The pure braid groups and their relatives

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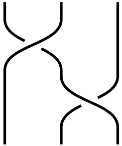
ARTIN'S BRAID GROUPS



- Let B_n be the *group of braids* on n strings (under concatenation).
- B_n is generated by $\sigma_1, \dots, \sigma_{n-1}$ subject to the relations $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ and $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| > 1$.
- Let $P_n = \ker(B_n \twoheadrightarrow S_n)$ be the *pure braid group* on n strings.
- P_n is generated by $A_{ij} = \sigma_{j-1}\cdots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\cdots\sigma_{j-1}^{-1}$ $(1 \leqslant i < j \leqslant n)$.
- $B_n = \mathrm{Mod}^1_{0,n}$, the mapping class group of D^2 with n marked points.
- Thus, $B_n < \mathrm{Aut}(F_n)$. In fact:

$$B_n = \{\beta \in \mathrm{Aut}(F_n) \mid \beta(x_i) = wx_{\tau(i)}w^{-1}, \beta(x_1\cdots x_n) = x_1\cdots x_n\}.$$

- The Torelli group

$$\mathrm{IA}_n = \ker(\mathrm{Aut}(F_n) \twoheadrightarrow \mathrm{GL}_n(\mathbb{Z}))$$

is generated by automorphisms α_{ij} and α_{ijk} ($1 \leqslant i \neq j \neq k \leqslant n$) which send x_i to $x_j x_i x_j^{-1}$ and $x_i x_j x_k x_j^{-1} x_k^{-1}$, and leave invariant the remaining generators of F_n.

- Clearly, $P_n = B_n \cap \mathrm{IA}_n$.

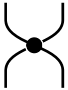
- A classifying space for P_n is the configuration space

$$\mathrm{Conf}_n(\mathbb{C}) = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j \text{ for } i \neq j\}.$$

- Thus, $B_n = \pi_1(\mathrm{Conf}_n(\mathbb{C})/S_n)$.

- Moreover, $P_n = F_{n-1} \rtimes_{\alpha_{n-1}} P_{n-1} = F_{n-1} \rtimes \cdots \rtimes F_2 \rtimes F_1$, where $\alpha_n \colon P_n \subset B_n \hookrightarrow \mathrm{Aut}(F_n)$. In particular, $P_2 = \mathbb{Z}$, $P_3 \cong F_2 \times \mathbb{Z}$.

WELDED BRAID GROUPS



- The set of all permutation-conjugacy automorphisms of F_n forms a subgroup $wB_n < \operatorname{Aut}(F_n)$, called the *welded braid group*.
- The *pure welded braid group* is the subgroup

$$wP_n = \ker(wB_n \twoheadrightarrow S_n) = \mathrm{IA}_n \cap wB_n$$

- McCool (1986) gave a finite presentation for wP_n. It is generated by the automorphisms α_{ij} ($1 \leqslant i \neq j \leqslant n$), subject to the relations

$$\begin{aligned} &\alpha_{ij}\alpha_{ik}\alpha_{jk} = \alpha_{jk}\alpha_{ik}\alpha_{ij} && \text{for } i, j, k \text{ distinct,} \\ &[\alpha_{ij}, \alpha_{st}] = 1 && \text{for } i, j, s, t \text{ distinct,} \\ &[\alpha_{ik}, \alpha_{jk}] = 1 && \text{for } i, j, k \text{ distinct.} \end{aligned}$$

- Brendle and Hatcher (2013): the welded braid group wB_n and the *McCool group* wP_n occur as fundamental groups of spaces of untwisted 'flying rings'.
- Let IA_n^+ be the subgroup of IA_n generated by α_{ij} and α_{ijk} with $i < j < k$.
- The *upper pure welded braid group* (or, *upper McCool group*) is the subgroup $wP_n^+ = wP_n \cap \mathrm{IA}_n^+$ generated by α_{ij} for $i < j$.
- $wP_n^+ \cong F_{n-1} \rtimes \cdots \rtimes F_2 \rtimes F_1$.
- $wP_2 = F_2, \quad wP_2^+ = \mathbb{Z}, \quad wP_3^+ \cong F_2 \times \mathbb{Z}$.

PROPOSITION (S.–WANG)

For $n \geqslant 4$, there is no epimorphism $wP_n \to wP_n^+$; in particular, the inclusion $wP_n^+ \hookrightarrow wP_n$ admits no splitting.

The proof uses the differing nature of the (first) resonance varieties of the two groups.

VIRTUAL BRAID GROUPS

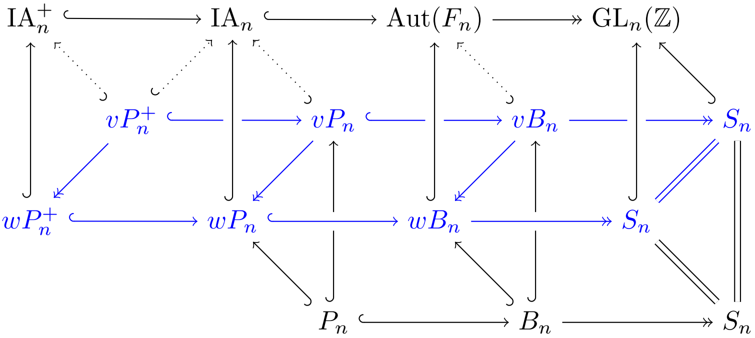
- The *virtual braid group* vB_n is obtained from wB_n by omitting certain commutation relations.
- Let $vP_n = \ker(vB_n \to S_n)$ be the *pure virtual braid group*.
- Bardakov (2004) gave a presentation for vP_n with generators x_{ij} for $1 \leqslant i \neq j \leqslant n$,

subject to the relations

$$x_{ij}x_{ik}x_{jk} = x_{jk}x_{ik}x_{ij}, \qquad \text{for } i, j, k \text{ distinct,}$$
$$[x_{ij}, x_{st}] = 1, \qquad \text{for } i, j, s, t \text{ distinct.}$$

- Let vP_n^+ be the subgroup of vP_n generated by x_{ij} for $i < j$.
- The inclusion $vP_n^+ \hookrightarrow vP_n$ is a split injection.
- Bartholdi, Enriquez, Etingof, and Rains (2006) constructed classifying spaces for vP_n and vP_n^+ by taking quotients of permutahedra by suitable actions of the symmetric groups.
- First few groups:
  - $vP_2^+ = \mathbb{Z}$
  - $vP_3^+ \cong \mathbb{Z} * \mathbb{Z}^2$
  - $vP_2 \cong F_2$
  - $vP_3 \cong \overline{P}_4 * \mathbb{Z}$, where $\overline{P}_4 = P_4/Z(P_4) = F_3 \rtimes F_2$

Summary diagram



COHOMOLOGY RINGS AND BETTI NUMBERS

The cohomology rings of the pure-braid like groups were computed by:

- P_n: V.I. Arnol'd (1969).
- wP_n: Jensen–McCammond–Meier (2006).
- wP_n^+: F. Cohen–Pakhianathan–Vershinin–Wu (2007).
- vP_n, vP_n^+: Bartholdi–Enriquez–Etingof–Rains (2006), Lee (2013).

The Betti numbers of the pure-braid like groups are given by:

	P_n	wP_n	wP_n^+	vP_n	vP_n^+
b_i	$s(n, n-i)$	$\binom{n-1}{i} n^i$	$s(n, n-i)$	$L(n, n-i)$	$S(n, n-i)$

Here $s(n, k)$ are the Stirling numbers of the first kind, $S(n, k)$ are the Stirling numbers of the second kind, and $L(n, k)$ are the Lah numbers.

	$H^*(P_n;\mathbb{C})$	$H^*(wP_n;\mathbb{C})$	$H^*(wP_n^+;\mathbb{C})$	$H^*(vP_n;\mathbb{C})$	$H^*(vP_n^+;\mathbb{C})$
Generators	u_{ij} $(i<j)$	a_{ij} $(i\neq j)$	e_{ij} $(i<j)$	a_{ij} $(i\neq j)$	e_{ij} $(i<j)$
Relations	(I1)	(I2) (I3)	(I5)	(I2)(I3)(I4)	(I5) (I6)
Koszul	Yes	No for $n\geqslant 4$	Yes	Yes	Yes

$$
\begin{array}{lll}
\text{(I1)} & u_{jk}u_{ik} = u_{ij}(u_{ik}-u_{jk}) & \text{for } i<j<k,\\
\text{(I2)} & a_{ij}a_{ji} = 0 & \text{for } i\neq j,\\
\text{(I3)} & a_{kj}a_{ik} = a_{ij}(a_{ik}-a_{jk}) & \text{for } i,j,k \text{ distinct},\\
\text{(I4)} & a_{ji}a_{ik} = (a_{ij}-a_{ik})a_{jk} & \text{for } i,j,k \text{ distinct},\\
\text{(I5)} & e_{ij}(e_{ik}-e_{jk}) = 0 & \text{for } i<j<k,\\
\text{(I6)} & (e_{ij}-e_{ik})e_{jk} = 0 & \text{for } i<j<k.
\end{array}
$$

- Koszulness for P_n: Kohno (1985).
- Koszulness for vP_n and vP_n^+: Bartholdi et al (2006), Lee (2013).
- Koszulness for wP_n^+: D. Cohen and G. Pruidze (2008).
- Non-Koszulness for wP_n: Conner and Goetz (2015).

RESONANCE VARIETIES

- Let G be a finitely presented group, and set $A = H^*(G, \mathbb{C})$.
- The (first) *resonance variety* of G is given by

$$\mathcal{R}_1(G) = \{a \in A^1 \mid \exists b \in A^1 \setminus \mathbb{C} \cdot a \text{ such that } a \cdot b = 0 \in A^2\}.$$

- For instance, $\mathcal{R}_1(F_n) = \mathbb{C}^n$ for $n \geqslant 2$, and $\mathcal{R}_1(\mathbb{Z}^n) = \{0\}$.

THEOREM (D. COHEN–S. 1999)

$\mathcal{R}_1(P_n)$ is a union of $\binom{n}{3} + \binom{n}{4}$ linear subspaces of dimension 2.

THEOREM (D. COHEN 2009)

$\mathcal{R}_1(wP_n)$ is a union of $\binom{n}{2}$ linear subspaces of dimension 2 and $\binom{n}{3}$ linear subspaces of dimension 3.

THEOREM (S.–WANG)

$$\mathcal{R}_1(wP_n^+) = \bigcup_{2\leqslant i<j\leqslant n} L_{ij},$$

where L_{ij} is a linear subspace of dimension i.

PROPOSITION (BARDAKOV–MIKHAILOV–VERSHININ–WU 2009, SW)

$\mathcal{R}_1(vP_3)$ coincides with $H^1(vP_3, \mathbb{C}) = \mathbb{C}^6$.

PROPOSITION (SW)

$\mathcal{R}_1(vP_4^+)$ is the subvariety of $H^1(vP_4^+, \mathbb{C}) = \mathbb{C}^6$ defined by

$$x_{12}x_{24}(x_{13}+x_{23}) + x_{13}x_{34}(x_{12}-x_{23}) - x_{24}x_{34}(x_{12}+x_{13}) = 0,$$
$$x_{12}x_{23}(x_{14}+x_{24}) + x_{12}x_{34}(x_{23}-x_{14}) + x_{14}x_{34}(x_{23}+x_{24}) = 0,$$
$$x_{13}x_{23}(x_{14}+x_{24}) + x_{14}x_{24}(x_{13}+x_{23}) + x_{34}(x_{13}x_{23}-x_{14}x_{24}) = 0,$$
$$x_{12}(x_{13}x_{14}-x_{23}x_{24}) + x_{34}(x_{13}x_{23}-x_{14}x_{24}) = 0.$$

ASSOCIATED GRADED LIE ALGEBRAS

- For a f.g. group G, define the *lower central series* inductively by $\gamma_1 G = G$ and $\gamma_{k+1} G = [\gamma_k G, G]$.
- The group commutator induces a graded Lie algebra structure on $\mathrm{gr}(G) = \bigoplus_{k \geqslant 1} (\gamma_k G / \gamma_{k+1} G) \otimes_{\mathbb{Z}} \mathbb{C}$.

	$\mathrm{gr}(P_n)$	$\mathrm{gr}(wP_n)$	$\mathrm{gr}(wP_n^+)$	$\mathrm{gr}(vP_n)$	$\mathrm{gr}(vP_n^+)$
Generators	x_{ij}, $i < j$	x_{ij}, $i \neq j$	x_{ij}, $i < j$	x_{ij}, $i \neq j$	x_{ij}, $i < j$
Relations	L2, L4	L1, L2, L3	L1, L2, L3	L1, L2	L1, L2
	Kohno, Falk–Randell	Jensen et al.	F. Cohen et al.	Bartholdi et al., Lee	Bartholdi et al., Lee

(L1) $[x_{ij}, x_{ik}] + [x_{ij}, x_{jk}] + [x_{ik}, x_{jk}] = 0$ for distinct i, j, k,

(L2) $[x_{ij}, x_{kl}] = 0$ for $\{i, j\} \cap \{k, l\} = \varnothing$,

(L3) $[x_{ik}, x_{jk}] = 0$ for distinct i, j, k,

(L4) $[x_{im}, x_{ij} + x_{ik} + x_{jk}] = 0$ for $m = j, k$ and i, j, m distinct.

- Let $\phi_k(G) = \dim \operatorname{gr}_k(G)$ be the *LCS ranks* of G.
- E.g.: $\phi_k(F_n) = \frac{1}{k}\sum_{d|k} \mu(\frac{k}{d}) n^d$.
- By the Poincaré–Birkhoff–Witt theorem,

$$\prod_{k=1}^{\infty} (1-t^k)^{-\phi_k(G)} = \operatorname{Hilb}(U(\operatorname{gr}(G)), t).$$

PROPOSITION (PAPADIMA–YUZVINSKY 1999)

Suppose $\operatorname{gr}(G)$ *is quadratic and* $A = H^*(G;\mathbb{C})$ *is Koszul. Then* $\operatorname{Hilb}(U(\operatorname{gr}(G)), t) \cdot \operatorname{Hilb}(A, -t) = 1$.

- If G is a pure braid-like group, then $\operatorname{gr}(G)$ is quadratic.
- Furthermore, if $G \neq wP_n$ ($n \geqslant 4$), then $H^*(G;\mathbb{C})$ is Koszul, in which case:

$$\prod_{k=1}^{\infty} (1-t^k)^{\phi_k(G)} = \sum_{i \geqslant 0} b_i(G)(-t)^i.$$

RESIDUAL PROPERTIES

- Let $\mathcal{P}$ be a group-theoretic property. A group G is said to be *residually* $\mathcal{P}$ if for any $g \in G$, $g \neq 1$, there exists a group Q with property $\mathcal{P}$, and an epimorphism $\psi\colon G \to Q$ such that $\psi(g) \neq 1$.
- G is *residually nilpotent* (RN) iff $\bigcap_{k\geqslant 1} \gamma_k G = \{1\}$.
- G is *residually torsion-free nilpotent* (RTFN) iff $\bigcap_{k\geqslant 1} \tau_k G = \{1\}$, where $\tau_k G = \{g \in G \mid g^n \in \gamma_k G, \text{ for some } n \in \mathbb{N}\}$.
- RTFN $\Rightarrow$ RN $\Rightarrow$ residually finite.
- RTFN $\Rightarrow$ torsion-free.
- If G is RN and $\operatorname{gr}_k(G, \mathbb{Z})$ is torsion-free, $\forall k \geqslant 1$, then G is RTFN.
- The property of being RN or RTFN is inherited by subgroups, and is preserved by taking direct products or free products (Malcev 1949, Baumslag 1999).

- Andreadakis–Johnson (descending) filtration of $\mathrm{Aut}(G)$:

$$\Phi_k(\mathrm{Aut}(G)) = \ker(\mathrm{Aut}(G) \to \mathrm{Aut}(G/\gamma_{k+1}(G)).$$

- The group $\mathcal{I}(G) = \Phi_1(\mathrm{Aut}(G))$ is called the *Torelli group* of G.
- Kaloujnine (1950): $\gamma_k(\mathcal{I}(G)) < \Phi_k(\mathrm{Aut}(G))$ for all $k \geqslant 1$.

THEOREM (ANDREADAKIS 1965)

If G is residually nilpotent, then $\mathcal{I}(G)$ is also residually nilpotent.

THEOREM (HAIN 1997, BERCEANU-PAPADIMA 2009)

Let G be a finitely generated, residually nilpotent group, and suppose $\mathrm{gr}_k(G, \mathbb{Z})$ is torsion-free for all $k \geqslant 1$. Then the Torelli group $\mathcal{I}(G)$ is residually torsion-free nilpotent.

- Magnus (1935): the free groups F_n are RTFN.
- Since $\mathrm{gr}(F_n, \mathbb{Z})$ is a free Lie algebra, and hence torsion-free, the Torelli group $\mathrm{IA}_n = \mathcal{I}(F_n)$ is RTFN.
- Hence, all subgroups, such as IA_n^+, P_n, wP_n, wP_n^+ are also RTFN.
- The braid group B_n are linear (Krammer, Bigelow), and thus residually finite. But they are *not* RN for $n \geqslant 3$ (Gorin–Lin).
- Hence, their supergroups, wB_n and vB_n, are *not* RN for $n \geqslant 3$.
- F. Cohen–Pakhianathan–Vershinin–Wu (2007), using Falk–Randell (1985): $\mathrm{gr}(wP_n^+, \mathbb{Z})$ is torsion-free.
- Metaftsis–Papistas (2015): $\mathrm{gr}(wP_3, \mathbb{Z})$ is torsion-free. Not known whether $\mathrm{gr}(wP_n, \mathbb{Z})$ is torsion-free for $n \geqslant 4$.
- Bardakov–Mikhailov–Vershinin–Wu (2016) and SW: vP_n and vP_n^+ are RTFN for $n \leqslant 3$. Not known whether they are RTFN for $n \geqslant 4$.

HOLONOMY AND MALCEV LIE ALGEBRAS

- Let G be a finitely generated group.
- (Chen 1977, Markl–Papadima 1992) The *holonomy Lie algebra* of G is
$$\mathfrak{h}(G) = \mathrm{Lie}(H_1(G,\mathbb{C}))/(\mathrm{im}(\cup_G^*)),$$
where $\cup_G\colon H^1(G,\mathbb{C}) \wedge H^1(G,\mathbb{C}) \to H^2(G,\mathbb{C})$ is the cup-product.
- The identification $H_1(G,\mathbb{C})) = \mathrm{gr}_1(G)$ extends to an epimorphism $\mathfrak{h}(G) \twoheadrightarrow \mathrm{gr}(G)$.
- (Quillen 1968) The *Malcev Lie algebra* of G is
$$\mathfrak{m}(G) = \mathrm{Prim}(\widehat{\mathbb{C}G}).$$
- This is a complete, filtered Lie algebra with $\mathrm{gr}(\mathfrak{m}(G)) \cong \mathrm{gr}(G)$.

FORMALITY PROPERTIES

- G is 1*-formal* if its Malcev Lie algebra is quadratic, that is, $\mathfrak{m}(G) \cong \widehat{\mathfrak{h}(G)}$.
- G is *graded-formal* if $\operatorname{gr}(G)$ is quadratic, that is, $\mathfrak{h}(G) \xrightarrow{\simeq} \operatorname{gr}(G)$.
- A filtered Lie algebra $\mathfrak{g}$ is *formal* if $\mathfrak{g} \cong \widehat{\operatorname{gr}(\mathfrak{g})}$.
- G is *filtered-formal* if $\mathfrak{m}(G)$ is formal, that is, $\mathfrak{m}(G) \cong \widehat{\operatorname{gr}(G)}$.
- G is 1-formal iff G is both graded formal and filtered formal.
- Formality properties of groups are preserved under (finite) direct products and free products, as well as retracts.
- If $\mathbb{Q} \subset \Bbbk$ is a field extension, $\Bbbk$-formality properties descend to $\mathbb{Q}$-formality properties.

THEOREM (DIMCA–PAPADIMA–S. 2009)

If G is 1-formal, then $\mathcal{R}_1(G)$ is a union of projectively disjoint, rationally defined linear subspaces of $H^1(G,\mathbb{C})$.

THEOREM (KOHNO 1983)

Fundamental groups of complements of complex projective hypersurfaces (e.g., F_n and P_n) are 1-formal.

THEOREM (BERCEANU–PAPADIMA 2009)

wP_n and wP_n^+ are 1-formal.

THEOREM (BARTHOLDI, ENRIQUEZ, ETINGOF, RAINS 2006, LEE 2013)

vP_n and vP_n^+ are graded formal.

THEOREM (S.–WANG)

vP_n and vP_n^+ are 1-formal if and only if $n \leqslant 3$.

PROOF.

- There are split monomorphisms

$$\begin{array}{ccccccccccc}
vP_2^+ & \hookrightarrow & vP_3^+ & \hookrightarrow & vP_4^+ & \hookrightarrow & vP_5^+ & \hookrightarrow & vP_6^+ & \hookrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
vP_2 & \hookrightarrow & vP_3 & \hookrightarrow & vP_4 & \hookrightarrow & vP_5 & \hookrightarrow & vP_6 & \hookrightarrow & \cdots
\end{array}$$

- $vP_2^+ = \mathbb{Z}$ and $vP_3^+ \cong \mathbb{Z} * \mathbb{Z}^2$. Thus, they are both 1-formal.
- $vP_3 \cong \overline{P}_4 * \mathbb{Z}$ and $P_4 \cong \overline{P}_4 \times \mathbb{Z}$. Thus, vP_3 is 1-formal.
- $\mathcal{R}_1(vP_4^+)$ is non-linear. Thus, vP_4^+ is not 1-formal.
- Hence, vP_n^+ and vP_n ($n \geqslant 4$) are also not 1-formal.

CHEN LIE ALGEBRAS

- The *Chen Lie algebra* of a f.g. group G is $\operatorname{gr}(G/G'')$, the associated graded Lie algebra of its maximal metabelian quotient.
- Let $\theta_k(G) = \dim \operatorname{gr}_k(G/G'')$ be the *Chen ranks* of G.
- Easy to see: $\theta_k(G) \leqslant \phi_k(G)$ and $\theta_k(G) = \phi_k(G)$ for $k \leqslant 3$.
- Chen(1951): $\theta_k(F_n) = (k-1)\binom{n+k-2}{k}$ for $k \geqslant 2$.

THEOREM (S.–WANG)

The projection $G \twoheadrightarrow G/G''$ induces an epimorphism of graded Lie algebras, $\operatorname{gr}(G)/\operatorname{gr}(G)'' \twoheadrightarrow \operatorname{gr}(G/G'')$.
Furthermore, if G is filtered-formal, this map is an isomorphism.

When G is 1-formal, this recovers a result of Papadima–S (2004) and Dimca–Papadima–S. (2009).

ALEXANDER INVARIANTS AND CHEN RANKS

- Let $B(G) = (G'/G'') \otimes \mathbb{C}$ be the *Alexander invariant* of G, viewed as a module over $R = \mathbb{C}[G/G']$.
- W.S. Massey (1980): $\sum_{k\geqslant 2} \theta_k(G) \cdot t^{k-2} = \mathrm{Hilb}(\mathrm{gr}_I(B(G)), t)$.
- Following Papadima–S. (2004), given a f.g., graded Lie algebra $\mathfrak{g}$, let $\mathfrak{B}(\mathfrak{g}) = \mathfrak{g}'/\mathfrak{g}''$ be its *infinitesimal Alexander invariant*, as a module over $S = \mathrm{Sym}(\mathfrak{g}_1)$.
- Set $\theta_k(\mathfrak{g}) = \dim(\mathfrak{g}/\mathfrak{g}'')_k$. Then: $\sum_{k\geqslant 2} \theta_k(\mathfrak{g}) t^{k-2} = \mathrm{Hilb}(\mathrm{gr}(\mathfrak{B}(\mathfrak{g})), t)$.
- Let $\mathfrak{B}(G) := \mathfrak{B}(\mathfrak{h}(G))$. Then the annihilator ideal of $\mathfrak{B}(G)$ defines the scheme structure for the resonance variety, and

$$\mathcal{R}_1(G) = V\big(\mathrm{Ann}(\mathfrak{B}(G)\big).$$

THEOREM (DIMCA–PAPADIMA–S. 2009, S.–WANG)

If G is 1-formal, then $\operatorname{gr}_I(B(G)) \cong \mathfrak{B}(\mathfrak{h}(G))$, as modules over $S = \operatorname{gr}_I(R)$.

PROPOSITION (S.–WANG)

1. $\theta_k(G) \leqslant \theta_k(\operatorname{gr}(G))$, *with equality if* $k \leqslant 3$, *or if* G *is filtered-formal.*
2. $\theta_k(\operatorname{gr}(G)) \leqslant \theta_k(\mathfrak{h}(G))$, *with equality if* $k \leqslant 2$, *or if* G *is graded-formal.*

QUESTION

Suppose G is graded-formal. Does the equality $\theta_k(G) = \theta_k(\operatorname{gr}(G))$ hold for all k?

Chen ranks of pure-braid like groups

Theorem (D. Cohen–S. 1993)

The Chen ranks of P_n *are given by* $\theta_1 = \binom{n}{2}$, $\theta_2 = \binom{n}{3}$, *and* $\theta_k = (k-1)\binom{n+1}{4}$ *for* $k \geqslant 3$.

Corollary

Let $\Pi_n = F_{n-1} \times \cdots \times F_1$. *Then* $P_n \not\cong \Pi_n$ *for* $n \geqslant 4$, *although both groups have the same Betti numbers and LCS ranks.*

Theorem (D. Cohen–Schenck 2015)

$\theta_k(wP_n) = (k-1)\binom{n}{2} + (k^2-1)\binom{n}{3}$, *for* $k \gg 0$.

Theorem (S.–Wang)

The Chen ranks of wP_n^+ *are given by* $\theta_1 = \binom{n}{2}$, $\theta_2 = \binom{n}{3}$, *and* $\theta_k = \sum_{i=3}^{k} \binom{n+i-2}{i+1} + \binom{n+1}{4}$ *for* $k \geqslant 3$.

COROLLARY (SW)

$wP_n^+ \not\cong P_n$ and $wP_n^+ \not\cong \Pi_n$ for $n \geqslant 4$, although all three groups have the same Betti numbers and LCS ranks.

- This answers a question of F. Cohen, Pakianathan, Vershinin, and Jie Wu (2008).
- For $n = 4$, an incomplete argument was given by Bardakov and Mikhailov (2008), using single-variable Alexander polynomials.

PROPOSITION (SW)

1. $\theta_k(\mathfrak{h}(vP_n^+)) = \theta_k(\mathrm{gr}(vP_n^+))$ *and* $\theta_k(\mathfrak{h}(vP_n)) = \theta_k(\mathrm{gr}(vP_n))$, $\forall n, k$.
2. $\theta_k(\mathfrak{h}(vP_n^+)) = \theta_k(vP_n^+)$ *for* $n \leqslant 6$ *and all* k.
3. $\theta_k(\mathfrak{h}(vP_n)) = \theta_k(vP_n)$ *for* $n \leqslant 3$ *and all* k.

We do not know if equality holds for all n and k in the last two formulas.

THE CHEN RANKS FORMULA

CONJECTURE (S. 2001)

Let G be a hyperplane arrangement group. Let $h_m(G)$ be the number of m-dimensional components of $\mathcal{R}_1(G)$. Then, for $k \gg 1$,

$$\theta_k(G) = \sum_{m \geqslant 2} h_m(G) \cdot \theta_k(F_m).$$

- The conjecture was known to hold for $G = P_n$ (D. Cohen–S.).
- It was verified for certain classes of arrangements by Papadima–S. (2006) and Schenck–S. (2006).
- Inequality $\geqslant$ was established in (SS 2006).

THEOREM (D. COHEN–SCHENCK 2015)

More generally, the conjecture holds if G is a 1-formal, commutator-relators group for which $\mathcal{R}_1(G)$ is isotropic, projectively disjoint, and reduced as a scheme.

- We show that the commutator-relators assumption can be dropped. (The other assumptions are necessary.)
- If both G_1 and G_2 satisfy the Chen ranks formula, then $G_1 \times G_2$ also satisfies the Chen ranks formula, but $G_1 * G_2$ may not.
- As noted by Cohen–Schenck, the groups wP_n satisfy the Chen ranks formula.
- However, we show that, for $n \geqslant 4$, the groups wP_n^+ do *not* satisfy the Chen ranks formula, and $\mathcal{R}_1(wP_n^+)$ is neither isotropic, nor reduced as a scheme.
- The groups vP_3^+ and vP_3 do *not* satisfy the Chen ranks formula, even though they are both 1-formal, and their first resonance varieties are projectively disjoint and reduced as schemes. But $\mathcal{R}_1(vP_3^+)$ and $\mathcal{R}_1(vP_3)$ are not isotropic.

G	Res variety $\mathcal{R}_1(G) \subseteq H^1(G;\mathbb{C})$	Chen ranks $\theta_k(G)$ for $k \geqslant 3$	ResChen formula
P_n	$\binom{n}{3} + \binom{n}{4}$ planes	$(k-1)\binom{n+1}{4}$	Yes
wP_n	$\binom{n}{2}$ planes and $\binom{n}{4}$ linear spaces of dim 3	$(k-1)\binom{n}{2} + (k^2-1)\binom{n}{3}$ for $k \gg 3$	Yes
wP_n^+	$(n-i)$ linear spaces of dim $2 \leqslant i \leqslant n-1$	$\sum_{i=3}^{k} \binom{n+i-2}{i+1} + \binom{n+1}{4}$	No (for $n \geqslant 4$)
vP_3	$H^1(vP_3, \mathbb{C}) = \mathbb{C}^6$	$\binom{k+3}{5} + \binom{k+2}{4} + \binom{k+1}{3} + 6\binom{k}{2} + k - 2$	No
vP_4^+	3-dim non-linear subvariety of deg 6	$(k^3-1) + \binom{k}{2}$	No

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