Cohomology jump loci and duality properties

Alex Suciu

Northeastern University

Topology Seminar

Institute of Mathematics of the Romanian Academy

June 1, 2018

1. JUMP LOCI
   - Support loci
   - Homology jump loci
   - Resonance varieties of a cdgm
2. POINCARÉ DUALITY
   - Poincaré duality algebras
   - 3-dimensional Poincaré duality algebras
3. CHARACTERISTIC VARIETIES
   - Characteristic varieties
   - The Tangent Cone theorem
4. CHARACTERISTIC VARIETIES OF 3-MANIFOLDS
   - Alexander polynomials
   - A Tangent Cone theorem for 3-manifolds
5. ABELIAN DUALITY
   - Duality spaces
   - Abelian duality spaces
   - Arrangements of smooth hypersurfaces

SUPPORT LOCI

- Let $\mathbb{k}$ be an (algebraically closed) field.
- Let S be a commutative, finitely generated $\mathbb{k}$-algebra.
- Let $\operatorname{Spec}(S) = \operatorname{Hom}_{\mathbb{k}\text{-alg}}(S, \mathbb{k})$ be the maximal spectrum of S.
- Let $E: \cdots \to E_i \xrightarrow{d_i} E_{i-1} \to \cdots \to E_0 \to 0$ be an S-chain complex.
- The *support varieties* of E are the subsets of $\operatorname{Spec}(S)$ given by

$$\mathcal{W}^i_d(E) = \operatorname{supp}\Big(\bigwedge^d H_i(E) \Big).$$

- They depend only on the chain-homotopy equivalence class of E.
- For each $i \geqslant 0$, $\operatorname{Spec}(S) = \mathcal{W}^i_0(E) \supseteq \mathcal{W}^i_1(E) \supseteq \mathcal{W}^i_2(E) \supseteq \cdots$.
- If all E_i are finitely generated S-modules, then the sets $\mathcal{W}^i_d(E)$ are Zariski closed subsets of $\operatorname{Spec}(S)$.

HOMOLOGY JUMP LOCI

- The *homology jump loci* of the S-chain complex E are defined as

$$\mathcal{V}^i_d(E) = \{\mathfrak{m} \in \operatorname{Spec}(S) \mid \dim_{\Bbbk} H_i(E \otimes_S S/\mathfrak{m}) \geqslant d\}.$$

- They depend only on the chain-homotopy equivalence class of E.
- Get stratifications $\operatorname{Spec}(S) = \mathcal{V}^i_0(E) \supseteq \mathcal{V}^i_1(E) \supseteq \mathcal{V}^i_2(E) \supseteq \cdots$.

THEOREM (PAPADIMA–S. 2014)

Suppose E is a chain complex of free, *finitely generated S-modules. Then:*

- *Each $\mathcal{V}^i_d(E)$ is a Zariski closed subset of $\operatorname{Spec}(S)$.*
- *For each q,*

$$\bigcup_{i\leqslant q} \mathcal{V}^i_1(E) = \bigcup_{i\leqslant q} \mathcal{W}^i_1(E).$$

RESONANCE VARIETIES OF A CDGM

- Let $A = (A^\bullet, d_A)$ be a connected, finite-type $\Bbbk$-CDGA ($\operatorname{char} \Bbbk \neq 2$).
- Let $M = (M^\bullet, d_M)$ be an A-CDGM.
- For each $a \in Z^1(A) \cong H^1(A)$, we have a cochain complex,

$$(M^\bullet, \delta_a)\colon\ M^0 \xrightarrow{\delta_a^0} M^1 \xrightarrow{\delta_a^1} M^2 \xrightarrow{\delta_a^2} \cdots,$$

  with differentials $\delta_a^i(m) = a \cdot m + d(m)$, for all $m \in M^i$.
- The *resonance varieties* of A are the affine varieties

$$\mathcal{R}_s^i(M) = \{a \in H^1(A) \mid \dim_{\Bbbk} H^i(M^\bullet, \delta_a) \geqslant s\}.$$

- If A is a CGA (that is, $d_A = 0$), the resonance varieties $\mathcal{R}_s^i(A)$ are *homogeneous* subvarieties of A^1.

- Fix a $\Bbbk$-basis $\{e_1, \dots, e_r\}$ for A^1, and let $\{x_1, \dots, x_r\}$ be the dual basis for $A_1 = (A^1)^*$.
- Identify $\mathrm{Sym}(A_1)$ with $S = \Bbbk[x_1, \dots, x_r]$, the coordinate ring of the affine space A^1.
- Cochain complex of free S-modules, $\mathbf{L}(M) := (M^\bullet \otimes S, \delta)$:

$$\cdots \longrightarrow M^i \otimes S \xrightarrow{\delta^i} M^{i+1} \otimes S \xrightarrow{\delta^{i+1}} M^{i+2} \otimes S \longrightarrow \cdots,$$

where $\delta^i(m \otimes f) = \sum_{j=1}^n e_j m \otimes f x_j + \mathrm{d}(m) \otimes f$.

- The specialization of $(M \otimes S, \delta)$ at $a \in Z^1(A)$ is (M, δ_a).
- Hence, $\mathcal{R}^i_s(M)$ is the zero-set of the ideal generated by all minors of size $b_i(M) - s + 1$ of the block-matrix $\delta^{i+1} \oplus \delta^i$.
- In particular, $\mathcal{R}^1_s(M) = V(I_{r-s}(\delta^1))$, the zero-set of the ideal of codimension s minors of δ^1.

EXAMPLE (EXTERIOR ALGEBRA)

Let $E = \bigwedge V$, where $V = \Bbbk^n$, and $S = \mathrm{Sym}(V)$. Then $\mathbf{L}(E)$ is the Koszul complex on V. E.g., for $n = 3$:

$$S \xrightarrow{\begin{pmatrix} x_3 & -x_2 & x_1 \end{pmatrix}} S^3 \xrightarrow{\begin{pmatrix} x_2 & -x_1 & 0 \\ x_3 & 0 & -x_1 \\ 0 & x_3 & -x_2 \end{pmatrix}} S^3 \xrightarrow{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}} S \,.$$

This chain complex provides a free resolution $\varepsilon \colon \mathbf{L}(E) \to \Bbbk$ of the trivial S-module $\Bbbk$. Hence,

$$\mathcal{R}^i_s(E) = \begin{cases} \{0\} & \text{if } s \leqslant \binom{n}{i}, \\ \emptyset & \text{otherwise.} \end{cases}$$

Example (Non-zero resonance)

Let $A = \bigwedge(e_1, e_2, e_3)/\langle e_1 e_2\rangle$, and set $S = \Bbbk[x_1, x_2, x_3]$. Then

$$\mathbf{L}(A):\quad S^2 \xrightarrow{\begin{pmatrix} x_3 & 0 & -x_1 \\ 0 & x_3 & -x_2 \end{pmatrix}} S^3 \xrightarrow{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}} S\,.$$

$$\mathcal{R}^1_s(A) = \begin{cases} \{x_3 = 0\} & \text{if } s = 1, \\ \{0\} & \text{if } s = 2 \text{ or } 3, \\ \emptyset & \text{if } s > 3. \end{cases}$$

Example (Non-linear resonance)

Let $A = \bigwedge(e_1, \dots, e_4)/\langle e_1 e_3, e_2 e_4, e_1 e_2 + e_3 e_4\rangle$. Then

$$\mathbf{L}(A):\quad S^3 \xrightarrow{\begin{pmatrix} x_4 & 0 & 0 & -x_1 \\ 0 & x_3 & -x_2 & 0 \\ -x_2 & x_1 & x_4 & -x_3 \end{pmatrix}} S^4 \xrightarrow{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}} S\,.$$

$$\mathcal{R}^1_1(A) = \{x_1 x_2 + x_3 x_4 = 0\}$$

THEOREM (DENHAM–S. 2018)

Let A be a connected $\mathbb{k}$-CDGA with locally finite cohomology. For every A-CDGM M and for every $i, s \geqslant 0$

$$\mathrm{TC}_0(\mathcal{R}^i_s(M)) \subseteq \mathcal{R}^i_s(H^{\boldsymbol{\cdot}}(M)).$$

In general, we cannot replace $\mathrm{TC}_0(\mathcal{R}^i(M))$ by $\mathcal{R}^i(M)$.

EXAMPLE

- Let $M = A = \bigwedge(a, b)$ with $\mathrm{d}\, a = 0$, $\mathrm{d}\, b = b \cdot a$.
- Then $\mathcal{R}^1(A) = \{0, 1\}$ is not contained in $\mathcal{R}^1(H^{\boldsymbol{\cdot}}(A)) = \{0\}$, though $\mathrm{TC}_0(\mathcal{R}^1(A)) = \{0\}$ is.

POINCARÉ DUALITY ALGEBRAS

- Let A be a graded, graded-commutative algebra over a field $\mathbb{k}$.
  - $A = \bigoplus_{i\geqslant 0} A^i$, where A^i are $\mathbb{k}$-vector spaces.
  - $\cdot\colon A^i \otimes A^j \to A^{i+j}$.
  - $ab = (-1)^{ij} ba$ for all $a \in A^i$, $b \in A^j$.
- We will assume that A is connected ($A^0 = \mathbb{k} \cdot 1$), and locally finite (all the Betti numbers $b_i(A) := \dim_{\mathbb{k}} A^i$ are finite).
- A is a *Poincaré duality* $\mathbb{k}$*-algebra* of dimension n if there is a $\mathbb{k}$-linear map $\varepsilon\colon A^n \to \mathbb{k}$ (called an *orientation*) such that all the bilinear forms $A^i \otimes_{\mathbb{k}} A^{n-i} \to \mathbb{k}$, $a \otimes b \mapsto \varepsilon(ab)$ are non-singular.
- Consequently,
  - $b_i(A) = b_{n-i}(A)$, and $A^i = 0$ for $i > n$.
  - ε is an isomorphism.
  - The maps $\mathrm{PD}\colon A^i \to (A^{n-i})^*, \mathrm{PD}(a)(b) = \varepsilon(ab)$ are isomorphisms.
  - Each $a \in A^i$ has a *Poincaré dual*, $a^\vee \in A^{n-i}$, such that $\varepsilon(aa^\vee) = 1$.
  - The *orientation class* is defined as $\omega_A = 1^\vee$, so that $\varepsilon(\omega_A) = 1$.

THE ASSOCIATED ALTERNATING FORM

- Associated to a $\Bbbk$-PD$_n$ algebra there is an alternating n-form,

$$\mu_A \colon \bigwedge^n A^1 \to \Bbbk, \quad \mu_A(a_1 \wedge \cdots \wedge a_n) = \varepsilon(a_1 \cdots a_n).$$

- Assume now that $n = 3$, and set $r = b_1(A)$. Fix a basis $\{e_1, \dots, e_r\}$ for A^1, and let $\{e_1^\vee, \dots, e_r^\vee\}$ be the dual basis for A^2.
- The multiplication in A, then, is given on basis elements by

$$e_i e_j = \sum_{k=1}^{r} \mu_{ijk}\, e_k^\vee, \quad e_i e_j^\vee = \delta_{ij}\omega,$$

  where $\mu_{ijk} = \mu(e_i \wedge e_j \wedge e_k)$.
- Alternatively, let $A_i = (A^i)^*$, and let $e^i \in A_1$ be the (Kronecker) dual of e_i. We may then view μ dually as a trivector,

$$\mu = \sum \mu_{ijk}\, e^i \wedge e^j \wedge e^k \in \bigwedge^3 A_1,$$

  which encodes the algebra structure of A.

Poincaré duality in orientable manifolds

- If M is a compact, connected, orientable, n-dimensional manifold, then the cohomology ring $A = H^{\cdot}(M, \Bbbk)$ is a PD_n algebra over $\Bbbk$.
- Sullivan (1975): for every finite-dimensional $\mathbb{Q}$-vector space V and every alternating 3-form $\mu \in \bigwedge^3 V^*$, there is a closed 3-manifold M with $H^1(M, \mathbb{Q}) = V$ and cup-product form $\mu_M = \mu$.
- Such a 3-manifold can be constructed via "Borromean surgery."

- If M bounds an oriented 4-manifold W such that the cup-product pairing on $H^2(W, M)$ is non-degenerate (e.g., if M is the link of an isolated surface singularity), then $\mu_M = 0$.

RESONANCE VARIETIES OF PD-ALGEBRAS

- Let A be a PD_n algebra.
- For all $0 \leqslant i \leqslant n$ and all $a \in A^1$, the square

$$\begin{array}{ccc} (A^{n-i})^* & \xrightarrow{(\delta_a^{n-i-1})^*} & (A^{n-i-1})^* \\ \mathrm{PD}\uparrow\cong & & \mathrm{PD}\uparrow\cong \\ A^i & \xrightarrow{\delta_a^i} & A^{i+1} \end{array}$$

commutes up to a sign of $(-1)^i$.
- Consequently,

$$\left(H^i(A, \delta_a)\right)^* \cong H^{n-i}(A, \delta_{-a}).$$

- Hence, for all i and s,

$$\mathcal{R}^i_s(A) = \mathcal{R}^{n-i}_s(A).$$

- In particular, $\mathcal{R}^n_1(A) = \{0\}$.

3-dimensional Poincaré duality algebras

- Let A be a PD_3-algebra with $b_1(A) = r > 0$. Then
  - $\mathcal{R}^3_1(A) = \mathcal{R}^0_1(A) = \{0\}$.
  - $\mathcal{R}^2_s(A) = \mathcal{R}^1_s(A)$ for $1 \leqslant s \leqslant r$.
  - $\mathcal{R}^i_s(A) = \emptyset$, otherwise.
- Write $\mathcal{R}_s(A) = \mathcal{R}^1_s(A)$. Then
  - $\mathcal{R}_{2k}(A) = \mathcal{R}_{2k+1}(A)$ if r is even.
  - $\mathcal{R}_{2k-1}(A) = \mathcal{R}_{2k}(A)$ if r is odd.
- If μ_A has rank $r \geqslant 3$, then $\mathcal{R}_{r-2}(A) = \mathcal{R}_{r-1}(A) = \mathcal{R}_r(A) = \{0\}$.
- If $r \geqslant 4$, and $\Bbbk = \bar{\Bbbk}$, then $\dim \mathcal{R}_1(A) \geqslant \text{null}(\mu_A) \geqslant 2$.
  - Here, the *rank* of a form $\mu\colon \bigwedge^3 V \to \Bbbk$ is the minimum dimension of a linear subspace $W \subset V$ such that μ factors through $\bigwedge^3 W$.
  - The *nullity* of μ is the maximum dimension of a subspace $U \subset V$ such that $\mu(a \wedge b \wedge c) = 0$ for all $a, b \in U$ and $c \in V$.

- If r is even, then $\mathcal{R}_1(A) = \mathcal{R}_0(A) = A^1$.
- If $r = 2g + 1 > 1$, then $\mathcal{R}_1(A) \neq A^1$ if and only if μ_A is "generic" (in the sense of [Berceanu–Papadima 1994]), that is, there is a $c \in A^1$ such that the 2-form $\gamma_c \in \bigwedge^2 A_1$,
$$\gamma_c(a \wedge b) = \mu_A(a \wedge b \wedge c)$$
has maximal rank, i.e., $\gamma_c^g \neq 0$ in $\bigwedge^{2g} A_1$.
- In that case, the principal minors of the skew-symmetric $r \times r$ matrix δ^1 satisfy $\operatorname{pf}(\delta^1(i;i)) = (-1)^{i+1} x_i \operatorname{Pf}(\mu_A)$, and so
$$\mathcal{R}_1(A) = \{\operatorname{Pf}(\mu_A) = 0\}.$$

EXAMPLE

Let $M = \Sigma_g \times S^1$, where $g \geqslant 2$. Then $\mu_M = \sum_{i=1}^{g} a_i b_i c$ is generic, and $\operatorname{Pf}(\mu_M) = x_{2g+1}^{g-1}$. Hence, $\mathcal{R}_1 = \cdots = \mathcal{R}_{2g-2} = \{x_{2g+1} = 0\}$ and $\mathcal{R}_{2g-1} = \mathcal{R}_{2g} = \mathcal{R}_{2g+1} = \{0\}$.

Using recent work of De Poi, Faenzi, Mezzetti, and Ranestad, I get:

Theorem

Let A be a PD_3-algebra, and set $n = \dim A^1$. Suppose $\operatorname{rank} \gamma_c > 2$, for all non-zero $c \in A^1$. Then:

- *If n is odd, then $\mathcal{R}^1_1(A)$ is a hypersurface of degree $(n-3)/2$ which is smooth if $n \leqslant 7$, and singular in codimension 5 if $n \geqslant 9$.*
- *If n is even, then $\mathcal{R}^1_2(A)$ is a subvariety of codimension 3 and degree $\frac{1}{4}\binom{n-1}{3} + 1$, which is smooth if $n \leqslant 10$, and is singular in codimension 7 if $n \geqslant 12$.*

Resonance varieties of 3-forms of low rank

n	μ	$\mathcal{R}_1$
3	123	0

n	μ	$\mathcal{R}_1=\mathcal{R}_2$	$\mathcal{R}_3$
5	125+345	$\{x_5=0\}$	0

n	μ	$\mathcal{R}_1$	$\mathcal{R}_2=\mathcal{R}_3$	$\mathcal{R}_4$
6	123+456	$\mathbb{C}^6$	$\{x_1=x_2=x_3=0\}\cup\{x_4=x_5=x_6=0\}$	0
	123+236+456	$\mathbb{C}^6$	$\{x_3=x_5=x_6=0\}$	0

n	μ	$\mathcal{R}_1=\mathcal{R}_2$	$\mathcal{R}_3=\mathcal{R}_4$	$\mathcal{R}_5$
7	147+257+367	$\{x_7=0\}$	$\{x_7=0\}$	0
	456+147+257+367	$\{x_7=0\}$	$\{x_4=x_5=x_6=x_7=0\}$	0
	123+456+147	$\{x_1=0\}\cup\{x_4=0\}$	$\{x_1=x_2=x_3=x_4=0\}\cup\{x_1=x_4=x_5=x_6=0\}$	0
	123+456+147+257	$\{x_1x_4+x_2x_5=0\}$	$\{x_1=x_2=x_4=x_5=x_7^2-x_3x_6=0\}$	0
	123+456+147+257+367	$\{x_1x_4+x_2x_5+x_3x_6=x_7^2\}$	0	0

n	μ	$\mathcal{R}_1$	$\mathcal{R}_2=\mathcal{R}_3$	$\mathcal{R}_4=\mathcal{R}_5$
8	147+257+367+358	$\mathbb{C}^8$	$\{x_7=0\}$	$\{x_3=x_5=x_7=x_8=0\}\cup\{x_1=x_3=x_4=x_5=x_7=0\}$
	456+147+257+367+358	$\mathbb{C}^8$	$\{x_5=x_7=0\}$	$\{x_3=x_4=x_5=x_7=x_1x_8+x_6^2=0\}$
	123+456+147+358	$\mathbb{C}^8$	$\{x_1=x_5=0\}\cup\{x_3=x_4=0\}$	$\{x_1=x_3=x_4=x_5=x_2x_6+x_7x_8=0\}$
	123+456+147+257+358	$\mathbb{C}^8$	$\{x_1=x_5=0\}\cup\{x_3=x_4=x_5=0\}$	$\{x_1=x_2=x_3=x_4=x_5=x_7=0\}$
	123+456+147+257+367+358	$\mathbb{C}^8$	$\{x_3=x_5=x_1x_4-x_7^2=0\}$	$\{x_1=x_2=x_3=x_4=x_5=x_6=x_7=0\}$
	147+268+358	$\mathbb{C}^8$	$\{x_1=x_4=x_7=0\}\cup\{x_8=0\}$	$\{x_1=x_4=x_7=x_8=0\}\cup\{x_2=x_3=x_5=x_6=x_8=0\}$
	147+257+268+358	$\mathbb{C}^8$	$L_1\cup L_2\cup L_3$	$L_1\cup L_2$
	456+147+257+268+358	$\mathbb{C}^8$	$C_1\cup C_2$	$L_1\cup L_2$
	147+257+367+268+358	$\mathbb{C}^8$	$L_1\cup L_2\cup L_3\cup L_4$	$L_1'\cup L_2'\cup L_3'$
	456+147+257+367+268+358	$\mathbb{C}^8$	$C_1\cup C_2\cup C_3$	$L_1\cup L_2\cup L_3$
	123+456+147+268+358	$\mathbb{C}^8$	$C_1\cup C_2$	L
	123+456+147+257+268+358	$\mathbb{C}^8$	$\{f_1=\cdots=f_{20}=0\}$	0
	123+456+147+257+367+268+358	$\mathbb{C}^8$	$\{g_1=\cdots=g_{20}=0\}$	0

CHARACTERISTIC VARIETIES

- Let X be a connected, finite-type CW-complex. Then $\pi = \pi_1(X, x_0)$ is a finitely presented group, with $\pi_{ab} \cong H_1(X, \mathbb{Z})$.
- The ring $R = \mathbb{C}[\pi_{ab}]$ is the coordinate ring of the character group, $\mathrm{Char}(X) = \mathrm{Hom}(\pi, \mathbb{C}^*) \cong (\mathbb{C}^*)^r \times \mathrm{Tors}(\pi_{ab})$, where $r = b_1(X)$.
- The *characteristic varieties* of X are the homology jump loci
$$\mathcal{V}^i_s(X) = \{\rho \in \mathrm{Char}(X) \mid \dim H_i(X, \mathbb{C}_\rho) \geqslant s\}.$$
- These varieties are homotopy-type invariants of X, with $\mathcal{V}^1_s(X)$ depending only on $\pi = \pi_1(X)$.
- Set $\mathcal{V}_1(\pi) := \mathcal{V}^1_1(K(\pi, 1))$; then $\mathcal{V}_1(\pi) = \mathcal{V}_1(\pi/\pi'')$.

EXAMPLE

Let $f \in \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ be a Laurent polynomial, $f(1) = 0$. There is then a finitely presented group π with $\pi_{ab} = \mathbb{Z}^n$ such that $\mathcal{V}_1(\pi) = V(f)$.

EXAMPLE (CIRCLE)

Let $X = S^1$. We have $(S^1)^{\text{ab}} = \mathbb{R}$. Identify $\pi_1(S^1, *) = \mathbb{Z} = \langle t \rangle$ and $\mathbb{Z}\mathbb{Z} = \mathbb{Z}[t^{\pm 1}]$. Then:

$$C_*((S^1)^{\text{ab}}) : \ 0 \longrightarrow \mathbb{Z}[t^{\pm 1}] \xrightarrow{t-1} \mathbb{Z}[t^{\pm 1}] \longrightarrow 0$$

For each $\rho \in \text{Hom}(\mathbb{Z}, \Bbbk^*) = \Bbbk^*$, get a chain complex

$$C_*(\widetilde{S^1}) \otimes_{\mathbb{Z}\mathbb{Z}} \Bbbk_\rho : \ 0 \longrightarrow \Bbbk \xrightarrow{\rho-1} \Bbbk \longrightarrow 0$$

which is exact, except for $\rho = 1$, when $H_0(S^1, \Bbbk) = H_1(S^1, \Bbbk) = \Bbbk$. Hence:

$$\mathcal{V}^0_1(S^1) = \mathcal{V}^1_1(S^1) = \{1\}$$

and $\mathcal{V}^i_s(S^1) = \emptyset$, otherwise.

EXAMPLE (TORUS)

Identify $\pi_1(T^n) = \mathbb{Z}^n$, and $\mathrm{Hom}(\mathbb{Z}^n, \Bbbk^*) = (\Bbbk^*)^n$. Then:

$$\mathcal{V}^i_s(T^n) = \begin{cases} \{1\} & \text{if } s \leqslant \binom{n}{i}, \\ \emptyset & \text{otherwise.} \end{cases}$$

EXAMPLE (WEDGE OF CIRCLES)

Identify $\pi_1(\bigvee^n S^1) = F_n$, and $\mathrm{Hom}(F_n, \Bbbk^*) = (\Bbbk^*)^n$. Then:

$$\mathcal{V}^1_s(\bigvee^n S^1) = \begin{cases} (\Bbbk^*)^n & \text{if } s < n, \\ \{1\} & \text{if } s = n, \\ \emptyset & \text{if } s > n. \end{cases}$$

EXAMPLE (ORIENTABLE SURFACE OF GENUS $g > 1$)

$$\mathcal{V}^1_s(\Sigma_g) = \begin{cases} (\Bbbk^*)^{2g} & \text{if } s < 2g-1, \\ \{1\} & \text{if } s = 2g-1, 2g, \\ \emptyset & \text{if } s > 2g. \end{cases}$$

TANGENT CONES

- Let $\exp\colon H^1(X,\mathbb{C}) \to H^1(X,\mathbb{C}^*)$ be the coefficient homomorphism induced by $\mathbb{C} \to \mathbb{C}^*$, $z \mapsto e^z$.
- Let $W = V(I)$, a Zariski closed subset of $\mathrm{Char}(G) = H^1(X,\mathbb{C}^*)$.
- The *tangent cone* at 1 to W is $\mathrm{TC}_1(W) = V(\mathrm{in}(I))$.
- The *exponential tangent cone* at 1 to W:

$$\tau_1(W) = \{z \in H^1(X,\mathbb{C}) \mid \exp(\lambda z) \in W,\ \forall \lambda \in \mathbb{C}\}.$$

- Both tangent cones are homogeneous subvarieties of $H^1(X,\mathbb{C})$; are non-empty iff $1 \in W$; depend only on the analytic germ of W at 1; commute with finite unions and arbitrary intersections.
- $\tau_1(W) \subseteq TC_1(W)$, with $=$ if all irred components of W are subtori, but $\neq$ in general.
- $\tau_1(W)$ is a finite union of rationally defined subspaces.

The tangent cone theorem

Let X be a connected CW-complex with finite q-skeleton. Suppose X admits a q-finite q-model A.

Theorem

For all $i \leqslant q$ and all s:

- *(DPS 2009, Dimca–Papadima 2014)* $\mathcal{V}^i_s(X)_{(1)} \cong \mathcal{R}^i_s(A)_{(0)}$.
- *(Budur–Wang 2017) All the irreducible components of $\mathcal{V}^i_s(X)$ passing through the origin of $\mathrm{Char}(X)$ are* algebraic subtori.

Consequently,

$$\tau_1(\mathcal{V}^i_s(X)) = \mathrm{TC}_1(\mathcal{V}^i_s(X)) = \mathcal{R}^i_s(A).$$

Theorem (Papadima–S. 2017)

A f.g. group G admits a 1-finite 1-model if and only if the Malcev Lie algebra $\mathfrak{m}(G)$ is the LCS completion of a finitely presented Lie algebra.

ALEXANDER POLYNOMIALS OF 3-MANIFOLDS

- Let $H = H_1(X, \mathbb{Z})/\mathrm{Tors}$.
- The *Alexander polynomial* $\Delta_X \in \mathbb{Z}[H]$ is the gcd of the codimension 1 minors of the Alexander matrix of $\pi_1(X)$.

PROPOSITION

Let λ be a Laurent polynomial in $n \leqslant 3$ variables such that $\bar{\lambda} = \lambda$ and $\lambda(1) \neq 0$. Then λ can be realized as the Alexander polynomial Δ_M of a closed, orientable 3-manifold M with $b_1(M) = n$.

Set $\mathcal{W}^1_1(M) = \mathcal{V}^1(M) \cap \mathrm{Char}^0(M)$.

PROPOSITION

Let M be a closed, orientable, 3-dimensional manifold. Then

$$\mathcal{W}^1_1(M) = V(\Delta_M) \cup \{1\}.$$

If, moreover, $b_1(M) \geqslant 4$, then $\mathcal{W}^1_1(M) = V(\Delta_M)$.

A TANGENT CONE THEOREM FOR 3-MANIFOLDS

THEOREM

Let M be a closed, orientable, 3-dimensional manifold. Suppose $b_1(M)$ is odd and μ_M is generic. Then $\mathrm{TC}_1(\mathcal{V}^1_1(M)) = \mathcal{R}^1_1(M)$.

- If $b_1(M)$ is even, the conclusion may or may not hold:
  - Let $M = S^1 \times S^2 \# S^1 \times S^2$; then $\mathcal{V}^1_1(M) = \mathrm{Char}(M) = (\mathbb{C}^*)^2$, and so $\mathrm{TC}_1(\mathcal{V}^1_1(M)) = \mathcal{R}^1_1(M) = \mathbb{C}^2$.
  - Let M be the Heisenberg nilmanifold; then $\mathrm{TC}_1(\mathcal{V}^1_1(M)) = \{0\}$, whereas $\mathcal{R}^1_1(M) = \mathbb{C}^2$.
- Let M be a closed, orientable 3-manifold with $b_1 = 7$ and $\mu = e_1e_3e_5 + e_1e_4e_7 + e_2e_5e_7 + e_3e_6e_7 + e_4e_5e_6$. Then μ is generic and $\mathrm{Pf}(\mu) = (x_5^2 + x_7^2)^2$. Hence, $\mathcal{R}^1_1(M) = \{x_5^2 + x_7^2 = 0\}$ splits as a union of two hyperplanes over $\mathbb{C}$, but not over $\mathbb{Q}$.

The above theorem does not hold in higher depth.

EXAMPLE

Let M be a closed, orientable 3-manifold with $b_1(M) = 10$ and intersection 3-form

$$\mu_M = e_1e_2e_5 + e_1e_3e_6 + e_2e_3e_7 + e_1e_4e_8 + e_2e_4e_9 + e_3e_4e_{10}.$$

- $\mathcal{R}^1_7(M) \cong \{z \in \mathbb{C}^6 \mid z_1z_6 - z_2z_5 + z_3z_4 = 0\}$, an irreducible quadric with an isolated singular point at 0.
- $\mathcal{V}^1_s(M) \subseteq \{1\}$, for all $s \geqslant 1$.
- Thus, $\mathrm{TC}_1(\mathcal{V}^1_7(M)) \neq \mathcal{R}^1_7(M)$, showing that M is not 1-formal.

DUALITY SPACES

A more general notion of duality is due to Bieri and Eckmann (1978).

Let X be a connected, finite-type CW-complex, and set $\pi = \pi_1(X, x_0)$.

- X is a *duality space* of dimension n if $H^i(X, \mathbb{Z}\pi) = 0$ for $i \neq n$ and $H^n(X, \mathbb{Z}\pi) \neq 0$ and torsion-free.
- Let $D = H^n(X, \mathbb{Z}\pi)$ be the dualizing $\mathbb{Z}\pi$-module. Given any $\mathbb{Z}\pi$-module A, we have $H^i(X, A) \cong H_{n-i}(X, D \otimes A)$.
- If $D = \mathbb{Z}$, with trivial $\mathbb{Z}\pi$-action, then X is a Poincaré duality space.
- If $X = K(\pi, 1)$ is a duality space, then π is a *duality group*.

ABELIAN DUALITY SPACES

We introduce in [Denham–S.–Yuzvinsky 2016/17] an analogous notion, by replacing $\pi \rightsquigarrow \pi_{ab}$.

- X is an *abelian duality space* of dimension n if $H^i(X, \mathbb{Z}\pi_{ab}) = 0$ for $i \neq n$ and $H^n(X, \mathbb{Z}\pi_{ab}) \neq 0$ and torsion-free.
- Let $B = H^n(X, \mathbb{Z}\pi_{ab})$ be the dualizing $\mathbb{Z}\pi_{ab}$-module. Given any $\mathbb{Z}\pi_{ab}$-module A, we have $H^i(X, A) \cong H_{n-i}(X, B \otimes A)$.
- The two notions of duality are independent:

EXAMPLE

- Surface groups of genus at least 2 are not abelian duality groups, though they are (Poincaré) duality groups.
- Let $\pi = \mathbb{Z}^2 * G$, where
$$G = \langle x_1, \dots, x_4 \mid x_1^{-2} x_2 x_1 x_2^{-1}, \dots, x_4^{-2} x_1 x_4 x_1^{-1} \rangle$$
is Higman's acyclic group. Then π is an abelian duality group (of dimension 2), but not a duality group.

THEOREM (DSY)

Let X be an abelian duality space of dimension n. Then:

- $b_1(X) \geqslant n-1$.
- $b_i(X) \neq 0$, *for* $0 \leqslant i \leqslant n$ *and* $b_i(X) = 0$ *for* $i > n$.
- $(-1)^n \chi(X) \geqslant 0$.
- *The characteristic varieties propagate, i.e.,* $\mathcal{V}_1^1(X) \subseteq \cdots \subseteq \mathcal{V}_1^n(X)$.

THEOREM (DENHAM–S. 2018)

Let U be a connected, smooth, complex quasi-projective variety of dimension n. Suppose U has a smooth compactification Y for which

1. *Components of $Y \backslash U$ form an arrangement of hypersurfaces $\mathcal{A}$;*
2. *For each submanifold X in the intersection poset $L(\mathcal{A})$, the complement of the restriction of $\mathcal{A}$ to X is a Stein manifold.*

Then U is both a duality space and an abelian duality space of dimension n.

LINEAR, ELLIPTIC, AND TORIC ARRANGEMENTS

THEOREM (DS18)

Suppose that $\mathcal{A}$ is one of the following:

1. *An affine-linear arrangement in $\mathbb{C}^n$, or a hyperplane arrangement in $\mathbb{CP}^n$;*
2. *A non-empty elliptic arrangement in E^n;*
3. *A toric arrangement in $(\mathbb{C}^*)^n$.*

Then the complement $M(\mathcal{A})$ is both a duality space and an abelian duality space of dimension $n-r$, $n+r$, and n, respectively, where r is the corank of the arrangement.

This theorem extends several previous results:

1. Davis, Januszkiewicz, Leary, and Okun (2011);
2. Levin and Varchenko (2012);
3. Davis and Settepanella (2013), Esterov and Takeuchi (2018).

REFERENCES

- G. Denham, A.I. Suciu, *Algebraic models and cohomology jump loci*, preprint 2018.
- A.I. Suciu, *Poincaré duality and resonance varieties*, arXiv:1809.01801.
- A.I. Suciu, *Cohomology jump loci of 3-manifolds*, arXiv:1901.01419.
- G. Denham, A.I. Suciu, and S. Yuzvinsky, *Abelian duality and propagation of resonance*, Selecta Math. **23** (2017), no. 4, 2331–2367.
- G. Denham, A.I. Suciu, *Local systems on arrangements of smooth, complex algebraic hypersurfaces*, Forum of Mathematics, Sigma **6** (2018), e6, 20 pages.