, C] \subseteq [\sqrt{L_n}, C] \subseteq [\sqrt{\gamma_n(A)}, C] \subseteq \sqrt{\gamma_{n+1}(A)}$.
- (c) By Lemma 9.3 again, we have that $[A, \gamma_n(C)] \subseteq \sqrt{\gamma_{n+1}(A)}$.

This shows that $\sqrt{L_{n+1}} \subseteq \sqrt{\gamma_{n+1}(A)}$, thus completing the induction, and finishing the proof of the first claim. The second claim now follows from the first one and Theorem 8.4.

9.3. The rational associated graded Lie algebra. As a consequence of Theorem 9.4, we obtain a description of the rational associated graded Lie algebra of a rational almost direct product of groups.

Theorem 9.5. Let $1 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 1$ be a split exact sequence of groups, with monodromy $\varphi \colon C \to \operatorname{Aut}(A)$. Suppose C acts trivially on A_{abf} . Then

(43)
$$0 \longrightarrow \operatorname{gr}^{\mathbb{Q}}(A) \xrightarrow{\operatorname{gr}^{\mathbb{Q}}(\alpha)} \operatorname{gr}^{\mathbb{Q}}(B) \xrightarrow{\operatorname{gr}^{\mathbb{Q}}(\beta)} \operatorname{gr}^{\mathbb{Q}}(C) \longrightarrow 0$$

is a split exact sequence of graded Lie algebras, and so $\operatorname{gr}^{\mathbb{Q}}(B) \cong \operatorname{gr}^{\mathbb{Q}}(A) \rtimes_{\bar{\varphi}} \operatorname{gr}^{\mathbb{Q}}(C)$, where $\bar{\varphi} \colon \operatorname{gr}^{\mathbb{Q}}(C) \to \operatorname{Der}(\operatorname{gr}^{\mathbb{Q}}(A))$ is the map of Lie algebras induced by φ .

Proof. Follows from Corollary 8.5 and Theorem 9.4.

Example 9.6. Let $K = \langle a, t | tat^{-1} = a^{-1} \rangle$ be the Klein bottle group, and consider the semidirect product $G = K \rtimes_{\varphi} \mathbb{Z}$ with monodromy action given by $t \mapsto ta$ and $a \mapsto a$. Clearly, \mathbb{Z} acts trivially on K_{abf} (though not on K_{ab}). Thus, we may apply Theorem 9.5; using the computation from Example 8.6, we conclude that $gr^{\mathbb{Q}}(G) = \mathbb{Z}^2$, concentrated in degree 1.

9.4. **The RTFN property.** We conclude this section with an application to the study of the residual torsion-free nilpotent (RTFN) property of groups arising as split extensions.

Theorem 9.7. Let $B = A \rtimes C$ be a split extension of RTFN groups. If C acts trivially on A_{abf} , then B is also RTFN.

Proof. Recall from Lemma 7.4 that a group *G* is residually torsion-free nilpotent precisely when $\gamma_{\omega}^{\mathbb{Q}}(G) = \{1\}$. The claim follows from Theorem 9.4 by an argument similar to the one employed in the proof of Corollary 6.6.

The next corollary follows at once from this theorem and Lemma 9.1.

Corollary 9.8. Let $B = A \rtimes C$ be a split extension of RTFN groups. Suppose A is finitely generated and C acts trivially on $H_1(A; \mathbb{Q})$. Then B is also RTFN.

If the group A is not finitely generated and we only assume that C acts trivially on $H_1(A; \mathbb{Q})$, then the conclusion of Corollary 9.8 may not hold.

Example 9.9. Let B = BS(1, n) be a metabelian Baumslag–Solitar group with $n \ge 2$. As noted in Example 9.2, this group is a semidirect product of the form $B = \mathbb{Z}[1/n] \rtimes \mathbb{Z}$, with \mathbb{Z} acting trivially on $H_1(\mathbb{Z}[1/n]; \mathbb{Q}) = \mathbb{Q}$, but non-trivially on the (non finitely generated) torsion-free abelian group $\mathbb{Z}[1/n]$. Clearly, both $\mathbb{Z}[1/n]$ and \mathbb{Z} are RTFN; yet, as we shall see in Remark 10.5, the group *B* is not RTFN.

Corollary 9.8 has the following topological consequence. Given a space X and a map $f: X \to X$, we let $T_f = X \times [0, 1]/(x, 0) \sim (f(x), 1)$ be the mapping torus of f.

Corollary 9.10. Let X be a finite, connected CW-complex, and let $f: X \to X$ be a map inducing the identity on $H_1(X; \mathbb{Q})$. If $\pi_1(X)$ is RTFN, then $\pi_1(T_f)$ is also RTFN.

Example 9.11. Let Σ be a closed, orientable surface, let $f: \Sigma \to \Sigma$ be a smooth map that belongs to the rational Torelli group (i.e., f induces the identity on $H_1(\Sigma; \mathbb{Q})$), and let $M = T_f$ be the 3-manifold which fibers over the circle with fiber Σ and monodromy f. As mentioned previously, the surface group $\pi_1(\Sigma)$ is known to be RTFN. By the above corollary, then, the 3-manifold group $\pi_1(M)$ is also RTFN.

10. Mod-p lower central series

10.1. *p*-Torsion series and N_p -series. Fix a prime *p*. For a subset *S* of a group *G*, we let S^p be the subgroup generated by $\{g^p \mid g \in S\}$. Clearly, if $\varphi \colon G \to H$ is a homomorphism, then $\varphi(S^p) \subseteq (\varphi(S))^p$.

A sequence $K = \{K_n\}_{n \ge 1}$ of subgroups of *G* is said to be a *p*-torsion series (or, *p*-central filtration) if $K_n^p \subseteq K_{n+1}$ for all $n \ge 1$, see [3, 12, 18]. Now suppose that *K* is also an *N*-series. Then the successive quotients, $\operatorname{gr}_n^K(G) = K_n/K_{n+1}$, are *p*-torsion abelian groups, that is, elementary abelian *p*-groups. Hence, the groups $\operatorname{gr}_n^K(G)$ can be viewed as \mathbb{Z}_p -vector spaces, and the associated graded Lie algebra, $\operatorname{gr}_n^K(G) = \bigoplus_{n \ge 1} \operatorname{gr}_n^K(G)$, becomes a Lie algebra over the prime field \mathbb{Z}_p .

A sequence $K = \{K_n\}_{n \ge 1}$ of subgroups of *G* is said to be an N_p -series (or, a *p*-restricted series) if *K* is an *N*-series and $K_n^p \subseteq K_{pn}$ for all $n \ge 1$. An N_p -series is both an *N*-series and a *p*-torsion series, but the converse does not hold. To every *N*-series *K* and prime *p*, there corresponds a canonical N_p -series, $K^{[p]}$; its terms are given by

(44)
$$K_n^{[p]} = \prod_{\substack{mp^j \ge n \\ m \ge 1, j \ge 0}} \left(K_m \right)^{p^j}.$$

This is the fastest descending N_p -series containing K, cf. [18].

As shown by Lazard in [25, Corollary 6.8], the associated graded Lie algebra of an N_p series K is a *p*-restricted Lie algebra, in the sense of Jacobson [24]. In addition to the usual Lie bracket, such a Lie algebra, $gr^K(G)$, comes endowed with a *p*-th power map, $gr_n^K(G) \to gr_{n+1}^K(G)$, which is induced from the map $G \to G$, $g \mapsto g^p$; moreover, the two operations satisfy the compatibility conditions listed in [24], see [12, 18].

10.2. The mod-*p* lower central series. In [38], Stallings introduced the *mod-p* lower central series, a subgroup series which has been much studied since then, see for instance [3, 5, 9, 11, 12, 18, 33]. Given a group *G*, this series, denoted $\gamma^p(G) = \{\gamma_n^p(G)\}_{n \ge 1}$, is defined inductively by setting $\gamma_1^p(G) = G$ and

(45)
$$\gamma_{n+1}^p(G) = \left\langle \left(\gamma_n^p(G)\right)^p, \left[G, \gamma_n^p(G)\right] \right\rangle.$$

By construction, the terms of this series satisfy $(\gamma_n^p(G))^p \subseteq \gamma_{n+1}^p(G)$; thus, $\gamma^p(G)$ is a *p*-torsion series. Clearly, the terms of $\gamma^p(G)$ are fully invariant subgroups of *G*. Note that $\gamma_2^p(G) = \langle G^p, G' \rangle$, and so $G/\gamma_2^p(G) = G_{ab} \otimes \mathbb{Z}_p = H_1(G; \mathbb{Z}_p)$. The series $\gamma^p(G)$ is a descending central series; in fact, it is the fastest descending such series among all *p*-torsion series for *G*. For completeness, we give a quick proof of this known fact. Lemma 10.1. Let G be a group, and p a prime. Then,

- (1) $\gamma^{p}(G)$ is an N-series for G.
- (2) If K is a descending, central, p-torsion series for G, then $\gamma_n^p(G) \leq K_n$ for all $n \geq 1$.

Proof. Part (1) is proved in [33, Lemma 2.2]. To prove part (2), we use induction on n, with the base case n = 1 being obvious. Assuming $\gamma_n^p(G) \subseteq K_n$, we have $\gamma_{n+1}^p(G) = \langle (\gamma_n^p(G))^p, [G, \gamma_n^p(G)] \rangle \subseteq \langle (K_n)^p, [G, K_n] \rangle \subseteq K_{n+1}$, and we are done.

The Stallings series $\gamma^p(G)$ is not in general an N_p -series. Rather, the canonical N_p -series associated to the lower central series, $\gamma^{[p]}(G)$, coincides with the series introduced by H. Zassenhaus in 1939, see [11, 12, 18]. The two filtrations are cofinal (and thus define the same topology on G), but they may differ quite a lot at a granular level. For instance, if G is abelian, then $\gamma_n^p(G) = G^{p^{n-1}}$, whereas $\gamma_n^{[p]}(G) = G^{p^j}$, where $j = \lfloor \log_p(n) \rfloor$.

10.3. Associated graded Lie algebras. Since both $\gamma^p(G)$ and $\gamma^{[p]}(G)$ are *p*-torsion *N*-series, the corresponding associated graded Lie algebras, $\operatorname{gr}^p(G)$ and $\operatorname{gr}^{[p]}(G)$, are Lie algebras over \mathbb{Z}_p . Clearly, the terms of both series are fully invariant subgroups of *G*; thus the constructions $G \rightsquigarrow \operatorname{gr}^p(G)$ and $G \rightsquigarrow \operatorname{gr}^{[p]}(G)$ are both functorial.

Example 10.2. For the cyclic group \mathbb{Z} , we have $gr_n^p(\mathbb{Z}) = \mathbb{Z}_p$, for all $n \ge 1$. On the other hand, $gr_n^{[p]}(\mathbb{Z})$ is equal to \mathbb{Z}_p if *n* is a non-negative power of *p*, and equal to 0, otherwise.

We will focus in §11 on $\operatorname{gr}^p(G)$, the *mod-p* associated graded Lie algebra of G. For now, let us make an observation. The power-*p* map $G \to G$, $x \mapsto x^p$ restricts to maps $\gamma_n^p(G) \to \gamma_{n+1}^p(G)$ for all $n \ge 1$, and thus induces maps $\operatorname{gr}_n^p(G) \to \operatorname{gr}_{n+1}^p(G)$. As noted in [3, §12], the \mathbb{Z}_p -Lie algebra $\operatorname{gr}^p(G)$ is generated—through Lie brackets and these power operations—by its degree 1 piece, $\operatorname{gr}_1^p(G) = H_1(G; \mathbb{Z}_p)$.

10.4. **Residually** *p* **groups.** A group *G* is said to be *residually p* if every nontrivial element of *G* can be detected in a finite *p*-group quotient. As shown by Paris in [33, Proposition 2.3], this property is detected by the intersection of the mod-*p* lower central series, $\gamma_{\omega}^{p}(G) \coloneqq \bigcap_{n \ge 1} \gamma_{n}^{p}(G)$.

Lemma 10.3 ([33]). Let G be a finitely generated group. Then G is residually p if and only if $\gamma_{\omega}^{p}(G) = \{1\}$.

Clearly, residually *p* groups are residually nilpotent (and thus residually finite), but this implication cannot be reversed.

Example 10.4. The Baumslag–Solitar groups BS(m, n) from Example 2.5 are residually p if and only if m = 1 and $n \equiv 1 \pmod{p}$; or $m = n = p^r > 1$; or $m = -n = 2^r > 1$ and p = 2; see [2, 23]. It follows that the groups BS(1, p + 1) are residually p, while the groups BS(1, p) are residually nilpotent (if $p \neq 2$) but not residually p.

Remark 10.5. Finitely generated RTFN groups are residually p for every prime p. Consequently, the groups BS(1, n) with n > 1 are not RTFN, since they may be residually p for some primes p, but not for all primes. Examples of finitely generated groups which are residually p for all primes p yet are not RTFN were given by Hartley [20].

11. The mod-p lower central series of a split extension

11.1. A *p*-torsion *N*-series. Given a split extension of groups, $B = A \rtimes_{\varphi} C$, and a prime *p*, we define by analogy with (45) and (20) a sequence of subgroups of *A*, denoted $\{L_n^p\}_{n \ge 1}$, as follows. We set $L_1^p = A$ and define the other terms inductively by

(46)
$$L_{n+1}^{p} = \left\langle \left(L_{n}^{p}\right)^{p}, \left[A, L_{n}^{p}\right], \left[A, \gamma_{n}^{p}(C)\right], \left[L_{n}^{p}, C\right] \right\rangle.$$

Lemma 11.1. The groups L_n^p are normal subgroups of A.

Proof. We establish the claim by induction on *n*. The base case $L_1^p = A$ is tautologically true. So assume $L_n^p \lhd A$. Since $ax^pa^{-1} = (axa^{-1})^p$, this immediately implies that $(L_n^p)^p \lhd A$. Since commutators of normal subgroups are again normal, $[A, L_n^p] \lhd A$; moreover, by Lemma 3.1, $[A, \gamma_n^p(C)] \lhd A$. Finally, if $x \in A$, then $x[L_n^p, C]x^{-1} \subseteq [L_n^p, C] \cdot [A, L_n^p]$, by formula (19). Therefore, $L_{n+1}^p \lhd A$, and the induction is complete.

Lemma 11.2. The subgroups $\{L_n^p\}_{n\geq 1}$ form a descending series for A.

Proof. We need to show that $L_{n+1}^p \subseteq L_n^p$ for all $n \ge 1$ or, equivalently, that $(L_n^p)^p$, $[A, L_n^p]$, $[A, \gamma_n^p(C)]$, and $[L_n^p, C]$ are all included in L_n^p .

- (a) $(L_n^p)^p \subseteq L_n^p$: this is obvious.
- (b) $[A, L_n^p] \subseteq L_n^p$: follows at once from Lemma 11.1.
- (c) $[A, \gamma_n^p(C)] \subseteq L_n^p$: we prove this by induction on *n*, with the base case, $[A, C] \subseteq A$, being obvious. We have $[[A, C], \gamma_n^p(C)] \subseteq [A, \gamma_n^p(C)] \subseteq L_{n+1}^p$, by (46) and $[[A, \gamma_n^p(C)], C] \subseteq [L_n^p, C] \subseteq L_{n+1}^p$, by the induction hypothesis and (46). Thus, $[A, [C, \gamma_n^p(C)]] \subseteq L_{n+1}^p$, by Lemma 2.1. By (46), we also have $[A, (\gamma_n^p(C))^p] \subseteq [A, \gamma_n^p(C)] \subseteq L_{n+1}^p$. Hence, $[A, \gamma_{n+1}^p(C)] \subseteq L_{n+1}^p$, and the induction is complete.
- (d) $[L_n^p, C] \subseteq L_n^p$: we prove this by induction on *n*, with the base case n = 1 being obvious. For the induction step, there are four cases to consider.
 - (i) Clearly, $[(L_n^p)^p, C] \subseteq [L_n^p, C] \subseteq L_{n+1}^p$.
 - (ii) Clearly, $[[A, C], L_n^p] \subseteq [A, L_n^p]$, while $[A, [L_n^p, C]] \subseteq [A, L_n^p]$ by the induction hypothesis. Therefore, $[[A, L_n^p], C] \subseteq [A, L_n^p] \subseteq L_{n+1}^p$, by Lemma 2.1 and (46).
 - (iii) We have $[[A, C], \gamma_n^p(C)] \subseteq [A, \gamma_n^p(C)] \subseteq L_{n+1}^p$, by (46), and $[A, [C, \gamma_n^p(C)]] \subseteq [A, \gamma_{n+1}^p(C)] \subseteq L_{n+1}^p$, by case (c). Therefore, $[[A, \gamma_n^p(C)], C] \subseteq L_{n+1}^p$, by Lemma 2.1.
 - (iv) Finally, $[[L_n^p, C], C] \subseteq [L_n^p, C] \subseteq L_{n+1}^p$, by the induction hypothesis and (46).

This shows that $[L_{n+1}^p, C] \subseteq L_{n+1}^p$, thereby completing the induction.

This ends the proof.

Lemma 11.3. $[L_n^p, \gamma_m^p(C)] \subseteq L_{n+m}^p$, for all $n, m \ge 1$.

Proof. We establish the claim by induction on *m*. The base case m = 1, which amounts to $[L_n^p, C] \subseteq L_{n+1}^p$ for all $n \ge 1$, follows directly from (46). Assume now that $[L_n^p, \gamma_m^p(C)] \subseteq L_{n+m}^p$, for all $n \ge 1$. We then have $[[L_n^p, C], \gamma_m^p(C)] \subseteq [L_{n+1}^p, \gamma_m^p(C)] \subseteq L_{n+m+1}^p$ and $[[L_n^p, \gamma_m^p(C)], C] \subseteq [L_{n+m}^p, C] \subseteq L_{n+m+1}^p$, and so $[L_n^p, [C, \gamma_m^p(C)]]$ is contained in L_{n+m+1}^p , by Lemma 2.1. We also have $[L_n^p, (\gamma_m^p(C))]^p \subseteq (L_{n+m}^p)^p \subseteq L_{n+m+1}^p$. Hence, $[L_n^p, \gamma_{m+1}^p(C)]$ is contained in L_{n+m+1}^p , and so the induction step is complete.

Theorem 11.4. The sequence of subgroups $\{L_n^p\}_{n\geq 1}$ forms a p-torsion N-series for A.

Proof. From the way the sequence was defined in (46), we have that $(L_n^p)^p \subseteq L_{n+1}^p$ for all $n \ge 1$; thus, L^p is a *p*-torsion series.

Next, we show that L^p is an *N*-series, that is, $[L_n^p, L_m^p] \subseteq L_{n+m}^p$ for all $n, m \ge 1$. We prove this claim by induction on *m*. The base case m = 1, which amounts to $[L_n^p, A] \subseteq L_{n+1}^p$, follows at once from definition (46). Assume now that $[L_n^p, L_m^p] \subseteq L_{n+m}^p$ for all $n \ge 1$. To prove the induction step, there are four cases to consider.

- (a) $[L_n^p, (L_m^p)^p] \subseteq L_{n+m+1}^p$: to prove this assertion, consider elements $a \in L_n^p$ and $b \in L_m^p$. Lemma 2.2 implies that $[a, b^p] \equiv [a, b]^p$ modulo $[[L_n^p, L_m^p], L_m^p]$, a subgroup of $L_{n+2m}^p \subseteq L_{n+m+1}^p$. But $[a, b] \in [L_n^p, L_m^p] \subseteq L_{n+m}^p$; thus, $[a, b]^p \in (L_{n+m}^p)^p \subseteq L_{n+m+1}^p$, showing that $[a, b^p] \in L_{n+m+1}^p$.
- (b) $[[L_n^p, A], L_m^p] \subseteq [L_{n+1}^p, L_m^p] \subseteq L_{n+1+m}^p$ and $[[L_n^p, L_m^p], A] \subseteq [L_{n+m}^p, A] \subseteq L_{n+1+m}^p$. Therefore, $[L_n^p, [A, L_m^p]] \subseteq L_{n+1+m}^p$.
- (c) Applying Lemma 11.3, we have that $[[L_n^p, A], \gamma_m^p(C)] \subseteq [L_{n+1}^p, \gamma_m^p(C)] \subseteq L_{n+1+m}^p$ and $[[L_n^p, \gamma_m^p(C)], A] \subseteq [L_{n+m}^p, A] \subseteq L_{n+1+m}^p$. Hence, $[L_n^p, [A, \gamma_m^p(C)]] \subseteq L_{n+1+m}^p$.
- (d) $[[L_{n}^{p}, C], L_{m}^{p}] \subseteq [L_{n+1}^{p}, L_{m}^{p}] \subseteq L_{n+1+m}^{p}$ and $[[L_{n}^{p}, L_{m}^{p}], C] \subseteq [L_{n+m}^{p}, C] \subseteq L_{n+1+m}^{p}$. Therefore, $[L_{n}^{p}, [L_{m}^{p}, C]] \subseteq L_{n+1+m}^{p}$.

This shows that $[L_n^p, L_{m+1}^p] \subseteq L_{n+m+1}^p$ for all $n \ge 1$, thereby completing the induction. \Box

We conclude this subsection with one more lemma.

Lemma 11.5. The inclusions $\gamma_n^p(A) \subseteq L_n^p \subseteq \gamma_n^p(B)$ hold for all $n \ge 1$.

Proof. The first claim follows at once from Lemma 10.1 and Theorem 11.4. The second claim is proved by induction on *n*, with the base case n = 1 being obvious. Assuming $L_n^p \subseteq \gamma_n^p(B)$, we have that all four subgroups generating L_{n+1}^p in (46) are included in either $(\gamma_n^p(B))^p$ or $[B, \gamma_n^p(B)]$, whence $L_{n+1}^p \subseteq \gamma_{n+1}^p(B)$.

11.2. Split extensions and the mod-*p* LCS. We are now in a position to state and prove the main result of this section.

Theorem 11.6. Let $B = A \rtimes_{\varphi} C$ be a split extension. For each $n \ge 1$, the homomorphism $\varphi: C \to \operatorname{Aut}(A)$ restricts to a homomorphism $\varphi: \gamma_n^p(C) \to \operatorname{Aut}(L_n^p)$. Moreover,

(47)
$$\gamma_n^p(B) = L_n^p \rtimes_{\varphi} \gamma_n^p(C) \,.$$

Proof. We prove the first claim by induction on *n*, the case n = 1 being tautological. So assume the map $\varphi \colon C \to \operatorname{Aut}(A)$ restricts to a map $\varphi \colon \gamma_{n-1}^p(C) \to \operatorname{Aut}(L_{n-1}^p)$. Let *g* be in $\gamma_n^p(C)$. For the induction step, there are four cases to consider.

- (a) Clearly, $\varphi(g)$ leaves $(L_{n-1}^p)^p$ invariant.
- (b) Since $g \in \gamma_{n-1}^{p}(C)$, too, $\varphi(g)$ leaves L_{n-1}^{p} invariant. Formula (16) now implies that $\varphi(g)$ leaves $[A, L_{n-1}^{p}]$ invariant.
- (c) Since $gc \in \gamma_{n-1}^{p}(C)$ for $c \in \gamma_{n-1}^{p}(C)$, formula (17) implies that $\varphi(g)$ leaves $[A, \gamma_{n-1}^{p}(C)]$ invariant.
- (d) By Lemma 3.1, $[L_{n-1}^{p}, C]$ is a normal subgroup of *A*. Formula (17) now implies that $\varphi(g)$ leaves $[L_{n-1}^{p}, C]$ invariant.

By definition, L_n^p is generated by the subgroups considered in the four cases above; hence, $\varphi(g)$ leaves L_n^p invariant, too.

Since the map $\varphi(g): A \to A$ is injective, its restriction to L_n^p is also injective. To show that the map $\varphi(g): L_n^p \to L_n^p$ is surjective, we go through the above four cases one more time. In case (a), the fact that the map $\varphi(g): L_{n-1}^p \to L_{n-1}^p$ is surjective implies that the map $\varphi(g): (L_{n-1}^p)^p \to (L_{n-1}^p)^p$ is also surjective. In case (b), the surjectivity of the maps $\varphi(g): A \to A$ and $\varphi(g): L_{n-1}^p \to L_{n-1}^p$ yield the claim. In cases (c) and (d) the claim follows from (17), by observing that $[c, a] = \varphi(g)([g^{-1}c, a] \cdot {}^a[g^{-1}, a^{-1}]).$

The second claim is also proved by induction on *n*, with the case n = 1 being tautological. Assume that $\gamma_{n-1}^{p}(B) = L_{n-1}^{p} \rtimes_{\varphi} \gamma_{n-1}^{p}(C)$. To show that $L_{n}^{p} \rtimes_{\varphi} \gamma_{n}^{p}(C) \subseteq \gamma_{n}^{p}(B)$, observe that $\gamma_{n}^{p}(C) \subseteq \gamma_{n}^{p}(B)$, and also $L_{n}^{p} \subseteq \gamma_{n}^{p}(B)$, by Lemma 11.5. For the reverse inclusion, definition (45), the induction hypothesis, and definition (46) show that

(48)

$$\gamma_{n}^{p}(B) = \left\langle \left(\gamma_{n-1}^{p}(B)\right)^{p}, \left[B, \gamma_{n-1}^{p}(B)\right] \right\rangle$$

$$= \left\langle \left(L_{n-1}^{p} \rtimes_{\varphi} \gamma_{n-1}^{p}(C)\right)^{p}, \left[A \rtimes_{\varphi} C, L_{n-1}^{p} \rtimes_{\varphi} \gamma_{n-1}^{p}(C)\right] \right\rangle$$

$$\subseteq L_{n}^{p} \rtimes_{\varphi} \gamma_{n}^{p}(C).$$

This completes the proof.

Let $\operatorname{gr}^{L^p}(A)$ be the graded Lie algebra associated to the *N*-series $L^p = \{L_n^p\}_{n \ge 1}$. Since L^p is a *p*-torsion series, the graded pieces of this Lie algebra are \mathbb{Z}_p -vector spaces.

Corollary 11.7. Let $1 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 1$ be a split exact sequence of groups. For each prime p, there is an induced split exact sequence of graded Lie algebras,

(49)
$$0 \longrightarrow \operatorname{gr}^{L^p}(A) \xrightarrow{\operatorname{gr}^{L^p}(\alpha)} \operatorname{gr}^p(B) \xrightarrow{\operatorname{gr}(\beta)} \operatorname{gr}^p(C) \longrightarrow 0$$

Consequently, $\operatorname{gr}^{p}(B) \cong \operatorname{gr}^{L^{p}}(A) \rtimes_{\bar{\varphi}} \operatorname{gr}^{p}(C)$, where $\bar{\varphi} \colon \operatorname{gr}^{p}(C) \to \operatorname{Der}(\operatorname{gr}^{L^{p}}(A))$ is the map of Lie algebras induced by the monodromy $\varphi \colon C \to \operatorname{Aut}(A)$ of the extension.

Proof. The proof is similar to that of Theorem 5.1.

12. TRIVIAL ACTION ON \mathbb{Z}_p -Homology

12.1. **Mod**-*p* **almost direct products.** Following [5], we say that a split extension $B = A \rtimes_{\varphi} C$ is a *mod*-*p almost direct product* if the group *C* acts trivially on $A_{ab} \otimes \mathbb{Z}_p = H_1(A;\mathbb{Z}_p)$. This condition may be expressed as $\varphi(c)(a) \cdot a^{-1} \in \gamma_2^p(A)$, for all $c \in C$ and $a \in A$. Alternatively, if we view *C* as a subgroup of *G* via the splitting $\sigma: C \to B$, the condition simply reads

(50)
$$[A,C] \subseteq \gamma_2^p(A).$$

As noted in [5, Proposition 3.2], the fact that a split extension $B = A \rtimes_{\varphi} C$ is a mod-*p* almost direct product does not depend on the choice of splitting σ .

Lemma 12.1. Assume that $[A, C] \subseteq \gamma_2^p(A)$. Then, for all $m, n \ge 1$, (51) $\left[\gamma_n^p(A), \gamma_m^p(C)\right] \subseteq \gamma_{m+n}^p(A)$.

Proof. We prove the claim by induction on *m*. The case m = 1, which amounts to $[\gamma_n^p(A), C] \subseteq \gamma_{n+1}^p(A)$ for all $n \ge 1$, is proved by induction on *n*. The base case, $[A, C] \subseteq \gamma_2^p(A)$, is our assumption. So assume that $[\gamma_n^p(A), C] \subseteq \gamma_{n+1}^p(A)$. For the induction step, we need to show that $[[A, \gamma_n^p(A)], C]$ and $[(\gamma_n^p(A))^p, C]$ are both included in $\gamma_{n+2}^p(A)$.

- (a) Our assumption, the induction hypothesis, and the fact that $\gamma(A)$ is an *N*-series imply that $[\gamma_n^p(A), [A, C]] \subseteq [\gamma_n^p(A), \gamma_2^p(A)] \subseteq \gamma_{n+2}^p(A)$ and $[A, [\gamma_n^p(A), C]] \subseteq [A, \gamma_{n+1}^p(A)] \subseteq \gamma_{n+2}^p(A)$. The first inclusion now follows from Lemma 2.1.
- (b) Next, let $a \in \gamma_n^p(A)$ and $c \in C$. Applying Lemma 2.2 shows that $[c, a^p] \equiv [c, a]^p \mod [\gamma_n^p(A), [\gamma_n^p(A), C]]$, a subgroup of $[\gamma_n^p(A), \gamma_{n+1}^p(A)] \subseteq \gamma_{2n+1}^p(A) \subseteq \gamma_{n+2}^p(A)$. But $[c, a] \in [\gamma_n^p(A), C] \subseteq \gamma_{n+1}^p(A)$, and so $[c, a]^p \in (\gamma_{n+1}^p(A))^p \subseteq \gamma_{n+2}^p(A)$. Thus, $[c, a^p] \in \gamma_{n+2}^p(A)$, and the second inclusion is also proved.

Assume now that $[\gamma_n^p(A), \gamma_m^p(C)] \subseteq \gamma_{m+n}^p(A)$ for all $n \ge 1$. For the induction step, we need to show that $[\gamma_n^p(A), [\gamma_m^p(C), C]]$ and $[(\gamma_n^p(A))^p, (\gamma_m^p(C))^p]$ are both included in $\gamma_{m+n+1}^p(A)$.

(a) The previous induction (on *n*) and our current induction hypothesis (on *m*) give $[[\gamma_n^p(A), C], \gamma_m^p(C]] \subseteq [\gamma_{n+1}^p(A), \gamma_m^p(C)] \subseteq \gamma_{m+n+1}^p(A)$ and $[[\gamma_n^p(A), \gamma_m^p(C)], C] \subseteq [\gamma_{m+n}^p(A), C] \subseteq \gamma_{m+n+1}^p(A)$. The first inclusion now follows from Lemma 2.1.

(b) Next, let $a \in \gamma_n^p(A)$ and $c \in \gamma_m^p(C)$. By Lemma 2.2, we have that $[a, c^p] \equiv [a, c]^p$ modulo $[[\gamma_n^p(A), \gamma_m^p(C)], \gamma_m^p(C)]]$, a subgroup of $[\gamma_{m+n}^p(A), \gamma_m^p(C)] \subseteq \gamma_{2m+n}^p(A) \subseteq$ $\gamma_{m+n+1}^p(A)$. But $[a, c] \in [\gamma_n^p(A), \gamma_m^p(C)] \subseteq \gamma_{m+n}^p(A)$, and so $[a, c]^p \in (\gamma_{m+n}^p(A))^p \subseteq$ $\gamma_{m+n+1}^p(A)$. Thus, $[a, c^p] \in \gamma_{m+n+1}^p(A)$, and the second inclusion is proved as well.

This completes the proof.

Theorem 12.2. Let $B = A \rtimes_{\varphi} C$ be a mod-p almost direct product. Then $L_n^p = \gamma_n^p(A)$ for all $n \ge 1$.

Proof. By Lemma 11.5, we have that $\gamma_n^p(A) \subseteq L_n^p$, for all $n \ge 1$. We prove the reverse inclusion by induction on *n*, with the base case n = 1 being obvious. Assume $L_n^p \subseteq \gamma_n^p(A)$, so that, in fact, $L_n^p = \gamma_n^p(A)$. For the induction step, it is enough to show that all four subgroups from (46) that generate L_{n+1}^p are included in $\gamma_{n+1}^p(A)$.

By the definition of $\gamma^p(A)$, we have that $(L_n^p)^p = (\gamma_n^p(A))^p \subseteq \gamma_{n+1}^p(A)$ and $[A, L_n^p] = [A, \gamma_n^p(A)] \subseteq \gamma_{n+1}^p(A)$. Moreover, by Lemma 12.1, $[A, \gamma_n^p(C)] \subseteq \gamma_{n+1}^p(A)$ and $[L_n^p, C] = [\gamma_n^p(A), C] \subseteq \gamma_{n+1}^p(A)$. This finishes the induction and completes the proof.

12.2. **Applications.** In [5], Bellingeri and Gervais proved mod-*p* versions of two of the Falk and Randell theorems. We recover their result by our method.

Theorem 12.3 ([5]). Let $B = A \rtimes_{\varphi} C$ be a mod-p almost direct product. Then,

- (1) $\gamma_n^p(B) = \gamma_n^p(A) \rtimes_{\omega} \gamma_n^p(C)$, for all $n \ge 1$.
- (2) If A and C are residually p, then B is also residually p.

Proof. The first claim follows at once from Theorems 11.6 and 12.2. The second claim follows from claim (1) and Lemma 10.3 by an argument entirely similar to the one given in the proof of Corollary 6.6. \Box

We conclude with a mod-p analogue of Falk and Randell's theorem on the decomposition of the associated graded Lie algebra of an almost direct product.

Theorem 12.4. Let $1 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 1$ be a split exact sequence of groups, with monodromy $\varphi: C \to \operatorname{Aut}(A)$. Suppose C acts trivially on $H_1(A; \mathbb{Z}_p)$. Then the sequence

(52)
$$0 \longrightarrow \operatorname{gr}^{p}(A) \xrightarrow{\operatorname{gr}^{p}(\alpha)} \operatorname{gr}^{p}(B) \xrightarrow{\operatorname{gr}^{p}(\beta)} \operatorname{gr}^{p}(C) \longrightarrow 0,$$

is a split exact sequence of graded Lie algebras; thus, $\operatorname{gr}^p(B) \cong \operatorname{gr}^p(A) \rtimes_{\bar{\varphi}} \operatorname{gr}^p(C)$, where $\bar{\varphi} \colon \operatorname{gr}^p(C) \to \operatorname{Der}(\operatorname{gr}^p(A))$ is the map of Lie algebras induced by φ .

Proof. By Theorem 12.2, we have that $L_n^p = \gamma_n^p(A)$ for all $n \ge 1$, and so $\operatorname{gr}^{L^p}(A) = \operatorname{gr}^p(A)$. The claim now follows from Corollary 11.7.

Example 12.5. Let $G = A \rtimes C = \langle a, t | tat^{-1} = a^{-1} \rangle$ be the Klein bottle group. Note that $C = \langle t \rangle$ acts trivially on $H_1(A; \mathbb{Z}_2) = \mathbb{Z}_2$, and thus Theorem 12.4 applies, with p = 2. From Example 10.2, we know that $\operatorname{gr}_n^2(A) = \mathbb{Z}_2 = \langle a^{2^{n-1}} \rangle$ and $\operatorname{gr}_n^2(C) = \mathbb{Z}_2 = \langle t^{2^{n-1}} \rangle$; therefore, $\operatorname{gr}_n^2(G) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ for all $n \ge 1$. Yet the extension (52) is nontrivial in this case, since $[t, a] = a^2$. Interestingly, $\operatorname{gr}^2(G) \not\cong \operatorname{gr}^2(\mathbb{Z}^2)$, although $H^*(G; \mathbb{Z}_2) \cong H^*(\mathbb{Z}^2; \mathbb{Z}_2)$, as graded rings.

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