Topology of complex arrangements

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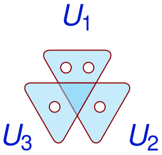
COMBINATORIAL COVERS

A *combinatorial cover* for a space X is a triple $(\mathscr{C}, \phi, \rho)$, where

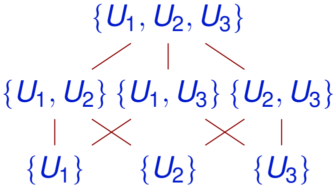
1. $\mathscr{C}$ is a countable cover which is either open, or closed and locally finite.
2. $\phi\colon N(\mathscr{C}) \to P$ is an order-preserving, surjective map from the nerve of the cover to a finite poset P, such that, if $S \leqslant T$ and $\phi(S) = \phi(T)$, then $\cap T \hookrightarrow \cap S$ admits a homotopy inverse.
3. If $S \leqslant T$ and $\bigcap S = \bigcap T$, then $\phi(S) = \phi(T)$.
4. $\rho\colon P \to \mathbb{Z}$ is an order-preserving map whose fibers are antichains.
5. ϕ induces a homotopy equivalence, $\phi\colon |N(\mathscr{C})| \to |P|$.

Example: $X = D^2 \setminus \{4 \text{ points}\}$.

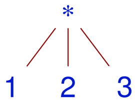
$\mathscr{C}$:



$N(\mathscr{C})$:



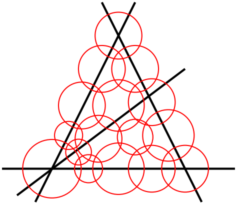
P :



- $\phi\colon N(\mathscr{C}) \to P$: $\phi(\{U_i\}) = i$ and $\phi(S) = *$ if $|S| \neq 1$.
- $\rho\colon P \to \mathbb{Z}$: $\rho(*) = 1$ and $\rho(i) = 0$.
- $\cap S = \cap T$ for any $S, T \in \phi^{-1}(*)$.
- Both $|N(\mathscr{C})|$ and $|P|$ are contractible.
- Thus, $\mathscr{C}$ is a combinatorial cover.

ARRANGEMENTS OF SUBMANIFOLDS

- Let $\mathcal{A}$ be an arrangement of submanifolds in a smooth, connected manifold. Assume each submanifold is either compact or open.
- Let $L(\mathcal{A})$ be the (ranked) intersection poset of $\mathcal{A}$.
- Assume that every element of $L(\mathcal{A})$ is smooth and contractible.



THEOREM (DENHAM–S.–YUZVINSKY 2014)

The complement $M(\mathcal{A})$ has a combinatorial cover $(\mathscr{C}, \phi, \rho)$ over $L(\mathcal{A})$.

A SPECTRAL SEQUENCE

THEOREM (DSY)

Suppose X has a combinatorial cover $(\mathscr{C}, \phi, \rho)$ over a poset P. For every locally constant sheaf $\mathcal{F}$ on X, there is a spectral sequence with

$$E_2^{pq} = \prod_{x \in P} \widetilde{H}^{p-\rho(x)-1}(\operatorname{lk}_{|P|}(x);\ H^{q+\rho(x)}(X, \mathcal{F}|_{U_x}))$$

converging to $H^{p+q}(X, \mathcal{F})$. Here, $U_x = \cap S$, where $S \in N(\mathscr{C})$ with $\phi(S) = x$.

DUALITY SPACES

Let X be a path-connected space, having the homotopy type of a finite-type CW-complex. Set $\pi = \pi_1(X)$.

Recall a notion due to Bieri and Eckmann (1978).

- X is a *duality space* of dimension n if $H^i(X, \mathbb{Z}\pi) = 0$ for $i \neq n$ and $H^n(X, \mathbb{Z}\pi) \neq 0$ and torsion-free.
- Let $D = H^n(X, \mathbb{Z}\pi)$ be the dualizing $\mathbb{Z}\pi$-module. Given any $\mathbb{Z}\pi$-module A, we have $H^i(X, A) \cong H_{n-i}(X, D \otimes A)$.
- If $D = \mathbb{Z}$, with trivial $\mathbb{Z}\pi$-action, then X is a Poincaré duality space.
- If $X = K(\pi, 1)$ is a duality space, then π is a *duality group*.

ABELIAN DUALITY SPACES

We introduce an analogous notion, by replacing $\pi \rightsquigarrow \pi_{\text{ab}}$.

- X is an *abelian duality space* of dimension n if $H^i(X, \mathbb{Z}\pi_{\text{ab}}) = 0$ for $i \neq n$ and $H^n(X, \mathbb{Z}\pi_{\text{ab}}) \neq 0$ and torsion-free.
- Let $B = H^n(X, \mathbb{Z}\pi_{\text{ab}})$ be the dualizing $\mathbb{Z}\pi_{\text{ab}}$-module. Given any $\mathbb{Z}\pi_{\text{ab}}$-module A, we have $H^i(X, A) \cong H_{n-i}(X, B \otimes A)$.
- The two notions of duality are independent.

Fix a field $\Bbbk$.

THEOREM (DENHAM–S.–YUZVINSKY 2015)

Let X be an abelian duality space of dimension n. If $\rho\colon \pi_1(X) \to \Bbbk^*$ satisfies $H^i(X, \Bbbk_\rho) \neq 0$, then $H^j(X, \Bbbk_\rho) \neq 0$, for all $i \leqslant j \leqslant n$.

CHARACTERISTIC VARIETIES

Consider the jump loci for cohomology with coefficients in rank-1 local systems on X,

$$\mathcal{V}^i_s(X,\Bbbk) = \{\rho \in \mathrm{Hom}(\pi_1(X),\Bbbk^*) \mid \dim_{\Bbbk} H_i(X,\Bbbk_\rho) \geqslant s\},$$

and set $\mathcal{V}^i(X,\Bbbk) = \mathcal{V}^i_1(X,\Bbbk)$.

COROLLARY (DSY)

Let X be an abelian duality space of dimension n. Then:

- *The characteristic varieties propagate:*

$$\mathcal{V}^1(X,\Bbbk) \subseteq \cdots \subseteq \mathcal{V}^n(X,\Bbbk).$$

- $\dim_{\Bbbk} H^1(X,\Bbbk) \geqslant n-1$.
- *If $n \geqslant 2$, then $H^i(X,\Bbbk) \neq 0$, for all $0 \leqslant i \leqslant n$.*

RESONANCE VARIETIES

- Assume $\text{char}(\Bbbk) \neq 2$, and set $A = H^*(X, \Bbbk)$.
- For each $a \in A^1$, we have a cochain complex

$$(A, \cdot a) \colon \; A^0 \xrightarrow{\;a\;} A^1 \xrightarrow{\;a\;} A^2 \longrightarrow \cdots$$

- The *resonance varieties* of X are the jump loci for the cohomology of these cochain complexes,

$$\mathcal{R}^i_s(X, \Bbbk) = \{a \in H^1(X, \Bbbk) \mid \dim_{\Bbbk} H^i(A, a) \geqslant s\}.$$

THEOREM (PAPADIMA–S. 2010)

Let X be a minimal CW-complex. Then the linearization of the cellular cochain complex $C^*(X^{\text{ab}}, \Bbbk)$, evaluated at $a \in A^1$ coincides with the cochain complex (A, a).

THEOREM (DSY)

Let X be an abelian duality space of dimension n which admits a minimal cell structure. Then the resonance varieties of X propagate:

$$\mathcal{R}^1(X,\Bbbk) \subseteq \cdots \subseteq \mathcal{R}^n(X,\Bbbk).$$

COROLLARY (DSY)

Let M be a compact, connected, orientable smooth manifold of dimension n. Suppose M admits a perfect Morse function, and $\mathcal{R}^1(M,\Bbbk) \neq 0$. Then M is not *an abelian duality space.*

EXAMPLE

- Let M be the 3-dimensional Heisenberg nilmanifold.
- M admits a perfect Morse function.
- Characteristic varieties propagate: $\mathcal{V}^i(M,\Bbbk) = \{1\}$ for $i \leqslant 3$.
- Resonance does not propagate: $\mathcal{R}^1(M,\Bbbk) = \Bbbk^2$ but $\mathcal{R}^3(M,\Bbbk) = 0$.

HYPERPLANE ARRANGEMENTS

- Let $\mathcal{A}$ be a central, essential hyperplane arrangement in $\mathbb{C}^n$.
- Its complement, $M(\mathcal{A})$, is a Stein manifold. It has the homotopy type of a minimal CW-complex of dimension n.
- $M(\mathcal{A})$ is a formal space.
- $M(\mathcal{A})$ admits a combinatorial cover.

THEOREM (DAVIS–JANUSZKIEWICZ–OKUN)

$M(\mathcal{A})$ is a duality space of dimension n.

Using the above spectral sequence, we prove:

THEOREM (DENHAM-S.-YUZVINSKY 2015)

$M(\mathcal{A})$ is an abelian duality space of dimension n. Furthermore, both the characteristic and resonance varieties of $M(\mathcal{A})$ propagate.

ELLIPTIC ARRANGEMENTS

- An *elliptic arrangement* is a finite collection, $\mathcal{A}$, of subvarieties in a product of elliptic curves E^n, each subvariety being a fiber of a group homomorphism $E^n \to E$.
- If $\mathcal{A}$ is essential, the complement $M(\mathcal{A})$ is a Stein manifold.
- $M(\mathcal{A})$ is minimal.
- $M(\mathcal{A})$ may be non-formal (examples by Bezrukavnikov and Berceanu–Măcinic–Papadima–Popescu).

THEOREM (DSY)

The complement of an essential, unimodular elliptic arrangement in E^n is both a duality space and an abelian duality space of dimension n.

In particular, the pure braid group of n strings on an elliptic curve is both a duality group and an abelian duality group.

RESONANCE VARIETIES AND MULTINETS

Let $\mathcal{R}_s(\mathcal{A}, \Bbbk) = \mathcal{R}^1_s(M(\mathcal{A}), \Bbbk)$. Work of Arapura, Falk, D.Cohen–A.S., Libgober–Yuzvinsky, and Falk–Yuzvinsky completely describes the varieties $\mathcal{R}_s(\mathcal{A}, \mathbb{C})$:

- $\mathcal{R}_1(\mathcal{A}, \mathbb{C})$ is a union of linear subspaces in $H^1(M(\mathcal{A}), \mathbb{C}) \cong \mathbb{C}^{|\mathcal{A}|}$.
- Each subspace has dimension at least 2, and each pair of subspaces meets transversely at 0.
- $\mathcal{R}_s(\mathcal{A}, \mathbb{C})$ is the union of those linear subspaces that have dimension at least $s + 1$.
- Each k*-multinet* on a sub-arrangement $\mathcal{B} \subseteq \mathcal{A}$ gives rise to a component of $\mathcal{R}_1(\mathcal{A}, \mathbb{C})$ of dimension $k - 1$. Moreover, all components of $\mathcal{R}_1(\mathcal{A}, \mathbb{C})$ arise in this way.

Definition (Falk and Yuzvinsky)

A *multinet* on $\mathcal{A}$ is a partition of the set $\mathcal{A}$ into $k \geqslant 3$ subsets $\mathcal{A}_1, \dots, \mathcal{A}_k$, together with an assignment of multiplicities, $m\colon \mathcal{A} \to \mathbb{N}$, and a subset $\mathcal{X} \subseteq L_2(\mathcal{A})$, called the base locus, such that:

1. There is an integer d such that $\sum_{H \in \mathcal{A}_\alpha} m_H = d$, for all $\alpha \in [k]$.
2. If H and H' are in different classes, then $H \cap H' \in \mathcal{X}$.
3. For each $X \in \mathcal{X}$, the sum $n_X = \sum_{H \in \mathcal{A}_\alpha : H \supset X} m_H$ is independent of α.
4. Each set $\left(\bigcup_{H \in \mathcal{A}_\alpha} H\right) \backslash \mathcal{X}$ is connected.

- A multinet as above is also called a (k, d)-multinet, or a k-multinet.
- The multinet is *reduced* if $m_H = 1$, for all $H \in \mathcal{A}$.
- A *net* is a reduced multinet with $n_X = 1$, for all $X \in \mathcal{X}$.

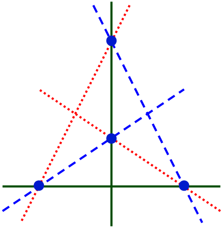


FIGURE : A $(3,2)$-net on the A_3 arrangement: $\mathcal{X}$ consists of 4 triple points ($n_X = 1$)

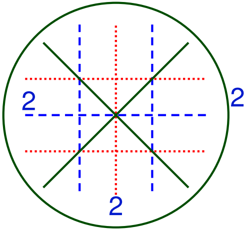
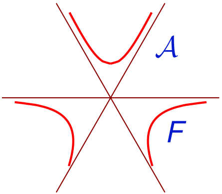
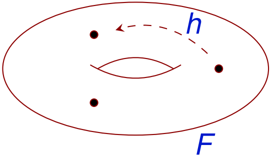


FIGURE : A $(3,4)$-multinet on the B_3 arrangement: $\mathcal{X}$ consists of 4 triple points ($n_X = 1$) and 3 triple points ($n_X = 2$)

MILNOR FIBRATION

- For each $H \in \mathcal{A}$ let α_H be a linear form with $\ker(\alpha_H) = H$, and let $Q = \prod_{H\in\mathcal{A}} \alpha_H$.
- $Q\colon \mathbb{C}^n \to \mathbb{C}$ restricts to a smooth fibration, $Q\colon M(\mathcal{A}) \to \mathbb{C}^*$.
- The typical fiber of this fibration, $Q^{-1}(1)$, is called the *Milnor fiber* of the arrangement, and is denoted by $F = F(\mathcal{A})$.
- F is neither formal, nor minimal, in general.
- The monodromy diffeomorphism, $h\colon F \to F$, is given by $h(z) = \exp(2\pi \mathrm{i}/m)z$, where $m = |\mathcal{A}|$.





Modular inequalities

- Let $\Delta(t)$ be the characteristic polynomial of the degree-1 algebraic monodromy, $h_*\colon H_1(F,\mathbb{C}) \to H_1(F,\mathbb{C})$.
- Since $h^m = \mathrm{id}$, we have

$$\Delta(t) = \prod_{d|m} \Phi_d(t)^{e_d(\mathcal{A})},$$

  where $\Phi_d(t)$ is the d-th cyclotomic polynomial, and $e_d(\mathcal{A}) \in \mathbb{Z}_{\geqslant 0}$.
- If there is a non-transverse multiple point on $\mathcal{A}$ of multiplicity not divisible by d, then $e_d(\mathcal{A}) = 0$.
- In particular, if $\mathcal{A}$ has only points of multiplicity 2 and 3, then $\Delta(t) = (t-1)^{m-1}(t^2+t+1)^{e_3}$.
- If multiplicity 4 appears, then also get factor of $(t+1)^{e_2} \cdot (t^2+1)^{e_4}$.

- Let $\sigma = \sum_{H\in\mathcal{A}} e_H \in A^1$ be the “diagonal" vector.
- Assume $\Bbbk$ has characteristic $p > 0$, and define

$$\beta_p(\mathcal{A}) = \dim_{\Bbbk} H^1(A, \cdot\sigma).$$

That is, $\beta_p(\mathcal{A}) = \max\{s \mid \sigma \in \mathcal{R}^1_s(A, \Bbbk)\}$.

THEOREM (COHEN–ORLIK 2000, PAPADIMA–S. 2010)

$e_{p^s}(\mathcal{A}) \leqslant \beta_p(\mathcal{A})$, *for all* $s \geqslant 1$.

THEOREM (PAPADIMA–S. 2014)

1. *Suppose* $\mathcal{A}$ *admits a* k*-net. Then* $\beta_p(\mathcal{A}) = 0$ *if* $p \nmid k$ *and* $\beta_p(\mathcal{A}) \geqslant k-2$, *otherwise.*
2. *If* $\mathcal{A}$ *admits a reduced* k*-multinet, then* $e_k(\mathcal{A}) \geqslant k-2$.

COMBINATORICS AND MONODROMY

THEOREM (PAPADIMA–S. 2014)

Suppose $\mathcal{A}$ has no points of multiplicity $3r$ with $r > 1$. Then, the following conditions are equivalent:

1. *$\mathcal{A}$ admits a reduced 3-multinet.*
2. *$\mathcal{A}$ admits a 3-net.*
3. $\beta_3(\mathcal{A}) \neq 0$.

Moreover, the following hold:

4. $\beta_3(\mathcal{A}) \leqslant 2$.
5. *$e_3(\mathcal{A}) = \beta_3(\mathcal{A})$, and thus $e_3(\mathcal{A})$ is combinatorially determined.*

THEOREM (PS)

Suppose $\mathcal{A}$ supports a 4-net and $\beta_2(\mathcal{A}) \leqslant 2$. Then

$$e_2(\mathcal{A}) = e_4(\mathcal{A}) = \beta_2(\mathcal{A}) = 2.$$

CONJECTURE (PS)

Let $\mathcal{A}$ be an arrangement which is not a pencil. Then $e_{p^s}(\mathcal{A}) = 0$ for all primes p and integers $s \geqslant 1$, with two possible exceptions:

$$e_2(\mathcal{A}) = e_4(\mathcal{A}) = \beta_2(\mathcal{A}) \text{ and } e_3(\mathcal{A}) = \beta_3(\mathcal{A}).$$

If $e_d(\mathcal{A}) = 0$ for all divisors d of $|\mathcal{A}|$ which are not prime powers, this conjecture would give:

$$\Delta_{\mathcal{A}}(t) = (t-1)^{|\mathcal{A}|-1}((t+1)(t^2+1))^{\beta_2(\mathcal{A})}(t^2+t+1)^{\beta_3(\mathcal{A})}.$$

The conjecture has been verified for several classes of arrangements:

- Complex reflection arrangements (Măcinic–Papadima–Popescu).
- Certain types of real arrangements (Yoshinaga, Bailet, Torielli).

TORSION IN HOMOLOGY

- A *pointed multinet* on an arrangement $\mathcal{A}$ is a multinet structure, together with a distinguished hyperplane $H \in \mathcal{A}$ for which $m_H > 1$ and $m_H \mid n_X$ for each $X \in \mathcal{X}$ such that $X \subset H$.
- We use a 'polarization' construction: $(\mathcal{A}, m) \rightsquigarrow \mathcal{A}\|m$, an arrangement of $N = \sum_{H\in\mathcal{A}} m_H$ hyperplanes, of rank equal to $\operatorname{rank}\mathcal{A} + |\{H \in \mathcal{A} \colon m_H \geqslant 2\}|$.

THEOREM (DENHAM–SUCIU 2014)

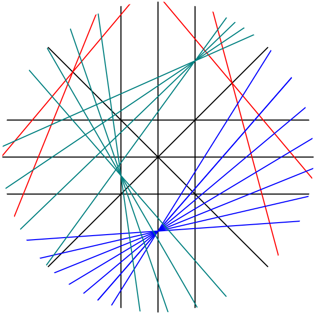
Suppose $\mathcal{A}$ admits a pointed multinet, with distinguished hyperplane H and multiplicity m. Let p be a prime dividing m_H.

There is then a choice of multiplicities m' on the deletion $\mathcal{A}' = \mathcal{A}\backslash\{H\}$ such that $H_q(F(\mathcal{B}), \mathbb{Z})$ has p-torsion, where $\mathcal{B} = \mathcal{A}'\|m'$ and $q = 1 + \left|\left\{K \in \mathcal{A}' : m'_K \geqslant 3\right\}\right|$.

In particular, $F(\mathcal{B})$ does not admit a minimal cell structure.

COROLLARY (DS)

For every prime $p \geqslant 2$, there is an arrangement $\mathcal{A}$ such that $H_q(F(\mathcal{A}), \mathbb{Z})$ has non-zero p-torsion, for some $q > 1$.



Simplest example: the arrangement of 27 hyperplanes in $\mathbb{C}^8$ with

$$Q(\mathcal{A}) = xy(x^2-y^2)(x^2-z^2)(y^2-z^2)w_1w_2w_3w_4w_5(x^2-w_1^2)(x^2-2w_1^2)(x^2-3w_1^2)(x-4w_1)\cdot$$

$$((x-y)^2-w_2^2)((x+y)^2-w_3^2)((x-z)^2-w_4^2)((x-z)^2-2w_4^2)\cdot((x+z)^2-w_5^2)((x+z)^2-2w_5^2).$$

Then $H_6(F(\mathcal{A}), \mathbb{Z})$ has 2-torsion (of rank 108).

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