Duality, finiteness, and cohomology jump loci

Alex Suciu

Northeastern University

Conference in Algebraic Geometry for
Alexandru Dimca's retirement

Ischia, Italy

October 12, 2021

Resonance varieties

- Let $A = (A^\bullet, \mathrm{d}_A)$ be a connected, locally finite, graded-commutative, differential graded algebra (cdga) over a field $\Bbbk$, and let $M = (M^\bullet, \mathrm{d}_M)$ be an A-cdgm.
- Since $A^0 = \Bbbk$, we have $Z^1(A) \cong H^1(A)$.
- Set $\mathcal{Q}(A) = \{a \in Z^1(A) \mid a^2 = 0 \in A^2\}$. For each $a \in \mathcal{Q}(A)$, we have a cochain complex,

$$(M^\bullet, \delta_a)\colon \; M^0 \xrightarrow{\delta_a^0} M^1 \xrightarrow{\delta_a^1} M^2 \xrightarrow{\delta_a^2} \cdots,$$

  with differentials $\delta_a^i(m) = a \cdot m + \mathrm{d}_M(m)$, for all $m \in M^i$.
- The *resonance varieties* of M (in degree $i \geq 0$ and depth $k \geq 0$):

$$\mathcal{R}_k^i(M) = \{a \in \mathcal{Q}(A) \mid \dim_\Bbbk H^i(M^\bullet, \delta_a) \geq k\}.$$

- Assume $\operatorname{char} \Bbbk \neq 2$. Since $a^2 = -a^2$ for all $a \in A^1$, we have $\mathcal{Q}(A) = Z^1(A)$, and so $\mathcal{R}_k^i(A)$ are subvarieties of $H^1(A)$.

Resonance varieties of graded algebras

- Assume further that $d = 0$ (i.e., A is a cga). Then the resonance varieties of A are homogenous subvarieties of $H^1(A) = A^1$.
- An element $a \in A^1$ belongs to $\mathcal{R}^i_k(A)$ if and only if there exist $u_1, \dots, u_k \in A^i$ such that $au_1 = \cdots = au_k = 0$ in A^{i+1}, and the set $\{au, u_1, \dots, u_k\}$ is linearly independent in A^i, for all $u \in A^{i-1}$.
- Set $b_j = b_j(A)$. For each $i \geq 0$, we have a descending filtration,

$$A^1 = \mathcal{R}^i_0(A) \supseteq \mathcal{R}^i_1(A) \supseteq \cdots \supseteq \mathcal{R}^i_{b_i}(A) = \{0\} \supset \mathcal{R}^i_{b_{i+1}}(A) = \emptyset.$$

- A linear subspace $U \subset A^1$ is *isotropic* if the restriction of $A^1 \wedge A^1 \xrightarrow{\cdot} A^2$ to $U \wedge U$ is the zero map (i.e., $ab = 0$, $\forall a, b \in U$).
- If $U \subseteq A^1$ is isotropic and $\dim U = k$, then $U \subseteq \mathcal{R}^1_{k-1}(A)$.
- $\mathcal{R}^1_1(A)$ is the union of all isotropic planes in A^1.
- If $\Bbbk \subset \mathbb{K}$ is a field extension, then the $\Bbbk$-points on $\mathcal{R}^i_k(A \otimes_{\Bbbk} \mathbb{K})$ coincide with $\mathcal{R}^i_k(A)$.

The BGG correspondence

- Let (A, d) be a connected, finite-type $\Bbbk$-cdga, where $\operatorname{char}(\Bbbk) \neq 2$.
- Fix a $\Bbbk$-basis $\{e_1, \dots, e_n\}$ for $H^1(A) \cong Z^1(A)$, and let $\{x_1, \dots, x_n\}$ be the dual basis for $H_1(A) = (H^1(A))^\vee$.
- Identify $\operatorname{Sym}(H_1(A))$ with $S = \Bbbk[x_1, \dots, x_n]$, the coordinate ring of the affine space $H^1(A)$.
- A BGG-type correspondence yields a cochain complex of finitely generated, free S-modules, $(A^\bullet \otimes S, \delta_A)$,

$$\cdots \longrightarrow A^i \otimes S \xrightarrow{\delta_A^i} A^{i+1} \otimes S \xrightarrow{\delta_A^{i+1}} A^{i+2} \otimes S \longrightarrow \cdots,$$

  where $\delta_A^i(u \otimes s) = \sum_{j=1}^r e_j u \otimes s x_j + \mathrm{d}\, u \otimes s$.
- The specialization of this complex at $a \in Z^1(A)$ is (A, δ_a).
- Hence, $\mathcal{R}_k^i(A)$ is the zero-set of the ideal generated by all minors of size $b_i(A) - k + 1$ of the block-matrix $\delta_A^{i+1} \oplus \delta_A^i$.

Characteristic varieties

- Let X be a connected CW-complex with finite q-skeleton ($q \geq 1$). Let $G = \pi_1(X)$, and set $n = \operatorname{rank}(G_{\mathrm{ab}}) = b_1(X)$.
- Let $\mathbb{T}_G := \operatorname{Hom}(G, \mathbb{C}^*)$ be the character group of $G = \pi_1(X)$, also denoted by $\operatorname{Char}(X) := H^1(X, \mathbb{C}^*)$. Then $\mathbb{T}_G$ is an algebraic group with coordinate ring $\mathbb{C}[G_{\mathrm{ab}}]$, and $\mathbb{T}_G \cong (\mathbb{C}^*)^n \times \operatorname{Tors}(G_{\mathrm{ab}})$.
- The *characteristic varieties* of X are the sets

$$\mathcal{V}^i_k(X) = \{\rho \in \mathbb{T}_G \mid \dim_{\mathbb{C}} H_i(X, \mathbb{C}_\rho) \geq k\}.$$

- These sets are Zariski closed for all $i \leq q$ and all $k \geq 0$.
- We may define similarly $\mathcal{V}^i_k(X, \Bbbk) \subset H^1(X, \Bbbk^*)$ for any field $\Bbbk$.
- These constructions are compatible with restriction and extension of the base field. Namely, if $\Bbbk \subset \mathbb{L}$ is a field extension, then

$$\mathcal{V}^i_k(X, \Bbbk) = \mathcal{V}^i_k(X, \mathbb{L}) \cap H^1(X, \Bbbk^\times),$$

$$\mathcal{V}^i_k(X, \mathbb{L}) = \mathcal{V}^i_k(X, \Bbbk) \times_{\Bbbk} \mathbb{L}.$$

- Let $\exp\colon \mathbb{C}^n \to (\mathbb{C}^*)^n$. Given a subvariety $W \subset (\mathbb{C}^*)^n$, define its *exponential tangent cone* at 1 (identity of $(\mathbb{C}^*)^n$) as

$$\tau_1(W) = \{z \in \mathbb{C}^n \mid \exp(\lambda z) \in W,\ \forall \lambda \in \mathbb{C}\}.$$

- (Dimca–Papadima–S. 2009) $\tau_1(W)$ is a finite union of rationally defined linear subspaces.
- Given a subfield $\Bbbk \subset \mathbb{C}$, we write $\tau_1^{\Bbbk}(W) = \tau_1(W) \cap \Bbbk^n$.
- Let A and B be two $\Bbbk$-cdgas. We say $A \simeq B$ if there is a zig-zag of quasi-isomorphisms connecting A to B. If those maps are isos in degrees $\leq q$ and injective in degree $q+1$, we say $A \simeq_q B$.
- A is *formal* (or just q*-formal*) if it is (q-) equiv. to $(H^\bullet(A), d=0)$.
- Given any (path-connected) space X, there is an associated Sullivan $\mathbb{Q}$-cdga, $A_{\mathrm{PL}}(X)$, such that $H^\bullet(A_{\mathrm{PL}}(X)) = H^\bullet(X, \mathbb{Q})$.
- An *algebraic (*q*-)model* for X (over $\Bbbk \supseteq \mathbb{Q}$) is a $\Bbbk$-cgda (A, d) which is (q-) equivalent to $A_{\mathrm{PL}}(X) \otimes_{\mathbb{Q}} \Bbbk$.

The tangent cone theorem

- Let X be a connected CW-complex with finite q-skeleton, and suppose X admits a q-finite q-model A (e.g., X q-formal).
- Set $\mathcal{R}^i_k(X,\Bbbk) := \mathcal{R}^i_k(H^\bullet(X,\Bbbk))$ and $\mathcal{R}^i_k(X) := \mathcal{R}^i_k(X,\mathbb{C})$.

Theorem

For all $i \le q$ and all $k \ge 0$:

- *(Dimca–Papadima–S. 2009, Dimca–Papadima 2014)*

$$\mathcal{V}^i_k(X)_{(1)} \cong \mathcal{R}^i_k(A)_{(0)}.$$

In particular, if X is q-formal, then $\mathcal{V}^i_k(X)_{(1)} \cong \mathcal{R}^i_k(X)_{(0)}$.

- *(Budur–Wang 2017) All the irreducible components of $\mathcal{V}^i_k(X)$ passing through the origin of $\mathrm{Char}(X)$ are* algebraic subtori.

- Consequently,

$$\tau_1(\mathcal{V}^i_k(X)) = \mathrm{TC}_1(\mathcal{V}^i_k(X)) = \mathcal{R}^i_k(A).$$

BIERI–NEUMANN–STREBEL–RENZ INVARIANTS

- Let G be a finitely generated group, $n = b_1(G) > 0$. Let $S(G) = S^{n-1}$ be the unit sphere in $\mathrm{Hom}(G, \mathbb{R}) = \mathbb{R}^n$.
- (Bieri–Neumann–Strebel 1987)

$$\Sigma^1(G) = \{\chi \in S(G) \mid \mathrm{Cay}_\chi(G) \text{ is connected}\},$$

  where $\mathrm{Cay}_\chi(G)$ is the induced subgraph of $\mathrm{Cay}(G)$ on vertex set $G_\chi = \{g \in G \mid \chi(g) \geq 0\}$.
- (Bieri–Renz 1988)

$$\Sigma^q(G, \mathbb{Z}) = \{\chi \in S(G) \mid \text{the monoid } G_\chi \text{ is of type } \mathrm{FP}_q\},$$

  i.e., there is a projective $\mathbb{Z}G_\chi$-resolution $P_\bullet \to \mathbb{Z}$, with P_i finitely generated for all $i \leq q$. Moreover, $\Sigma^1(G, \mathbb{Z}) = -\Sigma^1(G)$.
- The BNSR-invariants of G form a descending chain of open subsets of $S(G)$.

- The Σ-invariants control the finiteness properties of normal subgroups $N \triangleleft G$ for which G/N is free abelian:

$$N \text{ is of type } \mathrm{FP}_q \iff S(G,N) \subseteq \Sigma^q(G,\mathbb{Z})$$

where $S(G,N) = \{\chi \in S(G) \mid \chi(N) = 0\}$. In particular, $\ker(\chi\colon G \twoheadrightarrow \mathbb{Z})$ is f.g. $\iff \{\pm\chi\} \subseteq \Sigma^1(G)$.

- More generally, let X be a connected CW-complex with finite q-skeleton, for some $q \geq 1$.

- Let $G = \pi_1(X, x_0)$. For each $\chi \in S(X) := S(G)$, let

$$\widehat{\mathbb{Z}G}_\chi = \Big\{\lambda \in \mathbb{Z}^G \mid \{g \in \operatorname{supp}\lambda \mid \chi(g) \geq c\} \text{ is finite, } \forall c \in \mathbb{R}\Big\}$$

be the Novikov–Sikorav completion of $\mathbb{Z}G$.

- (Farber–Geoghegan–Schütz 2010)

$$\Sigma^q(X,\mathbb{Z}) = \{\chi \in S(X) \mid H_i(X, \widehat{\mathbb{Z}G}_{-\chi}) = 0,\ \forall\, i \leq q\}.$$

- (Bieri 2007) If G is FP_k, then $\Sigma^q(G,\mathbb{Z}) = \Sigma^q(K(G,1),\mathbb{Z})$, $\forall q \leq k$.

Tropical varieties

- Let $\mathbb{K} = \mathbb{C}\{\{t\}\} = \bigcup_{n\geq 1} \mathbb{C}((t^{1/n}))$ be the field of Puiseux series.
- Elements of $\mathbb{K}^*$: $c(t) = c_1 t^{a_1} + c_2 t^{a_2} + \cdots$, where $c_i \in \mathbb{C}^*$ and $a_1 < a_2 < \cdots$ are rationals with a common denominator.
- $\mathbb{K}$ is an algebraically closed field; it admits a non-Archimedean valuation, $v\colon \mathbb{K}^* \to \mathbb{Q}$, given by $v(c(t)) = a_1$.
- Let $v\colon (\mathbb{K}^*)^n \to \mathbb{Q}^n \subset \mathbb{R}^n$ be the n-fold product of the valuation.
- The *tropicalization* of a subvariety $W \subset (\mathbb{K}^*)^n$, denoted $\operatorname{Trop}(W)$, is the closure (in the Euclidean topology) of $v(W)$ in $\mathbb{R}^n$.
- This is a rational polyhedral complex in $\mathbb{R}^n$. For instance, if W is a curve, then $\operatorname{Trop}(W)$ is a graph with rational edge directions.
- For a variety $W \subset (\mathbb{C}^*)^n$, we may define its tropicalization by setting $\operatorname{Trop}(W) = \operatorname{Trop}(W \times_{\mathbb{C}} \mathbb{K})$. This is a polyhedral fan in $\mathbb{R}^n$.

Tropicalizing the characteristic varieties

- Let X be a space as above, and set $n = b_1(X)$. We define $\nu_X\colon \mathrm{Char}_{\mathbb{K}}(X) \to \mathbb{Q}^n \subset \mathbb{R}^n$ to be the composite

$$H^1(X, \mathbb{K}^*) \xrightarrow{\nu_*} H^1(X, \mathbb{Q}) \longrightarrow H^1(X, \mathbb{R}).$$

- Given an algebraic subvariety $W \subset H^1(X, \mathbb{C}^*)$ we define its *tropicalization* as the closure in $H^1(X, \mathbb{R}) \cong \mathbb{R}^n$ of the image of $W \times_{\mathbb{C}} \mathbb{K} \subset H^1(X, \mathbb{K}^*)$ under ν_X:

$$\mathrm{Trop}(W) := \overline{\nu_X(W \times_{\mathbb{C}} \mathbb{K})}.$$

- Applying this definition to the varieties $\mathcal{V}^i(X) := \mathcal{V}^i_1(X)$ and recalling that $\mathcal{V}^i(X, \mathbb{K}) = \mathcal{V}^i(X) \times_{\mathbb{C}} \mathbb{K}$, we get

$$\mathrm{Trop}(\mathcal{V}^i(X)) = \overline{\nu_X(\mathcal{V}^i(X, \mathbb{K}))}.$$

Lemma

Let $W \subset (\mathbb{C}^*)^n$ be an algebraic variety. Then $\tau_1^{\mathbb{R}}(W) \subseteq \mathrm{Trop}(W)$.

A TROPICAL BOUND FOR THE Σ-INVARIANTS

THEOREM (PAPADIMA–S.-2010, S-2021)

Let $\rho\colon \pi_1(X) \to \Bbbk^*$ be a character such that $\rho \in \mathcal{V}^{\le q}(X, \Bbbk)$. Let $v\colon \Bbbk^* \to \mathbb{R}$ be the homomorphism defined by a valuation on $\Bbbk$, and write $\chi = v \circ \rho$. If the homomorphism $\chi\colon \pi_1(X) \to \mathbb{R}$ is non-zero, then $\chi \notin \Sigma^q(X, \mathbb{Z})$.

THEOREM (S-2021)

$$\Sigma^q(X, \mathbb{Z}) \subseteq S(\operatorname{Trop}(\mathcal{V}^{\le q}(X)))^{\complement}$$

Hence:

- (Papadima-S.-2010) $\Sigma^q(X, \mathbb{Z}) \subseteq S(\tau_1^{\mathbb{R}}(\mathcal{V}^{\le q}(X)))^{\complement}$
- If X is q-formal, then $\Sigma^i(X, \mathbb{Z}) \subseteq S\left(\mathcal{R}^{\le i}(X)\right)^{\complement}$ for all $i \le q$.
- If $\mathcal{V}^{\le q}(X)$ contains a component of $\operatorname{Char}(X)$, then $\Sigma^q(X, \mathbb{Z}) = \emptyset$.

Kähler manifolds

- Let M be a compact Kähler manifold. Then M is formal.
- (Beauville, Catanese, Green–Lazarsfeld, Simpson, Arapura, Budur, B. Wang) The varieties $\mathcal{V}^i_k(M)$ are finite unions of torsion translates of algebraic subtori of $H^1(M, \mathbb{C}^*)$.

Theorem (Delzant 2010)

$$\Sigma^1(M) = S(M) \setminus \bigcup_\alpha S(f_\alpha^*(H^1(C_\alpha, \mathbb{R}))),$$

where the union is taken over those orbifold fibrations $f_\alpha \colon M \to C_\alpha$ with the property that either $\chi(C_\alpha) < 0$, or $\chi(C_\alpha) = 0$ and f_α has some multiple fiber.

In degree 1, we may recast this result in the tropical setting, as follows.

Corollary

$$\Sigma^1(M) = S(\operatorname{Trop}(\mathcal{V}^1(M))^{\complement}.$$

Hyperplane arrangements

- Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an (essential, central) arrangement of hyperplanes in $\mathbb{C}^d$, with intersection lattice $L(\mathcal{A})$.
- The complement, $M(\mathcal{A}) := (\mathbb{C}^*)^d \setminus \bigcup_{i=1}^n H_i$, is a smooth, quasi-projective Stein manifold; thus, it has the homotopy type of a finite, d-dimensional CW-complex.
- (Orlik–Solomon) The cohomology ring $H^*(M(\mathcal{A}), \mathbb{Z})$ is determined by $L(\mathcal{A})$.
- Thus, the resonance varieties $\mathcal{R}^i(\mathcal{A}) := \mathcal{R}^i(M(\mathcal{A})) \subset \mathbb{C}^n$ depend only on $L(\mathcal{A})$.
- (Arapura) The characteristic varieties $\mathcal{V}^i(\mathcal{A}) := \mathcal{V}^i(M(\mathcal{A})) \subset (\mathbb{C}^*)^n$ are unions of translated subtori.
- Consequently, $\operatorname{Trop}(\mathcal{V}^i(\mathcal{A})) = -\operatorname{Trop}(\mathcal{V}^i(\mathcal{A}))$.
- (Denham–S.–Yuzvinsky 2016/17) $M(\mathcal{A})$ is an "abelian duality space", and so $\mathcal{V}^1(\mathcal{A}) \subseteq \mathcal{V}^2(\mathcal{A}) \subseteq \cdots \subseteq \mathcal{V}^{d-1}(\mathcal{A})$.
- (Arnol'd, Brieskorn) $M(\mathcal{A})$ is formal. Thus, $\tau_1(\mathcal{V}^i(\mathcal{A})) = \mathcal{R}^i(\mathcal{A})$.

Theorem

Let M be the complement of an arrangement of n hyperplanes in $\mathbb{C}^d$. Then, for each $1 \leq q \leq d-1$:

- $\operatorname{Trop}(\mathcal{V}^q(M))$ *is the union of a subspace arrangement in* $\mathbb{R}^n$.
- $\Sigma^q(M, \mathbb{Z}) \subseteq S(\operatorname{Trop}(\mathcal{V}^q(M)))^{\complement}$.

Question (MFO Miniworkshop 2007)

Given an arrangement $\mathcal{A}$, do we have

$$\Sigma^1(M(\mathcal{A})) = S(\mathcal{R}^1(\mathcal{A}, \mathbb{R}))^{\complement}? \qquad (\star)$$

Example (Koban–McCammond–Meier 2013)

- Let $\mathcal{A}$ be the braid arrangement in $\mathbb{C}^n$, defined by $\prod_{1\le i<j\le n}(z_i - z_j) = 0$. Then $M(\mathcal{A}) = \mathrm{Conf}(n, \mathbb{C}) \simeq K(P_n, 1)$.
- Answer to ($\star$) is yes: $\Sigma^1(M(\mathcal{A}))$ is the complement of the union of $\binom{n}{3} + \binom{n}{4}$ planes in $\mathbb{C}^{\binom{n}{2}}$, intersected with the unit sphere.

Example

- Let $\mathcal{A}$ be the "deleted B_3" arrangement, defined by $z_1 z_2 (z_1^2 - z_2^2)(z_1^2 - z_3^2)(z_2^2 - z_3^2) = 0$.
- (S. 2002) $\mathcal{V}^1(\mathcal{A})$ contains a (1-dimensional) translated torus $\rho \cdot T$.
- Thus, $\mathrm{Trop}(\rho \cdot T) = \mathrm{Trop}(T)$ is a line in $\mathbb{C}^8$ which is *not* contained in $\mathcal{R}^1(\mathcal{A}, \mathbb{R})$. Hence, the answer to ($\star$) is no.

Question (Revised)

$$\Sigma^1(M(\mathcal{A})) = S(\mathrm{Trop}(\mathcal{V}^1(\mathcal{A}))^{\complement}? \qquad (\star\star)$$

Classification of alternating forms

(Following J. Schouten, G. Gurevich, D. Djoković, A. Cohen–A. Helminck, ...)

- Let V be a $\Bbbk$-vector space of dimension n. The group $\mathrm{GL}(V)$ acts on $\bigwedge^m(V^*)$ by $(g\cdot\mu)(a_1\wedge\cdots\wedge a_m)=\mu\left(g^{-1}a_1\wedge\cdots\wedge g^{-1}a_m\right)$.
- The orbits of this action are the equivalence classes of alternating m-forms on V. (We write $\mu\sim\mu'$ if $\mu'=g\cdot\mu$.)
- Over $\overline{\Bbbk}$, the closures of these orbits are affine algebraic varieties.
- There are finitely many orbits over $\overline{\Bbbk}$ only if $n^2\geq\binom{n}{m}$, that is, $m\leq 2$ or $m=3$ and $n\leq 8$.
- For $\overline{\Bbbk}=\mathbb{C}$, each complex orbit has only finitely many real forms.
- When $m=3$, and $n=8$, there are 23 complex orbits, which split into either 1, 2, or 3 real orbits, for a total of 35 real orbits.

Poincaré duality algebras

- Let A be a connected, locally finite $\mathbb{k}$-cga.
- A is a *Poincaré duality* $\mathbb{k}$*-algebra* of dimension m if there is a $\mathbb{k}$-linear map $\varepsilon\colon A^m \to \mathbb{k}$ (called an *orientation*) such that all the bilinear forms $A^i \otimes_{\mathbb{k}} A^{m-i} \to \mathbb{k}$, $a \otimes b \mapsto \varepsilon(ab)$ are non-singular.
- If A is a PD_m algebra, then:
  - $b_i(A) = b_{m-i}(A)$, and $A^i = 0$ for $i > m$.
  - ε is an isomorphism.
  - The maps $\mathrm{PD}\colon A^i \to (A^{m-i})^*$, $\mathrm{PD}(a)(b) = \varepsilon(ab)$ are isomorphisms.
- Each $a \in A^i$ has a *Poincaré dual*, $a^\vee \in A^{m-i}$, such that $\varepsilon(aa^\vee) = 1$.
- The *orientation class* is $\omega_A := 1^\vee$. We have $\varepsilon(\omega_A) = 1$, and thus $aa^\vee = \omega_A$.

THE ASSOCIATED ALTERNATING FORM

- Associated to a $\Bbbk$-PD_m algebra there is an alternating m-form,
$$\mu_A\colon \bigwedge^m A^1 \to \Bbbk, \quad \mu_A(a_1 \wedge \cdots \wedge a_m) = \varepsilon(a_1 \cdots a_m).$$
- A and B are isomorphic as PD_m algebras if and only if they are isomorphic as graded algebras, in which case $\mu_A \sim \mu_B$.
- Assume now that $m = 3$, and set $n = b_1(A)$. Fix a basis $\{e_1, \dots, e_n\}$ for A^1, and let $\{e_1^\vee, \dots, e_n^\vee\}$ be the dual basis for A^2.
- The multiplication in A, then, is given on basis elements by
$$e_i e_j = \sum_{k=1}^{r} \mu_{ijk}\, e_k^\vee, \quad e_i e_j^\vee = \delta_{ij}\omega,$$
where $\mu_{ijk} = \mu(e_i \wedge e_j \wedge e_k)$.
- Two PD_3 algebras A and B are isomorphic if and only if $\mu_A \sim \mu_B$. We thus have a bijection, $A \leftrightsquigarrow \mu_A$, between isomorphism classes of PD_3 algebras and equivalence classes of alternating 3-forms.

POINCARÉ DUALITY IN ORIENTABLE MANIFOLDS

- If M is a compact, connected, orientable, m-dimensional manifold, then the cohomology ring $A = H^{\bullet}(M, \Bbbk)$ is a PD_m algebra over $\Bbbk$.
- Sullivan (1975): for every finite-dimensional $\mathbb{Q}$-vector space V and every alternating 3-form $\mu \in \bigwedge^3 V^*$, there is a closed 3-manifold M with $H^1(M, \mathbb{Q}) = V$ and cup-product form $\mu_M = \mu$.
- Such a 3-manifold can be constructed via “Borromean surgery."

- E.g., 0-surgery on the Borromean rings in S^3 yields $M = T^3$, with $\mu_M = e^1 e^2 e^3$.
- If M is the link of an isolated surface singularity (e.g., if $M = \Sigma(p, q, r)$ is a Brieskorn manifold), then $\mu_M = 0$.

Resonance varieties of PD-algebras

- Let A be a PD_m algebra. For $0 \le i \le m$ and $a \in A^1$, we have

$$\left(H^i(A,\delta_a)\right)^\vee \cong H^{m-i}(A,\delta_{-a}).$$

- Hence, $\mathcal{R}^i_k(A) = \mathcal{R}^{m-i}_k(A)$ for all i and k, In particular, $\mathcal{R}^m_1(A) = \mathcal{R}^0_k(A) = \{0\}$.

Theorem

Let A be a PD_3 algebra with $b_1(A) = n$. Then $\mathcal{R}^i_k(A) = \emptyset$, except for:

1) $\mathcal{R}^i_0(A) = A^1$ *for all* $i \ge 0$.
2) $\mathcal{R}^3_1(A) = \mathcal{R}^0_1(A) = \{0\}$ *and* $\mathcal{R}^2_n(A) = \mathcal{R}^1_n(A) = \{0\}$.
3) $\mathcal{R}^2_k(A) = \mathcal{R}^1_k(A)$ *for* $0 < k < n$.

Moreover, $\mathcal{R}^1_k(A) = A^1$ *for all* $k < \operatorname{corank} \mu_A$

(The *rank* of $\mu\colon \bigwedge^3 V \to \mathbb{k}$ is the minimum dimension of a linear subspace $W \subset V$ such that μ factors through $\bigwedge^3 W$.)

- A linear subspace $U \subset V$ is 2-*singular* with respect to a 3-form $\mu\colon \bigwedge^3 V \to \Bbbk$ if $\mu(a \wedge b \wedge c) = 0$ for all $a, b \in U$ and $c \in V$.
- If $\dim U = 2$, we simply say U is a *singular plane*.
- The *nullity* of μ, denoted $\operatorname{null}(\mu)$, is the maximum dimension of a 2-singular subspace $U \subset V$.
- Clearly, V contains a singular plane if and only if $\operatorname{null}(\mu) \geq 2$.
- Let A be a PD_3 algebra. A linear subspace $U \subset A^1$ is 2-singular (with respect to μ_A) if and only if U is isotropic.
- Using a result of A. Sikora [2005], we obtain:

THEOREM

Let A be a PD_3 algebra over an algebraically closed field $\Bbbk$ with $\operatorname{char}(\Bbbk) \neq 2$, and let $\nu = \operatorname{null}(\mu_A)$. If $b_1(A) \geq 4$, then

$$\dim \mathcal{R}^1_{\nu-1}(A) \geq \nu \geq 2.$$

In particular, $\dim \mathcal{R}^1_1(A) \geq \nu$.

Real forms and resonance

- Sikora made the following conjecture: If $\mu\colon \bigwedge^3 V \to \Bbbk$ is a 3-form with $\dim V \geq 4$ and if $\operatorname{char}(\Bbbk) \neq 2$, then $\operatorname{null}(\mu) \geq 2$.
- Conjecture holds if $n := \dim V$ is even or equal to 5, or if $\Bbbk = \overline{\Bbbk}$.
- Work of J. Draisma and R. Shaw [2010, 2014] implies that the conjecture does not hold for $\Bbbk = \mathbb{R}$ and $n = 7$. We obtain:

Theorem

Let A be a PD_3 algebra over $\mathbb{R}$. Then $\mathcal{R}^1_1(A) \neq \{0\}$, except when

- $n = 1$, $\mu_A = 0$.
- $n = 3$, $\mu_A = e^1 e^2 e^3$.
- $n = 7$, $\mu_A = -e^1e^3e^5 + e^1e^4e^6 + e^2e^3e^6 + e^2e^4e^5 + e^1e^2e^7 + e^3e^4e^7 + e^5e^6e^7$.

Sketch: If $\mathcal{R}^1_1(A) = \{0\}$, then the formula $(x \times y) \cdot z = \mu_A(x, y, z)$ defines a cross-product on $A^1 = \mathbb{R}^n$, and thus a division algebra structure on $\mathbb{R}^{n+1}$, forcing $n = 1, 3$ or 7 by Bott–Milnor/Kervaire [1958].

Pfaffians and resonance

- For a $\Bbbk$-PD$_3$ algebra A, the complex $(A \otimes_{\Bbbk} S, \delta_A)$ looks like

$$A^0 \otimes_{\Bbbk} S \xrightarrow{\delta_A^0} A^1 \otimes_{\Bbbk} S \xrightarrow{\delta_A^1} A^2 \otimes_{\Bbbk} S \xrightarrow{\delta_A^2} A^3 \otimes_{\Bbbk} S\,,$$

  where $\delta_A^0 = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}$ and $\delta_A^2 = (\delta_A^0)^\top$, while δ_A^1 is the skew-symmetric matrix whose are entries linear forms in S given by

$$\delta_A^1(e_i) = \textstyle\sum_{j=1}^{n} \sum_{k=1}^{n} \mu_{jik} e_k^\vee \otimes x_j\,.$$

- Recall that $\mathcal{R}_k^1(A) = V(I_{n-k}(\delta_A^1))$. Using work of Buchsbaum and Eisenbud [1977] on Pfaffians of skew-symmetric matrices, we get:

Theorem

$\mathcal{R}_{2k}^1(A) = \mathcal{R}_{2k+1}^1(A) = V(\mathrm{Pf}_{n-2k}(\delta_A^1))$, *if n is even,*
$\mathcal{R}_{2k-1}^1(A) = \mathcal{R}_{2k}^1(A) = V(\mathrm{Pf}_{n-2k+1}(\delta_A^1))$, *if n is odd.*

Theorem

If μ_A has maximal rank $n \geq 3$, then

$$\mathcal{R}_{n-2}^1(A) = \mathcal{R}_{n-1}^1(A) = \mathcal{R}_n^1(A) = \{0\}.$$

Lemma (Turaev 2002)

Suppose $n \geq 3$. There is then a polynomial $\operatorname{Det}(\mu_A) \in \operatorname{Sym}(A_1)$ such that, if $\delta^1_A(i;j)$ is the sub-matrix obtained from δ^1_A by deleting the i-th row and j-th column, then $\det \delta^1_A(i;j) = (-1)^{i+j} x_i x_j \operatorname{Det}(\mu_A)$.

Moreover, if n is even, then $\operatorname{Det}(\mu_A) = 0$, while if n is odd, then $\operatorname{Det}(\mu_A) = \operatorname{Pf}(\mu_A)^2$, where $\operatorname{pf}(\delta^1_A(i;i)) = (-1)^{i+1} x_i \operatorname{Pf}(\mu_A)$.

Suppose $\dim_{\Bbbk} V = 2g+1 > 1$. We say $\mu\colon \bigwedge^3 V \to \Bbbk$ is *generic* (in the sense of Berceanu–Papadima [1994]) if there is a $v \in V$ such that the 2-form $\gamma_v \in V^\vee \wedge V^\vee$ given by $\gamma_v(a \wedge b) = \mu_A(a \wedge b \wedge v)$ for $a, b \in V$ has rank $2g$, that is, $\gamma_v^g \neq 0$ in $\bigwedge^{2g} V^\vee$.

Theorem

Let A be a PD_3 algebra with $b_1(A) = n$. Then

$$\mathcal{R}^1_1(A) = \begin{cases} \emptyset & \text{if } n = 0;\\ \{0\} & \text{if } n = 1 \text{ or } n = 3 \text{ and } \mu \text{ has rank } 3;\\ V(\operatorname{Pf}(\mu_A)) & \text{if } n \text{ is odd, } n > 3, \text{ and } \mu_A \text{ is BP-generic;}\\ A^1 & \text{otherwise.} \end{cases}$$

As a corollary, we recover a closely related result, proved by Draisma and Shaw [2010] by very different methods.

Corollary

Let V be a $\mathbb{k}$-vector space of odd dimension $n \geq 5$ and let $\mu \in \bigwedge^3 V^\vee$. Then the union of all singular planes is either all of V or a hypersurface defined by a homogeneous polynomial in $\mathbb{k}[V]$ of degree $(n-3)/2$.

For $\mu \in \bigwedge^3 V^\vee$, there is another genericity condition, due to P. De Poi, D. Faenzi, E. Mezzetti, and K. Ranestad [2017]: $\operatorname{rank}(\gamma_v) > 2$, for all non-zero $v \in V$. We may interpret some of their results, as follows.

Theorem (DFMR)

Let A be a PD_3 algebra over $\mathbb{C}$, and suppose μ_A is generic. Then:

- *If n is odd, then $\mathcal{R}^1_1(A)$ is a hypersurface of degree $(n-3)/2$ which is smooth if $n \leq 7$, and singular in codimension 5 if $n \geq 9$.*
- *If n is even, then $\mathcal{R}^1_2(A)$ has codim 3 and degree $\frac{1}{4}\binom{n-2}{3} + 1$; it is smooth if $n \leq 10$, and singular in codimension 7 if $n \geq 12$.*

ALEXANDER POLYNOMIALS OF 3-MANIFOLDS

- Let $H = H_1(X, \mathbb{Z})/\mathrm{Tors}$. Let $X^H \to X$ be the maximal torsion-free abelian cover of X, with cellular chain complex $C_\bullet(X^H, \partial^H)$.
- The *Alexander polynomial* $\Delta_X \in \mathbb{Z}[H]$ is the gcd of the codimension 1 minors of the Alexander matrix ∂_1^H.

Set $\mathcal{W}_1^1(M) = \mathcal{V}_1^1(M) \cap \mathrm{Char}^0(M)$. Using work of McMullen [2002] and Turaev [2002], as well as Dimca–Papadima–S. [2008], we find:

PROPOSITION

Let M be a closed, orientable, 3-dimensional manifold. Then $\mathcal{W}_1^1(M) = V(\Delta_M) \cup \{1\}$. If, moreover, $b_1(M) \geq 4$, then $\Delta_M(1) = 0$, and so $\mathcal{W}_1^1(M) = V(\Delta_M)$.

A Tangent Cone theorem for 3-manifolds

Let M be a closed, orientable, 3-manifold, and set $n = b_1(M)$.

Theorem

1) *If either $n \leq 1$, or n is odd, $n \geq 3$, and μ_M is BP-generic, then*

$$\mathrm{TC}_1(\mathcal{V}^1_1(M)) = \mathcal{R}^1_1(M).$$

2) *If n is even, $n \geq 2$, then $\mathcal{R}^1(M) = H^1(M, \mathbb{C})$. Moreover,*

$$\mathrm{TC}_1(\mathcal{V}^1_1(M)) = \mathcal{R}^1_1(M) \iff \Delta_M = 0.$$

Remark

For n even, the equality $\mathcal{R}^1(M) = H^1(M, \mathbb{C})$ was first proved in [Dimca–S, 2009], where it was used to show that the only 3-manifold groups which are also Kähler groups are the finite subgroups of $\mathrm{O}(4)$.

Theorem

1) *If $n \leq 1$, then M is formal, and $M \simeq_{\mathbb{Q}} S^3$ or $S^1 \times S^2$.*
2) *If n is even, $n \geq 2$, and $\Delta_M \neq 0$, then M is not 1-formal.*
3) *If $\Delta_M \neq 0$, yet $\Delta_M(1) = 0$ and $\operatorname{TC}_1(V(\Delta_M))$ is not a finite union of $\mathbb{Q}$-linear subspaces, then M admits no 1-finite 1-model.*

- (BNS 1987) The BNS invariant of $G = \pi_1(M)$ is the projection onto $S(G)$ of the open fibered faces of the Thurston norm ball B_T; in particular, $\Sigma^1(G) = -\Sigma^1(G)$.

Theorem

Let M be a compact, connected, orientable, 3-manifold with empty or toroidal boundary. Set $G = \pi_1(M)$ and assume $b_1(M) \geq 2$. Then

1) $\operatorname{Trop}\left(\mathcal{V}^1(G) \cap \mathbb{T}^0_G\right)$ *is the positive-codimension skeleton of $\mathcal{F}(B_A)$, the face fan of the unit ball in the Alexander norm.*
2) $\Sigma^1(G)$ *is contained in the union of the open cones on the facets of B_A.*

References

- A.I. Suciu, *Poincaré duality and resonance varieties*, Proceedings of the Royal Society of Edinburgh Section A: Mathematics **150** (2020), nr. 6, 3001–3027. doi:10.1017/prm.2019.55

- A.I. Suciu, *Cohomology jump loci of 3-manifolds*, Manuscripta Mathematica, doi:10.1007/s00229-020-01264-5

- A.I. Suciu, *Sigma-invariants and tropical varieties*, Mathematische Annalen **380** (2021), 1427–1463. doi:10.1007/s00208-021-02172-z