Topology and combinatorics of Milnor fibrations of hyperplane arrangements

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HYPERPLANE ARRANGEMENTS

- $\mathcal{A}$: A (central) arrangement of hyperplanes in $\mathbb{C}^\ell$.
- Intersection lattice: $L(\mathcal{A})$.
- Complement: $M(\mathcal{A}) = \mathbb{C}^\ell \setminus \bigcup_{H\in\mathcal{A}} H$.
- The Boolean arrangement $\mathcal{B}_n$
  - $\mathcal{B}_n$: all coordinate hyperplanes $z_i = 0$ in $\mathbb{C}^n$.
  - $L(\mathcal{B}_n)$: lattice of subsets of $\{0, 1\}^n$.
  - $M(\mathcal{B}_n)$: complex algebraic torus $(\mathbb{C}^*)^n$.
- The braid arrangement $\mathcal{A}_n$ (or, reflection arr. of type A_{n-1})
  - $\mathcal{A}_n$: all diagonal hyperplanes $z_i - z_j = 0$ in $\mathbb{C}^n$.
  - $L(\mathcal{A}_n)$: lattice of partitions of $[n] = \{1, \dots, n\}$.
  - $M(\mathcal{A}_n)$: configuration space of n ordered points in $\mathbb{C}$ (a classifying space for the pure braid group on n strings).

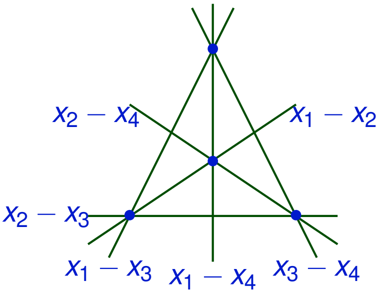


FIGURE : A planar slice of the braid arrangement $\mathcal{A}_4$

- Let $\mathcal{A}$ be an arrangement of planes in $\mathbb{C}^3$. Its projectivization, $\bar{\mathcal{A}}$, is an arrangement of lines in $\mathbb{CP}^2$.
- $L_1(\mathcal{A}) \longleftrightarrow$ lines of $\bar{\mathcal{A}}$, $L_2(\mathcal{A}) \longleftrightarrow$ intersection points of $\bar{\mathcal{A}}$. Poset structure of $L_{\leqslant 2}(\mathcal{A}) \longleftrightarrow$ incidence structure of $\bar{\mathcal{A}}$.
- A flat $X \in L_2(\mathcal{A})$ has multiplicity q if $\mathcal{A}_X = \{H \in \mathcal{A} \mid X \supset H\}$ has size q, i.e., there are exactly q lines from $\bar{\mathcal{A}}$ passing through $\bar{X}$.

- If $\mathcal{A}$ is essential, then $M = M(\mathcal{A})$ is a (very affine) subvariety of $(\mathbb{C}^*)^n$, where $n = |\mathcal{A}|$.
- M has the homotopy type of a connected, finite CW-complex of dimension ℓ. In fact, M admits a minimal cell structure.
- In particular, $H_*(M, \mathbb{Z})$ is torsion-free. The Betti numbers $b_q(M) := \operatorname{rank} H_q(M, \mathbb{Z})$ are given by

$$\sum_{q=0}^{\ell} b_q(M) t^q = \sum_{X \in L(\mathcal{A})} \mu(X)(-t)^{\operatorname{rank}(X)}.$$

- The Orlik–Solomon algebra $A = H^*(M, \mathbb{Z})$ is the quotient of the exterior algebra on generators dual to the meridians, by an ideal determined by the circuits in the matroid of $\mathcal{A}$.
- On the other hand, the group $\pi_1(M)$ is not determined by $L(\mathcal{A})$.

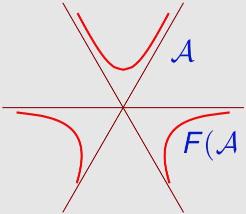
THE MILNOR FIBRATION OF AN ARRANGEMENT

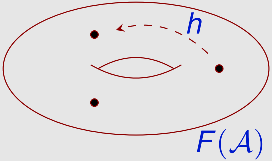
- For each $H \in \mathcal{A}$, let $f_H \colon \mathbb{C}^\ell \to \mathbb{C}$ be a linear form with kernel H
- Let $Q(\mathcal{A}) = \prod_{H \in \mathcal{A}} f_H$, a homogeneous polynomial of degree n.
- The map $Q \colon \mathbb{C}^\ell \to \mathbb{C}$ restricts to a map $Q \colon M(\mathcal{A}) \to \mathbb{C}^*$.
- This is the projection of a smooth, locally trivial bundle, known as the *Milnor fibration* of the arrangement.
- The typical fiber, $F(\mathcal{A}) = Q^{-1}(1)$, is a very affine variety, with the homotopy type of a connected, finite CW-complex of dim $\ell - 1$.
- The monodromy of the bundle is the diffeomorphism

$$h \colon F \to F, \quad z \mapsto e^{2\pi \mathrm{i}/n} z.$$

EXAMPLE

Let $\mathcal{A}$ be a pencil of 3 lines through the origin of $\mathbb{C}^2$. Then $F(\mathcal{A})$ is a thrice-punctured torus, and h is an automorphism of order 3:





More generally, if $\mathcal{A}$ is a pencil of n lines in $\mathbb{C}^2$, then $F(\mathcal{A})$ is a Riemann surface of genus $\binom{n-1}{2}$, with n punctures.

EXAMPLE

Let $\mathcal{B}_n$ be the Boolean arrangement, with $Q = z_1 \cdots z_n$. Then $M(\mathcal{B}_n) = (\mathbb{C}^*)^n$ and $F(\mathcal{B}_n) = \ker(\mathbb{Q}) \cong (\mathbb{C}^*)^{n-1}$.

Two basic questions about the Milnor fibration of an arrangement:

(Q1) Are the Betti numbers $b_q(F(\mathcal{A}))$ and the characteristic polynomial of the algebraic monodromy, $h_q\colon H_q(F(\mathcal{A}),\mathbb{C}) \to H_q(F(\mathcal{A}),\mathbb{C})$, determined by $L(\mathcal{A})$?

(Q2) Are the homology groups $H_*(F(\mathcal{A}),\mathbb{Z})$ torsion-free? If so, does $F(\mathcal{A})$ admit a minimal cell structure?

Recent progress on both questions:

- A partial, positive answer to (Q1).
- A negative answer to (Q2).

Let $\Delta_{\mathcal{A}}(t) := \det(h_1 - t \cdot \mathrm{id})$. Then $b_1(F(\mathcal{A})) = \deg \Delta_{\mathcal{A}}$.

THEOREM (PAPADIMA–S. 2013)

Suppose all flats $X \in L_2(\mathcal{A})$ have multiplicity 2 or 3. Then $\Delta_{\mathcal{A}}(t)$, and thus $b_1(F(\mathcal{A}))$, are combinatorially determined.

THEOREM (DENHAM–S. 2013)

For every prime $p \geqslant 2$, there is an arrangement $\mathcal{A}$ such that $H_q(F(\mathcal{A}), \mathbb{Z})$ has non-zero p-torsion, for some $q > 1$.

- In both results, we relate the cohomology jump loci of $M(\mathcal{A})$ in characteristic p with those in characteristic 0.
- In the first result, the bridge between the two goes through the representation variety $\mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{h}(\mathcal{A}), \mathfrak{sl}_2)$.
- A key combinatorial ingredient in both proofs is the notion of multinet.

RESONANCE VARIETIES AND THE β_p-INVARIANTS

- Let $A = H^*(M(\mathcal{A}), \Bbbk)$ — an algebra that depends only on $L(\mathcal{A})$ (and the field $\Bbbk$).
- For each $a \in A^1$, we have $a^2 = 0$. Thus, we get a cochain complex, $(A, \cdot a)\colon\ A^0 \xrightarrow{a} A^1 \xrightarrow{a} A^2 \longrightarrow \cdots$
- The (degree 1) *resonance varieties* of $\mathcal{A}$ are the cohomology jump loci of this "Aomoto complex":

$$\mathcal{R}_s(\mathcal{A}, \Bbbk) = \{a \in A^1 \mid \dim_{\Bbbk} H^1(A, \cdot a) \geqslant s\},$$

- In particular, $a \in A^1$ belongs to $\mathcal{R}_1(\mathcal{A}, \Bbbk)$ iff there is $b \in A^1$ not proportional to a, such that $a \cup b = 0$ in A^2.

- Now assume $\Bbbk$ has characteristic $p > 0$.
- Let $\sigma = \sum_{H \in \mathcal{A}} e_H \in A^1$ be the "diagonal" vector, and define

$$\beta_p(\mathcal{A}) = \dim_{\Bbbk} H^1(A, \cdot\sigma).$$

That is, $\beta_p(\mathcal{A}) = \max\{s \mid \sigma \in \mathcal{R}^1_s(A, \Bbbk)\}$.

- Clearly, $\beta_p(\mathcal{A})$ depends only on $L(\mathcal{A})$ and p. Moreover, $0 \leqslant \beta_p(\mathcal{A}) \leqslant |\mathcal{A}| - 2$.

THEOREM (PS)

If $L_2(\mathcal{A})$ has no flats of multiplicity $3r$ with $r > 1$, then $\beta_3(\mathcal{A}) \leqslant 2$.

- For each $m \geqslant 1$, there is a matroid $\mathcal{M}_m$ with all rank 2 flats of multiplicity 3, and such that $\beta_3(\mathcal{M}_m) = m$.
- $\mathcal{M}_1$: pencil of 3 lines. $\mathcal{M}_2$: Ceva arrangement.
- $\mathcal{M}_m$ with $m > 2$: not realizable over $\mathbb{C}$.

THE HOMOLOGY OF THE MILNOR FIBER

- The monodromy $h\colon F(\mathcal{A}) \to F(\mathcal{A})$ has order $n = |\mathcal{A}|$. Thus,

$$\Delta_{\mathcal{A}}(t) = \prod_{d|n} \Phi_d(t)^{e_d(\mathcal{A})},$$

where $\Phi_1 = t-1$, $\Phi_2 = t+1$, $\Phi_3 = t^2+t+1$, $\Phi_4 = t^2+1$, ... are the cyclotomic polynomials, and $e_d(\mathcal{A}) \in \mathbb{Z}_{\geqslant 0}$.

- Easy to see: $e_1(\mathcal{A}) = n-1$. Hence, $H_1(F(\mathcal{A}), \mathbb{C})$, when viewed as a module over $\mathbb{C}[\mathbb{Z}_n]$, decomposes as

$$(\mathbb{C}[t]/(t-1))^{n-1} \oplus \bigoplus_{1<d|n} (\mathbb{C}[t]/\Phi_d(t))^{e_d(\mathcal{A})}.$$

- In particular, $b_1(F(\mathcal{A})) = n - 1 + \sum_{1<d|n} \varphi(d) e_d(\mathcal{A})$.

- Thus, in degree 1, question (Q1) is equivalent to: are the integers $e_d(\mathcal{A})$ determined by $L_{\leqslant 2}(\mathcal{A})$?
- Not all divisors of n appear in the above formulas: If d does *not* divide $|\mathcal{A}_X|$, for some $X \in L_2(\mathcal{A})$, then $e_d(\mathcal{A}) = 0$ (Libgober).
- In particular, if $L_2(\mathcal{A})$ has only flats of multiplicity 2 and 3, then $\Delta_{\mathcal{A}}(t) = (t-1)^{n-1}(t^2+t+1)^{e_3}$.
- If multiplicity 4 appears, then also get factor of $(t+1)^{e_2} \cdot (t^2+1)^{e_4}$.

THEOREM (COHEN–ORLIK 2000, PAPADIMA–S. 2010)

$e_{p^s}(\mathcal{A}) \leqslant \beta_p(\mathcal{A})$, for all $s \geqslant 1$.

THEOREM (PS13)

Suppose $L_2(\mathcal{A})$ has no flats of multiplicity $3r$, with $r > 1$. Then $e_3(\mathcal{A}) = \beta_3(\mathcal{A})$, and thus $e_3(\mathcal{A})$ is combinatorially determined.

A similar result holds for $e_2(\mathcal{A})$ and $e_4(\mathcal{A})$, under some additional hypothesis.

COROLLARY

If $\bar{\mathcal{A}}$ is an arrangement of n lines in $\mathbb{P}^2$ with only double and triple points, then $\Delta_{\mathcal{A}}(t) = (t-1)^{n-1}(t^2+t+1)^{\beta_3(\mathcal{A})}$ is combinatorially determined.

COROLLARY (LIBGOBER 2012)

If $\bar{\mathcal{A}}$ is an arrangement of n lines in $\mathbb{P}^2$ with only double and triple points, then the question whether $\Delta_{\mathcal{A}}(t) = (t-1)^{n-1}$ or not is combinatorially determined.

CONJECTURE

Let $\mathcal{A}$ be an essential arrangement in $\mathbb{C}^3$. Then

$$\Delta_{\mathcal{A}}(t) = (t-1)^{|\mathcal{A}|-1}(t^2+t+1)^{\beta_3(\mathcal{A})}[(t+1)(t^2+1)]^{\beta_2(\mathcal{A})}.$$

MULTINETS

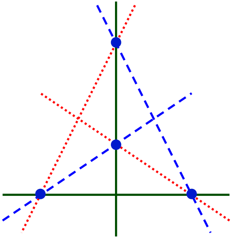
DEFINITION (FALK AND YUZVINSKY)

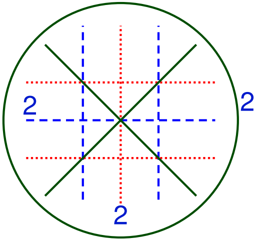
A *multinet* on $\mathcal{A}$ is a partition of the set $\mathcal{A}$ into $k \geqslant 3$ subsets $\mathcal{A}_1, \dots, \mathcal{A}_k$, together with an assignment of multiplicities, $m\colon \mathcal{A} \to \mathbb{N}$, and a subset $\mathcal{X} \subseteq L_2(\mathcal{A})$, called the base locus, such that:

1. There is an integer d such that $\sum_{H \in \mathcal{A}_\alpha} m_H = d$, for all $\alpha \in [k]$.
2. If H and H' are in different classes, then $H \cap H' \in \mathcal{X}$.
3. For each $X \in \mathcal{X}$, the sum $n_X = \sum_{H \in \mathcal{A}_\alpha : H \supset X} m_H$ is independent of α.
4. Each set $\left(\bigcup_{H \in \mathcal{A}_\alpha} H\right) \backslash \mathcal{X}$ is connected.

- A similar definition can be made for any (rank 3) matroid.
- A multinet as above is also called a (k, d)-multinet, or a k-multinet.
- The multinet is *reduced* if $m_H = 1$, for all $H \in \mathcal{A}$.

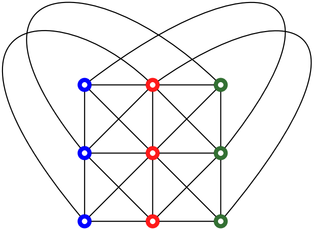
- A *net* is a reduced multinet with $n_X = 1$, for all $X \in \mathcal{X}$.
- In this case, $|\mathcal{A}_\alpha| = |\mathcal{A}| / k = d$, for all α.
- Moreover, $\bar{\mathcal{X}}$ has size d^2, and is encoded by a $(k-2)$-tuple of orthogonal Latin squares.

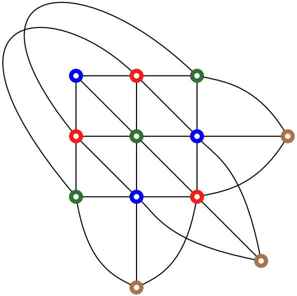




A $(3,2)$-net on the A_3 arrangement
$\bar{\mathcal{X}}$ consists of 4 triple points ($n_X = 1$)

A $(3,4)$-multinet on the B_3 arrangement
$\bar{\mathcal{X}}$ consists of 4 triple points ($n_X = 1$)
and 3 triple points ($n_X = 2$)





A $(3,3)$-net on the Ceva matroid. A $(4,3)$-net on the Hessian matroid.

- If $\mathcal{A}$ has no flats of multiplicity kr, for some $r > 1$, then every reduced k-multinet is a k-net.
- (Kawahara): given any Latin square, there is a matroid $\mathcal{M}$ with a 3-net $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3)$ realizing it, such that each $\mathcal{M}_\alpha$ is uniform.
- (Yuzvinsky and Pereira–Yuz): If $\mathcal{A}$ supports a k-multinet with $|\mathcal{X}| > 1$, then $k = 3$ or 4; if the multinet is not reduced, then $k = 3$.
- (Wakefield & al): The only $(4, 3)$-net in $\mathbb{CP}^2$ is the Hessian; there are no $(4, 4)$, $(4, 5)$, or $(4, 6)$ nets in $\mathbb{CP}^2$.
- Conjecture (Yuz): The only 4-multinet is the Hessian $(4, 3)$-net.

LEMMA (PS)

If $\mathcal{A}$ supports a 3-net with parts $\mathcal{A}_\alpha$, then:

1. $1 \leqslant \beta_3(\mathcal{A}) \leqslant \beta_3(\mathcal{A}_\alpha) + 1$, *for all* α.
2. *If* $\beta_3(\mathcal{A}_\alpha) = 0$, *for some* α, *then* $\beta_3(\mathcal{A}) = 1$.
3. *If* $\beta_3(\mathcal{A}_\alpha) = 1$, *for some* α, *then* $\beta_3(\mathcal{A}) = 1$ *or* 2.

All possibilities do occur:

- Braid arrangement: has a $(3,2)$-net from the Latin square of $\mathbb{Z}_2$. $\beta_3(\mathcal{A}_\alpha) = 0$ $(\forall \alpha)$ and $\beta_3(\mathcal{A}) = 1$.
- Pappus arrangement: has a $(3,3)$-net from the Latin square of $\mathbb{Z}_3$. $\beta_3(\mathcal{A}_1) = \beta_3(\mathcal{A}_2) = 0$, $\beta_3(\mathcal{A}_3) = 1$ and $\beta_3(\mathcal{A}) = 1$.
- Ceva arrangement: has a $(3,3)$-net from the Latin square of $\mathbb{Z}_3$. $\beta_3(\mathcal{A}_\alpha) = 1$ $(\forall \alpha)$ and $\beta_3(\mathcal{A}) = 2$.

COMPLEX RESONANCE VARIETIES

Let $\mathcal{A}$ be an arrangement in $\mathbb{C}^3$. Work of Arapura, Falk, Cohen–S., Libgober–Yuz, Falk–Yuz completely describes the varieties $\mathcal{R}_s(\mathcal{A}, \mathbb{C})$:

- $\mathcal{R}_1(\mathcal{A}, \mathbb{C})$ is a union of linear subspaces in $H^1(M(\mathcal{A}), \mathbb{C}) = \mathbb{C}^{|\mathcal{A}|}$.
- Each subspace has dimension at least 2, and each pair of subspaces meets transversely at 0.
- $\mathcal{R}_s(\mathcal{A}, \mathbb{C})$ is the union of those linear subspaces that have dimension at least $s+1$.

- Each flat $X \in L_2(\mathcal{A})$ of multiplicity $k \geqslant 3$ gives rise to a *local* component of $\mathcal{R}_1(\mathcal{A}, \mathbb{C})$, of dimension $k - 1$.
- More generally, every k-multinet on a sub-arrangement $\mathcal{B} \subseteq \mathcal{A}$ gives rise to a component of dimension $k - 1$, and all components of $\mathcal{R}_1(\mathcal{A}, \mathbb{C})$ arise in this way.
- Note: the varieties $\mathcal{R}_1(\mathcal{A}, \mathbb{k})$ with $\mathrm{char}(\mathbb{k}) > 0$ can be more complicated: components may be non-linear, and they may intersect non-transversely.

THEOREM (PS)

Suppose $L_2(\mathcal{A})$ has no flats of multiplicity $3r$, with $r > 1$. Then $\mathcal{R}_1(\mathcal{A}, \mathbb{C})$ has at least $(3^{\beta_3(\mathcal{A})} - 1)/2$ essential components, all corresponding to 3-nets.

CHARACTERISTIC VARIETIES

- Let X be a connected, finite cell complex, and let $\pi = \pi_1(X, x_0)$.
- Let $\Bbbk$ be an algebraically closed field, and let $\operatorname{Hom}(\pi, \Bbbk^*) = H^1(X, \Bbbk^*)$ be the character group of π.
- The (degree 1) *characteristic varieties* of X are the jump loci for homology with coefficients in rank-1 local systems on X:

$$\mathcal{V}_s(X, \Bbbk) = \{\rho \in \operatorname{Hom}(\pi, \Bbbk^*) \mid \dim_{\Bbbk} H_1(X, \Bbbk_\rho) \geqslant s\}.$$

- Let $X = M(\mathcal{A})$, and identify $\operatorname{Hom}(\pi, \Bbbk^*) = (\Bbbk^*)^n$, where $n = |\mathcal{A}|$.
- The characteristic varieties $\mathcal{V}_s(\mathcal{A}, \Bbbk) := \mathcal{V}_s(M(\mathcal{A}), \Bbbk)$ lie in the subtorus $\{t \in (\Bbbk^*)^n \mid t_1 \cdots t_n = 1\}$.

Work of Arapura, Libgober, Cohen–S., S., Libgober–Yuz, Falk–Yuz, Dimca, Dimca–Papadima–S., Artal–Cogolludo–Matei, Budur–Wang ... provides a fairly explicit description of the varieties $\mathcal{V}_s(\mathcal{A},\mathbb{C})$:

- Each variety $\mathcal{V}_s(\mathcal{A},\mathbb{C})$ is a finite union of torsion-translates of algebraic subtori of $(\mathbb{C}^*)^n$.
- If a linear subspace $L \subset \mathbb{C}^n$ is a component of $\mathcal{R}_s(\mathcal{A},\mathbb{C})$, then the algebraic torus $T = \exp(L)$ is a component of $\mathcal{V}_s(\mathcal{A},\mathbb{C})$.
- Moreover, $T = f^*(H^1(S,\mathbb{C}^*))$, where $f\colon M(\mathcal{A}) \to S$ is an orbifold fibration, with base $S = \mathbb{CP}^1 \setminus \{k \text{ points}\}$, for some $k \geqslant 3$.
- All components of $\mathcal{V}_s(\mathcal{A},\mathbb{C})$ passing through the origin $1 \in (\mathbb{C}^*)^n$ arise in this way (and thus, are combinatorially determined).

BACK TO THE MILNOR FIBRATION

- The Milnor fiber $F(\mathcal{A})$ is a regular $\mathbb{Z}_n$-cover of the projectivized complement $U = M(\mathcal{A})/\mathbb{C}^*$.
- This cover classified by the homomorphism $\delta\colon \pi_1(U) \twoheadrightarrow \mathbb{Z}_n$ that sends each meridian to 1.
- Let $\hat{\delta}\colon \operatorname{Hom}(\mathbb{Z}_n, \Bbbk^*) \to \operatorname{Hom}(\pi_1(U), \Bbbk^*)$. If $\operatorname{char}(\Bbbk) \nmid n$, then

$$\dim_{\Bbbk} H_1(F(\mathcal{A}), \Bbbk) = \sum_{s\geqslant 1} \left|\mathcal{V}_s(U, \Bbbk) \cap \operatorname{im}(\hat{\delta})\right|.$$

The available information on $\mathcal{V}_s(U, \mathbb{C}) \cong \mathcal{V}_s(\mathcal{A}, \mathbb{C})$ implies:

THEOREM

If $\mathcal{A}$ admits a reduced k-multinet, then $e_k(\mathcal{A}) \geqslant k - 2$.

THEOREM (PS)

Suppose $L_2(\mathcal{A})$ has no flats of multiplicity $3r$ with $r > 1$. Then, the following conditions are equivalent:

1. $L_{\leqslant 2}(\mathcal{A})$ *admits a reduced* 3*-multinet.*
2. $L_{\leqslant 2}(\mathcal{A})$ *admits a* 3*-net.*
3. $\beta_3(\mathcal{A}) \neq 0$.
4. $e_3(\mathcal{A}) \neq 0$.

Moreover, $\beta_3(\mathcal{A}) \leqslant 2$ *and* $\beta_3(\mathcal{A}) = e_3(\mathcal{A})$.

- (2) $\Rightarrow$ (1): obvious.
- (1) $\Rightarrow$ (4): by above theorem.
- (4) $\Rightarrow$ (3): by modular bound $e_p(\mathcal{A}) \leqslant \beta_p(\mathcal{A})$.
- (3) $\Rightarrow$ (2): use flat, $\mathfrak{sl}_2$-valued connections on the OS-algebra.
- $\beta_3(\mathcal{A}) \leqslant 2$: a previous theorem.
- Last assertion: put things together, and use [ACM].

TORSION IN THE HOMOLOGY OF THE MILNOR FIBER

- Let $(\mathcal{A}, m)$ be a multi-arrangement, with defining polynomial

$$Q_m(\mathcal{A}) = \prod_{H\in\mathcal{A}} f_H^{m_H},$$

- Let $F_m(\mathcal{A}) = Q_m^{-1}(1)$ be the corresponding Milnor fiber.

THEOREM (COHEN–DENHAM–S. 2003)

For every prime $p \geqslant 2$, there is a multi-arrangement $(\mathcal{A}, m)$ such that $H_1(F_m(\mathcal{A}), \mathbb{Z})$ has non-zero p-torsion.

Simplest example: the arrangement of 8 hyperplanes in $\mathbb{C}^3$ with

$$Q_m(\mathcal{A}) = x^2 y(x^2 - y^2)^3 (x^2 - z^2)^2 (y^2 - z^2)$$

Then $H_1(F_m(\mathcal{A}), \mathbb{Z}) = \mathbb{Z}^7 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

We now can generalize and reinterpret these examples, as follows.

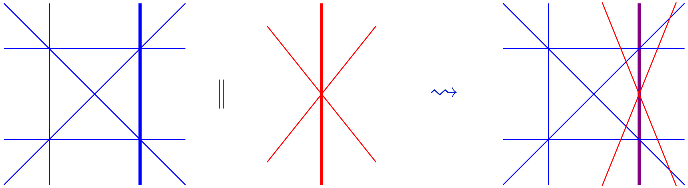
A *pointed multinet* on an arrangement $\mathcal{A}$ is a multinet structure, together with a distinguished hyperplane $H \in \mathcal{A}$ for which $m_H > 1$ and $m_H \mid n_X$ for each $X \in \mathcal{X}$ such that $X \subset H$.

THEOREM (DENHAM–S. 2013)

Suppose $\mathcal{A}$ admits a pointed multinet, with distinguished hyperplane H and multiplicity m. Let p be a prime dividing m_H. There is then a choice of multiplicities m' on the deletion $\mathcal{A}' = \mathcal{A} \setminus \{H\}$ such that $H_1(F_{m'}(\mathcal{A}'), \mathbb{Z})$ has non-zero p-torsion.

This torsion is explained by the fact that the geometry of $\mathcal{V}_1(\mathcal{A}', \Bbbk)$ varies with $\operatorname{char}(\Bbbk)$.

To produce p-torsion in the homology of the usual Milnor fiber, we use a “polarization" construction:

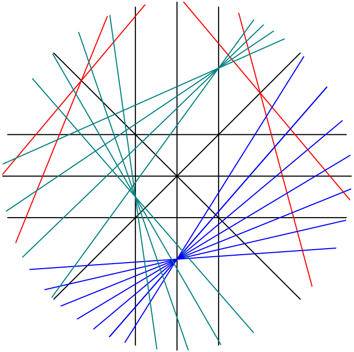


$(\mathcal{A}, m) \rightsquigarrow \mathcal{A} \| m$, an arrangement of $N = \sum_{H \in \mathcal{A}} m_H$ hyperplanes, of rank equal to $\operatorname{rank} \mathcal{A} + |\{H \in \mathcal{A} \colon m_H \geqslant 2\}|$.

THEOREM (DS)

Suppose $\mathcal{A}$ admits a pointed multinet, with distinguished hyperplane H and multiplicity m. Let p be a prime dividing m_H.

There is then a choice of multiplicities m' on the deletion $\mathcal{A}' = \mathcal{A} \backslash \{H\}$ such that $H_q(F(\mathcal{B}), \mathbb{Z})$ has p-torsion, where $\mathcal{B} = \mathcal{A}' \| m'$ and $q = 1 + \left|\left\{K \in \mathcal{A}' : m'_K \geqslant 3\right\}\right|$.



Simplest example: the arrangement of 27 hyperplanes in $\mathbb{C}^8$ with defining polynomial

$$Q(\mathcal{A}) = xy(x^2-y^2)(x^2-z^2)(y^2-z^2)w_1w_2w_3w_4w_5(x^2-w_1^2)(x^2-2w_1^2)(x^2-3w_1^2)(x-4w_1)\cdot$$

$$((x-y)^2-w_2^2)((x+y)^2-w_3^2)((x-z)^2-w_4^2)((x-z)^2-2w_4^2)\cdot((x+z)^2-w_5^2)((x+z)^2-2w_5^2).$$

Then $H_6(F(\mathcal{A}),\mathbb{Z})$ has 2-torsion (of rank 108).