MILNOR FIBRATIONS OF HYPERPLANE ARRANGEMENTS

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References

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HYPERPLANE ARRANGEMENTS

- An *arrangement of hyperplanes* is a finite set $\mathcal{A}$ of codimension-1 linear subspaces in $\mathbb{C}^{d+1}$.
- Let $M(\mathcal{A}) = \mathbb{C}^{d+1} \setminus \bigcup_{H\in\mathcal{A}} H$ be the complement of $\mathcal{A}$.
- This is a smooth, quasi-projective variety, with the homotopy type of a connected, finite CW-complex of dimension $d+1$.
- In fact, as conjectured by Papadima–S. (2000) and proved by Dimca–Papadima (2003), $M(\mathcal{A})$ admits a *minimal* cell structure (the Morse inequalities are satisfied on the nose).
- In particular, $H_*(M(\mathcal{A}), \mathbb{Z})$ is torsion-free.
- $\operatorname{Poin}(M(\mathcal{A}), t) = \sum_{X\in L(\mathcal{A})} \mu(X)(-t)^{\operatorname{rank}(X)}$, where $\mu\colon L(\mathcal{A}) \to \mathbb{Z}$ is given by $\mu(\mathbb{C}^{d+1}) = 1$ and $\mu(X) = -\sum_{Y\supsetneq X} \mu(Y)$.
- The ring $H^*(M(\mathcal{A}), \mathbb{Z})$ is the quotient of the exterior algebra on $\mathcal{A}$ (gens in deg 1) by an ideal (gens in deg $\geqslant 2$) determined by $L(\mathcal{A})$.

THE MILNOR FIBRATION(S) OF AN ARRANGEMENT

- For each $H \in \mathcal{A}$, let $f_H \colon \mathbb{C}^{d+1} \to \mathbb{C}$ be a linear form with kernel H.
- For each $m \in \mathbb{N}^{|\mathcal{A}|}$, let $Q_m = \prod_{H\in\mathcal{A}} f_H^{m_H}$: a homogeneous polynomial of degree $N = \sum_{H\in\mathcal{A}} m_H$.
- The map $Q_m \colon \mathbb{C}^{d+1} \to \mathbb{C}$ restricts to a map $Q_m \colon M(\mathcal{A}) \to \mathbb{C}^*$.
- This is the projection of a smooth, locally trivial bundle, known as the *(global) Milnor fibration* of the multi-arrangement $(\mathcal{A}, m)$,

$$F_m(\mathcal{A}) \longrightarrow M(\mathcal{A}) \xrightarrow{Q_m} \mathbb{C}^*.$$

- The typical fiber, $F_m(\mathcal{A}) = Q_m^{-1}(1)$, is called the *Milnor fiber* of the multi-arrangement; it has $\gcd(m)$ connected components.
- The *(geometric) monodromy* is the diffeomorphism

$$h \colon F_m(\mathcal{A}) \to F_m(\mathcal{A}), \quad z \mapsto e^{2\pi \mathrm{i}/N} z.$$

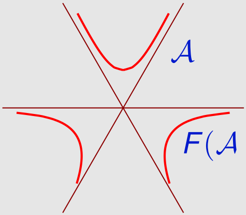
- If all $m_H = 1$, the polynomial $Q = Q_m$ is the usual defining polynomial, and $F(\mathcal{A}) = F_m(\mathcal{A})$ is the usual Milnor fiber of $\mathcal{A}$.

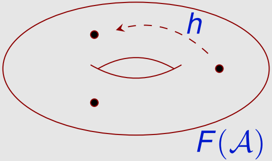
EXAMPLE

Let $\mathcal{B}_n$ be the Boolean arrangement in $\mathbb{C}^n$, with $Q = z_1 \cdots z_n$. Then $M(\mathcal{B}_n) = (\mathbb{C}^*)^n$ and $F(\mathcal{B}_n) = (\mathbb{C}^*)^{n-1}$.

EXAMPLE

Let $\mathcal{A}$ be a pencil of 3 lines through the origin of $\mathbb{C}^2$. Then $F(\mathcal{A})$ is a thrice-punctured torus, and h is an automorphism of order 3:



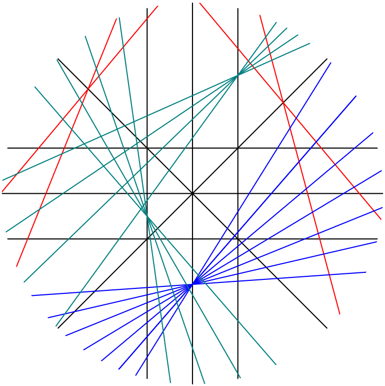


- $F_m(\mathcal{A})$ is a Stein domain of complex dimension d; thus, it has the homotopy type of a finite CW-complex of dimension d.
- Question 1 (folklore): Is $H_*(F(\mathcal{A}), \mathbb{Q})$ determined by $L(\mathcal{A})$? Answer still not known, even for $* = 1$.
- Question 2 (Dimca–Némethi, Randell): Is $H_*(F(\mathcal{A}), \mathbb{Z})$ torsion-free?
- In previous work (Cohen–Denham–S. 2003), we showed that the answer to Q2 is no, but only for $F_m(\mathcal{A})$, for non-trivial m.

THEOREM (DS12)

For every prime $p \geqslant 2$, there is a hyperplane arrangement $\mathcal{A}$ whose Milnor fiber $F(\mathcal{A})$ has non-trivial p-torsion in homology.

In particular, Milnor fibers of arrangements may not admit a minimal cell structure.



Simplest example: the arrangement of 27 hyperplanes in $\mathbb{C}^8$ with

$$Q(\mathcal{A}) = xy(x^2-y^2)(x^2-z^2)(y^2-z^2)w_1w_2w_3w_4w_5(x^2-w_1^2)\cdot$$
$$(x^2-2w_1^2)(x^2-3w_1^2)(x-4w_1)((x-y)^2-w_2^2)((x+y)^2-w_3^2)\cdot$$
$$((x-z)^2-w_4^2)((x-z)^2-2w_4^2)\cdot((x+z)^2-w_5^2)((x+z)^2-2w_5^2).$$

Then $H_6(F(\mathcal{A}),\mathbb{Z})$ has 2-torsion (of rank 108).

Basic idea of proof:

- $F_m(\mathcal{A})$ is the regular $\mathbb{Z}_N$-cover of $U = \mathbb{P}M(\mathcal{A})$ defined by the homomorphism $\delta_m\colon \pi_1(U) \twoheadrightarrow \mathbb{Z}_N$, taking each meridian x_H to m_H.
- If $\Bbbk = \overline{\Bbbk}$ and $\operatorname{char}(\Bbbk) \nmid N$, then $H_q(F_m(\mathcal{A}), \Bbbk) \cong \bigoplus_\rho H_q(U, \Bbbk_\rho)$, where $\rho\colon \pi_1(U) \to \Bbbk^*$ factors through δ_m.
- Thus, if there is such a character ρ for which $H_q(U, \Bbbk_\rho) \neq 0$, where $p = \operatorname{char}(\Bbbk) \nmid N$, but there is no corresponding character in characteristic 0, then $H_q(F_m(\mathcal{A}), \mathbb{Z})$ will have p-torsion.
- Fix a prime p. Starting with an arrangement $\mathcal{A}$ with a suitable multinet structure, we find a deletion $\mathcal{A}' = \mathcal{A} \backslash \{H\}$, and a choice of multiplicities m' on $\mathcal{A}'$ such that $H_1(F_{m'}(\mathcal{A}'), \mathbb{Z})$ has p-torsion.
- Finally, we construct a "polarized" arrangement, $\mathcal{B} = \mathcal{A}' \parallel m'$, and show that $H_*(F(\mathcal{B}), \mathbb{Z})$ has p-torsion.

CHARACTERISTIC VARIETIES

- Let X be a connected, finite-type CW-complex. Then $G = \pi_1(X, x_0)$ is a finitely generated group, with $G_{ab} \cong H_1(X, \mathbb{Z})$.
- The character group $\widehat{G} = \widehat{G}_{\Bbbk} := \mathrm{Hom}(G, \Bbbk^*) \cong H^1(X, \Bbbk^*)$ is an abelian algebraic group, with $\widehat{G} \cong \widehat{G_{ab}}$.
- Characteristic varieties:

$$\mathcal{V}^i_s(X, \Bbbk) := \{\rho \in \widehat{G} \mid \dim_{\Bbbk} H_i(X, \Bbbk_\rho) \geqslant s\}.$$

Here, $\Bbbk_\rho$ is the local system defined by ρ, i.e, $\Bbbk$ viewed as a $\Bbbk G$-module, via $g \cdot x = \rho(g)x$, and $H_i(X, \Bbbk_\rho) = H_i(C_*(\widetilde{X}, \Bbbk) \otimes_{\Bbbk G} \Bbbk_\rho)$.

- Product formula: $\mathcal{V}^i_1(X_1 \times X_2, \Bbbk) = \bigcup_{p+q=i} \mathcal{V}^p_1(X_1, \Bbbk) \times \mathcal{V}^q_1(X_2, \Bbbk)$.
- The sets $\mathcal{V}^1_s(X, \Bbbk)$ depend only on $G = \pi_1(X)$—in fact, only on G/G''. Write them as $\mathcal{V}^1_s(G, \Bbbk)$, and set $\mathcal{V}^1(G, \Bbbk) = \mathcal{V}^1_s(G, \Bbbk)$.
- If $\varphi\colon G \twoheadrightarrow Q$ is an epimorphism, then the induced morphism $\hat{\varphi}_{\Bbbk}\colon \widehat{Q} \hookrightarrow \widehat{G}$ restricts to an embedding $\mathcal{V}^1_s(Q, \Bbbk) \hookrightarrow \mathcal{V}^1_s(G, \Bbbk)$, $\forall s$.

EXAMPLE

Identify $\pi_1(\mathbb{C}\setminus\{n \text{ points}\}) = F_n$, and $\widehat{F_n} = (\Bbbk^*)^n$. Then:

$$\mathcal{V}^1_s(\mathbb{C}\setminus\{n \text{ points}\}, \Bbbk) = \begin{cases} (\Bbbk^*)^n & \text{if } s < n, \\ \{1\} & \text{if } s = n, \\ \varnothing & \text{if } s > n. \end{cases}$$

In general, though, the characteristic varieties depend on $\operatorname{char}(\Bbbk)$.

EXAMPLE

- Let $\Gamma = F_n * \mathbb{Z}_p$, and identify $\widehat{\Gamma} = \widehat{F_n} \times \widehat{\mathbb{Z}_p}$.
- If $\operatorname{char}(\Bbbk) \neq p$, then $\widehat{\mathbb{Z}_p} = \operatorname{Hom}(\mathbb{Z}_p, \Bbbk^*)$ has order p, and

$$\mathcal{V}^1(\Gamma, \Bbbk) = \begin{cases} \widehat{\Gamma} & \text{if } n \geqslant 2, \\ (\widehat{\Gamma}\setminus\widehat{\Gamma}^\circ) \cup \{\mathbf{1}\} & \text{if } n = 1\ . \end{cases}$$

- If $\operatorname{char}(\Bbbk) = p$, then $\widehat{\mathbb{Z}_p} = \{\mathbf{1}\}$, and $\mathcal{V}^1(\Gamma, \Bbbk) = \widehat{\Gamma}$, for all $n \geqslant 1$.

HOMOLOGY OF FINITE ABELIAN COVERS

- Let X be a connected, finite-type CW-complex, and $G = \pi_1(X)$.
- Let A be a finite abelian group.
- Every epimorphism $\chi\colon G \to A$ determines a regular, connected A-cover $X^\chi \to X$.
- Let $\Bbbk$ be a field, $p = \operatorname{char}(\Bbbk)$. Assume $p = 0$ or $p \nmid |A|$. Then

$$H_q(X^\chi, \Bbbk) \cong H_q(X, \Bbbk[A]) \cong \bigoplus_{\rho \in \widehat{A}} H_q(X, \Bbbk_\rho).$$

- Hence,

$$\dim_{\Bbbk} H_q(X^\chi, \Bbbk) = \sum_{s \geqslant 1} \left| \mathcal{V}^q_s(X, \Bbbk) \cap \operatorname{im}(\hat{\chi}_{\Bbbk}) \right|,$$

where $\hat{\chi}_{\Bbbk}\colon \widehat{A}_{\Bbbk} \to \widehat{G}_{\Bbbk}$ is the induced morphism between character groups.

- Now suppose A is a finite cyclic group. Choose a generator $\alpha \in A$.
- Let $h = h_\alpha \colon X^\chi \to X^\chi$ be the monodromy automorphism.
- Let $h_* \colon H_q(X^\chi, \Bbbk) \to H_q(X^\chi, \Bbbk)$ be the induced homomorphism.
- Note: if $\rho \colon G \to \Bbbk^*$ is a character belonging to $\operatorname{im}(\hat{\chi}_\Bbbk)$, there is a unique character $\iota_\rho \colon A \to \Bbbk^*$ such that $\rho = \iota_\rho \circ \chi$.
- Assume $\operatorname{char}(\Bbbk) \nmid |A|$. Then, the characteristic polynomial of the algebraic monodromy, $\Delta^\Bbbk_{\chi,q}(t) = \det(t \cdot \mathrm{id} - h_*)$, is given by

$$\Delta^\Bbbk_{\chi,q}(t) = \prod_{s\geqslant 1} \prod_{\rho \in \operatorname{im}(\hat{\chi}) \cap \mathcal{V}^q_s(X,\Bbbk)} (t - \iota_\rho(\alpha)).$$

THEOREM

Let $X^{\chi} \to X$ be a regular, finite cyclic cover, defined by an epimorphism $\chi\colon \pi_1(X) \twoheadrightarrow \mathbb{Z}_r$. Suppose that $\operatorname{im}(\hat{\chi}_{\mathbb{C}}) \nsubseteq \mathcal{V}^q_1(X,\mathbb{C})$, but $\operatorname{im}(\hat{\chi}_{\Bbbk}) \subseteq \mathcal{V}^q_1(X,\Bbbk)$, for some field $\Bbbk$ of characteristic p not dividing r. Then $H_q(X^{\chi},\mathbb{Z})$ has non-zero p-torsion.

- In order to apply this theorem, one needs to know both the characteristic varieties over $\mathbb{C}$ and over $\Bbbk$, and exploit the qualitative differences between the two.
- In a special situation, we only need to verify that a certain condition on the first characteristic variety over $\mathbb{C}$ holds.

THEOREM

Suppose there is a character $\rho\colon \pi_1(X) \to \mathbb{C}^*$ which factors as $\pi_1(X) \twoheadrightarrow \mathbb{Z} * \mathbb{Z}_p \twoheadrightarrow \mathbb{Z} \to \mathbb{C}^*$, for some prime p, but $\rho \notin \mathcal{V}^1(X,\mathbb{C})$. Then, for all sufficiently large integers r not divisible by p, there is a regular, r-fold cyclic cover $Y \to X$ such that $H_1(Y,\mathbb{Z})$ has non-zero p-torsion.

ORBIFOLD FIBRATIONS

- Let $\Sigma_{g,r}$ be a Riemann surface of genus $g \geqslant 0$ with $r \geqslant 0$ points removed. Fix points $q_1, \dots, q_s$ on the surface, and assign integer weights $\mu_1, \dots, \mu_s$ with $\mu_i \geqslant 2$.
- The orbifold $\Sigma = (\Sigma_{g,r}, \mu)$ is *hyperbolic* if $\chi^{\text{orb}}(\Sigma) := 2 - 2g - r - \sum_{i=1}^{s}(1 - 1/\mu_i)$ is negative.
- A hyperbolic orbifold Σ is *small* if either $\Sigma = S^1 \times S^1$ and $s \geqslant 2$, or $\Sigma = \mathbb{C}^*$ and $s \geqslant 1$; otherwise, Σ is *large*.
- The orbifold fundamental group is defined as

$$\Gamma = \left\langle \begin{matrix} x_1, \dots, x_g, y_1, \dots, y_g \\ z_1, \dots, z_s \end{matrix} \;\middle|\; \begin{matrix} [x_1, y_1] \cdots [x_g, y_g] z_1 \cdots z_s = 1 \\ z_1^{\mu_1} = \cdots = z_t^{\mu_s} = 1 \end{matrix} \right\rangle (r = 0)$$

$$\Gamma = F_{2g+r-1} * \mathbb{Z}_{\mu_1} * \cdots * \mathbb{Z}_{\mu_s} \qquad (r > 0)$$

- From (Artal–Cogolludo–Matei 2013), we get:

$$\mathcal{V}^1(\Gamma) = \begin{cases} \widehat{\Gamma} & \text{if } \Sigma \text{ is a large hyperbolic orbifold,} \\ (\widehat{\Gamma} \setminus \widehat{\Gamma}^\circ) \cup \{1\} & \text{if } \Sigma \text{ is a small hyperbolic orbifold,} \\ \{1\} & \text{otherwise.} \end{cases}$$

- Let X be a smooth, quasi-projective variety, and $G = \pi_1(X)$.
- A surjective, holomorphic map $f\colon X \to (\Sigma_{g,r}, \mu)$ is called an *orbifold fibration* if the generic fiber is connected; the multiplicity of the fiber over each marked point q_i equals μ_i; and f admits an extension $\bar{f}\colon \overline{X} \to \Sigma_g$ with the same properties.
- Such a map induces an epimorphism $f_\sharp\colon G \twoheadrightarrow \Gamma$, where $\Gamma = \pi_1^{\text{orb}}(\Sigma_{g,s}, \mu)$, and thus a monomorphism $\hat{f}_\sharp\colon \widehat{\Gamma} \hookrightarrow \widehat{G}$.

THEOREM (ARAPURA + ACM + BW)

$$\mathcal{V}^1(X) = \bigcup_{f \text{ large}} \operatorname{im}(\hat{f}_\sharp) \cup \bigcup_{f \text{ small}} \left(\operatorname{im}(\hat{f}_\sharp) \setminus \operatorname{im}(\hat{f}_\sharp)^\circ\right) \cup Z,$$

where Z is a finite set of torsion characters.

THEOREM

Suppose there is a small orbifold fibration $f\colon X \to (\Sigma, \mu)$ and a prime p dividing each μ_i. Then, for all $r \gg 0$ with $p \nmid r$, there is a regular, r-fold cyclic cover $Y \to X$ such that $H_1(Y, \mathbb{Z})$ has non-zero p-torsion.

MULTINETS

DEFINITION (FALK AND YUZVINSKY)

A (k, ℓ)-*multinet* ($k \geqslant 3$, $\ell \geqslant 1$) on an arrangement $\mathcal{A}$ consists of:

- A partition $\mathcal{A} = (\mathcal{A}_1, \dots, \mathcal{A}_k)$.
- An assignment of multiplicities on the hyperplanes, $m\colon \mathcal{A} \to \mathbb{N}$.
- A subset $\mathcal{X} \subseteq L_2(\mathcal{A})$, called the base locus.

Moreover,

- $\sum_{H \in \mathcal{A}_i} m_H = \ell$, for all $i \in [k]$.
- For any hyperplanes H and H' in different classes, $H \cap H' \in \mathcal{X}$.
- For each $X \in \mathcal{X}$, the sum $n_X = \sum_{H \in \mathcal{A}_i \colon H \supset X} m_H$ is independent of i.
- For each $i \in [k]$, the space $\big(\bigcup_{H \in \mathcal{A}_i} H\big) \setminus \mathcal{X}$ is connected.

WLOG, we may assume $\gcd\{m_H \mid H \in \mathcal{A}\} = 1$. If all $m_H = 1$, the multinet is *reduced*. If, furthermore, every flat in $\mathcal{X}$ is contained in precisely one hyperplane from each class, this is a *net*.

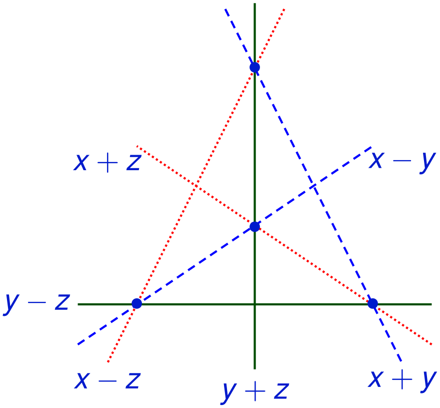


FIGURE : A $(3,2)$-net on the A_3 arrangement

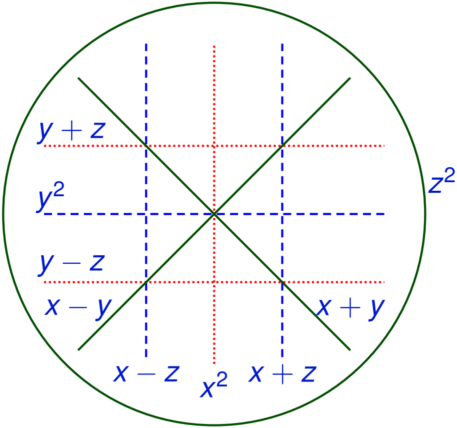


FIGURE : A $(3, 4)$-multinet on the B_3 arrangement

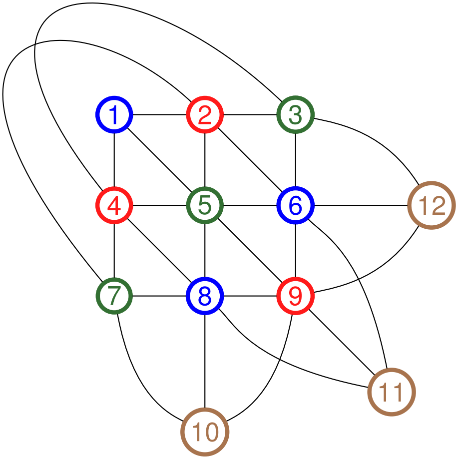


FIGURE : A $(4,3)$-net on the Hessian matroid

- Let $\mathcal{A}$ be a (central) arrangement of hyperplanes in $\mathbb{C}^{d+1}$.
- Let $M = \mathbb{C}^{d+1} \setminus \bigcup_{H \in \mathcal{A}} H$, and $U = \mathbb{P}M$. Then $M \cong U \times \mathbb{C}^*$.
- Identify $H_1(M, \mathbb{Z}) = \mathbb{Z}^n$, with basis the meridians $\{x_H \mid H \in \mathcal{A}\}$.
- Let $\text{Hom}(\pi_1(M), \Bbbk^*) = (\Bbbk^*)^n$ be the character torus.
- Then $\mathcal{V}^1(\mathcal{A}) := \mathcal{V}^1(M) \subset (\Bbbk^*)^n$ is isomorphic to $\mathcal{V}^1(U) \subseteq \{t \in (\Bbbk^*)^n \mid t_1 \cdots t_n = 1\} \cong (\Bbbk^*)^{n-1}$.

THEOREM (FALK–YUZ, PEREIRA–YUZ, YUZVINSKY)

Each positive-dimensional, non-local component of $\mathcal{V}^1(\mathcal{A})$ is of the form ρT, where ρ is a torsion character, $T = f^*(H^1(\Sigma_{0,k}, \mathbb{C}^*))$, for some orbifold fibration $f \colon M(\mathcal{A}) \to (\Sigma_{0,k}, \mu)$, and either

- $k = 2$, *and* f *has at least one multiple fiber, or*
- $k = 3$ *or* 4, *and* f *corresponds to a multinet with* k *classes on the multiarrangement* $(\mathcal{A}, m)$, *for some* m.

MILNOR FIBERS

- Let $(\mathcal{A}, m)$ be a multi-arrangement with $\gcd\{m_H \mid H \in \mathcal{A}\} = 1$. Set $N = \sum_{H\in\mathcal{A}} m_H$.
- The Milnor fiber $F_m(\mathcal{A})$ is the regular $\mathbb{Z}_N$-cover of $U(\mathcal{A})$ defined by the homomorphism $\delta_m \colon \pi_1(U(\mathcal{A})) \twoheadrightarrow \mathbb{Z}_N$, $x_H \mapsto m_H \bmod N$.
- If $\operatorname{char}(\Bbbk) \nmid N$, then

$$\dim_{\Bbbk} H_q(F_m(\mathcal{A}), \Bbbk) = \sum_{s\geqslant 1} \left| \mathcal{V}^q_s(U(\mathcal{A}), \Bbbk) \cap \operatorname{im}(\widehat{\delta_m}) \right| .$$

- The characteristic polynomial of the algebraic monodromy, $h_* \colon H_q(F_m(\mathcal{A}), \Bbbk) \to H_q(F_m(\mathcal{A}), \Bbbk)$, is given by

$$\Delta^{\Bbbk}_{h,q}(t) = \prod_{s\geqslant 1} \prod_{\substack{\zeta\in\Bbbk^* \colon \zeta^N=1, \\ \zeta^m\in\mathcal{V}^q_s(U(\mathcal{A}),\Bbbk)}} (t-\zeta).$$

Not every regular, cyclic cover of a projective arrangement complement arises through the Milnor fiber construction. Nevertheless, the Milnor fibers dominate all other cyclic covers:

LEMMA

Let $(\mathcal{A}, m)$ be a multiarrangement, and let $U^\chi \to U$ be a regular, r-fold cyclic cover of $U = \mathbb{P}M(\mathcal{A})$. There exist then infinitely many multiplicity vectors m such that

$$\begin{array}{ccc} F(\mathcal{A}, m) & \dashrightarrow & U^\chi \\ & \searrow & \downarrow \\ & & U. \end{array}$$

Moreover, for any prime $p \nmid r$, we may choose m so that the degree of $Q(\mathcal{A}, m)$ is not divisible by p.

Deletion

A *pointed multinet* $(\mathcal{M}, H)$ on an arrangement $\mathcal{A}$ is a multinet structure $\mathcal{M} = ((\mathcal{A}_1, \dots, \mathcal{A}_k), m, \mathcal{X})$, together with a distinguished hyperplane $H \in \mathcal{A}$ for which $m_H > 1$ and $m_H \mid n_X$ for each $X \in \mathcal{X}$ such that $X \subset H$.

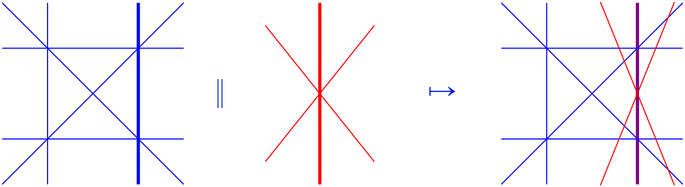
Lemma

Suppose $\mathcal{A}$ admits a pointed multinet, and set $\mathcal{A}' = \mathcal{A} \backslash \{H\}$. Then $U(\mathcal{A}')$ supports a small pencil, and $\mathcal{V}^1(\mathcal{A}')$ has a component which is a 1-dimensional subtorus, translated by a character of order m_H.

Theorem

Suppose $\mathcal{A}$ admits a pointed multinet, with distinguished hyperplane H and multiplicity m. Let p be a prime dividing m_H. There is then a choice of multiplicities m' on the deletion $\mathcal{A}' = \mathcal{A} \backslash \{H\}$ such that $H_1(F_{m'}(\mathcal{A}'), \mathbb{Z})$ has non-zero p-torsion.

POLARIZATION



- Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be arrangements in V_1 and V_2, defined by polynomials $Q_1 = \prod_{H \in \mathcal{A}_1} f_H$ and $Q_2 = \prod_{H \in \mathcal{A}_2} g_H$.
- Fix $H_1 \in \mathcal{A}_1$ and $H_2 \in \mathcal{A}_2$. The *parallel connection*, $\mathcal{A}_1 \, {}_{H_1}\!\|_{H_2} \mathcal{A}_2$, is the arrangement in $V_1 \times V_2$ defined by the polynomial

$$f_{H_1} \cdot \prod_{H \in \mathcal{A}_1 \setminus \{H_1\}} f_H \cdot \prod_{H \in \mathcal{A}_2 \setminus \{H_2\}} g_H \in \Bbbk[V_1^*] \otimes_{\Bbbk} \Bbbk[V_2^*] / (f_{H_1} - g_{H_2}).$$

- Falk and Proudfoot (2002): $\mathbb{P}(M_1) \times \mathbb{P}(M_2) \cong \mathbb{P}(M_1 \, {}_{H_1}\!\|_{H_2} M_2)$.

- If $(\mathcal{A}_i, H_i)$ are pointed arrangements and $H \in \mathcal{A}_1$, we let

$$(\mathcal{A}_1, H_1) \circ_H (\mathcal{A}_2, H_2) = (\mathcal{A}_1 {}_H\|_{H_2} \mathcal{A}_2, H_1).$$

- Let $\mathcal{A} = (\{H_1, \dots, H_n\}, m)$ be a multi-arrangement. The *polarization* $\mathcal{A} \,\|\, m$ is the iterated parallel connection

$$\mathcal{A} \circ_{H_1} \mathcal{P}_{m_{H_1}} \circ_{H_2} \cdots \circ_{H_n} \mathcal{P}_{m_{H_n}},$$

  where $\mathcal{P}_k$ denotes a pencil of k lines in $\mathbb{C}^2$.

- Note: $\operatorname{rank}(\mathcal{A} \,\|\, m) = \operatorname{rank} \mathcal{A} + |\{H \in \mathcal{A} \colon m_H \geqslant 2\}|$.

- Let $P_k = U(\mathcal{P}_k) \cong \mathbb{C} \setminus \{k-1 \text{ points}\}$. Write $P(\mathcal{A}) = \prod_{H \in \mathcal{A}} P_{m_H}$. Then

$$U(\mathcal{A}) \times P(\mathcal{A}) \cong U(\mathcal{A} \,\|\, m).$$

- A special class of arrangements obtained by iterated parallel connection was studied in (Choudary–Dimca–Papadima 2005).

The polarization construction works well with respect to Milnor fibrations:

$$\begin{array}{ccc} F_m(\mathcal{A}) & \longrightarrow & F(\mathcal{A} \,\|\, m) \\ \downarrow & & \downarrow \\ U(\mathcal{A}) & \xrightarrow{\ j\ } & U(\mathcal{A} \,\|\, m), \end{array}$$

i.e., $j^*(\delta_{\mathcal{A}\,\|\,m}) = \delta_{\mathcal{A},m}$.

THEOREM

Suppose $\mathcal{A}$ admits a pointed multinet, with distinguished hyperplane H and multiplicity m. Let p be a prime dividing m_H.

There is then a choice of multiplicities m' on the deletion $\mathcal{A}' = \mathcal{A} \backslash \{H\}$ such that $H_q(F(\mathcal{B}), \mathbb{Z})$ has p-torsion, where $\mathcal{B} = \mathcal{A}' \| m'$ and $q = 1 + \left|\{K \in \mathcal{A}' : m'_K \geqslant 3\}\right|$.

EXAMPLE

- Let $\mathcal{A}$ be reflection arrangement of type B_3, with $Q(\mathcal{A}) = xyz(x-y)(x+y)(x-z)(x+z)(y-z)(y+z)$.
- Recall $\mathcal{A}$ supports a $(3,4)$-multinet.
- Let $\mathcal{A}' = \mathcal{A} \backslash \{z=0\}$ be the deleted B_3 arrangement.
- Pick $m' = (8,1,3,3,5,5,1,1)$.
- Let $\mathcal{B} = \mathcal{A}' \| m'$, an arrangement of 27 hyperplanes in $\mathbb{C}^8$
- Then $H_6(F(\mathcal{B}), \mathbb{Z})$ has 2-torsion of rank 108.
- $\Delta_6^{\mathbb{C}}(t) = (t-1)^{11968}$, yet $\Delta_6^{\overline{\mathbb{F}_2}}(t) = (t-1)^{11968}(t^2+t+1)^{54}$.

THE BOUNDARY MANIFOLD

- Let $\mathcal{A}$ be a (central) arrangement of hyperplanes in $\mathbb{C}^{d+1}$ ($d \geqslant 1$).
- Let $\mathbb{P}(\mathcal{A}) = \{\mathbb{P}(H)\}_{H\in\mathcal{A}}$, and let $\nu(W)$ be a regular neighborhood of the algebraic hypersurface $W = \bigcup_{H\in\mathcal{A}} \mathbb{P}(H)$ inside $\mathbb{CP}^d$.
- Let $\overline{U} = \mathbb{CP}^d \setminus \operatorname{int}(\nu(W))$ be the *exterior* of $\mathbb{P}(\mathcal{A})$.
- The *boundary manifold* of $\mathcal{A}$ is $\partial\overline{U} = \partial\nu(W)$: a compact, orientable, smooth manifold of dimension $2d-1$.

EXAMPLE

Let $\mathcal{A}$ be a pencil of n hyperplanes in $\mathbb{C}^{d+1}$, defined by $Q = z_1^n - z_2^n$. If $n = 1$, then $\partial\overline{U} = S^{2d-1}$. If $n > 1$, then $\partial\overline{U} = \sharp^{n-1} S^1 \times S^{2(d-1)}$.

EXAMPLE

Let $\mathcal{A}$ be a near-pencil of n planes in $\mathbb{C}^3$, defined by $Q = z_1(z_2^{n-1} - z_3^{n-1})$. Then $\partial\overline{U} = S^1 \times \Sigma_{n-2}$, where $\Sigma_g = \sharp^g S^1 \times S^1$.

- By Lefschetz duality: $H_q(\partial\overline{U},\mathbb{Z}) \cong H_q(U,\mathbb{Z}) \oplus H_{2d-q-1}(U,\mathbb{Z})$
- Let $A = H^*(U,\mathbb{Z})$; then $\check{A} = \operatorname{Hom}_{\mathbb{Z}}(A,\mathbb{Z})$ is an A-bimodule, with $(a \cdot f)(b) = f(ba)$ and $(f \cdot a)(b) = f(ab)$.

THEOREM (COHEN–S. 2006)

The ring $\widehat{A} = H^*(\partial\overline{U},\mathbb{Z})$ is the "double" of A, that is: $\widehat{A} = A \oplus \check{A}$, with multiplication given by $(a,f)\cdot(b,g) = (ab, ag + fb)$, and grading $\widehat{A}^q = A^q \oplus \check{A}^{2d-q-1}$.

- Now assume $d = 2$. Then $\partial\overline{U}$ is a graph-manifold of dimension 3, modeled on a graph Γ based on the poset $L_{\leqslant 2}(\mathcal{A})$.

THEOREM (COHEN–S. 2008)

The manifold $\partial\overline{U}$ admits a minimal cell structure. Moreover,

$$\mathcal{V}_1^1(\partial\overline{U}) = \bigcup_{v\in \mathrm{V}(\Gamma)\,:\, d_v\geqslant 3} \{t_v - 1 = 0\},$$

where d_v denotes the degree of the vertex v, and $t_v = \prod_{i\in v} t_i$.

THE BOUNDARY OF THE MILNOR FIBER

- Let $(\mathcal{A}, m)$ be a multi-arrangement in $\mathbb{C}^{d+1}$.
- Define $\overline{F}_m(\mathcal{A}) = F_m(\mathcal{A}) \cap D^{2(d+1)}$ to be the *closed Milnor fiber* of $(\mathcal{A}, m)$. Clearly, $F_m(\mathcal{A})$ deform-retracts onto $\overline{F}_m(\mathcal{A})$.
- The *boundary of the Milnor fiber* of $(\mathcal{A}, m)$ is the compact, smooth, orientable, $(2d-1)$-manifold $\partial\overline{F}_m(\mathcal{A}) = F_m(\mathcal{A}) \cap S^{2d+1}$.
- The pair $(\overline{F}_m, \partial\overline{F}_m)$ is $(d-1)$-connected. In particular, if $d \geqslant 2$, then $\partial\overline{F}_m$ is connected, and $\pi_1(\partial\overline{F}_m) \to \pi_1(\overline{F}_m)$ is surjective.

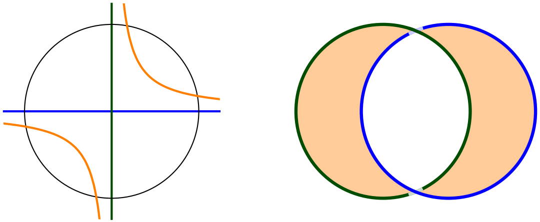


FIGURE : Closed Milnor fiber for $Q(\mathcal{A}) = xy$

EXAMPLE

- Let $\mathcal{B}_n$ be the Boolean arrangement in $\mathbb{C}^n$. Recall $F = (\mathbb{C}^*)^{n-1}$. Hence, $\overline{F} = T^{n-1} \times D^{n-1}$, and so $\partial\overline{F} = T^{n-1} \times S^{n-2}$.
- Let $\mathcal{A}$ be a near-pencil of n planes in $\mathbb{C}^3$. Then $\partial\overline{F} = S^1 \times \Sigma_{n-2}$.

The Hopf fibration $\pi\colon \mathbb{C}^{d+1}\backslash\{0\} \to \mathbb{CP}^d$ restricts to regular, cyclic n-fold covers, $\pi\colon \overline{F} \to \overline{U}$ and $\pi\colon \partial\overline{F} \to \partial\overline{U}$, which fit into the ladder

$$\begin{array}{ccccccccc}
\mathbb{Z}_n & = & \mathbb{Z}_n & = & \mathbb{Z}_n & \longrightarrow & \mathbb{C}^* & = & \mathbb{C}^* \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\partial\overline{F} & \longrightarrow & \overline{F} & \xrightarrow{\simeq} & F & \longrightarrow & M & \longrightarrow & \mathbb{C}^{d+1}\backslash\{0\} \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\
\partial\overline{U} & \longrightarrow & \overline{U} & \xrightarrow{\simeq} & U & = & U & \longrightarrow & \mathbb{CP}^d
\end{array}$$

Assume now that $d = 2$. The group $\pi_1(\partial\overline{U})$ has generators $x_1, \dots, x_{n-1}$ corresponding to the meridians around the first $n-1$ lines in $\mathbb{P}(\mathcal{A})$, and generators $y_1, \dots, y_s$ corresponding to the cycles in the associated graph Γ.

PROPOSITION (S13)

The $\mathbb{Z}_n$-cover $\pi\colon \partial\overline{F} \to \partial\overline{U}$ is classified by the homomorphism $\pi_1(\partial\overline{U}) \twoheadrightarrow \mathbb{Z}_n$ given by $x_i \mapsto 1$ and $y_i \mapsto 0$.

EXAMPLE

Let $\mathcal{A}$ be a pencil of $n+1$ planes in $\mathbb{C}^3$. Since $\partial\overline{U} = \sharp^n S^1 \times S^2$, and $\partial\overline{F} \to \partial\overline{U}$ is a cover with $n+1$ sheets, we see that $\partial\overline{F} = \sharp^{n^2} S^1 \times S^2$.

THEOREM (NÉMETHI–SZILARD 2012)

Let $\mathcal{A}$ be an arrangement of n planes in $\mathbb{C}^3$. The characteristic polynomial of the algebraic monodromy acting on $H_1(\partial\overline{F}, \mathbb{C})$ is given by

$$\Delta(t) = \prod_{X \in L_2(\mathcal{A})} (t-1)(t^{\gcd(\mu(X)+1,n)} - 1)^{\mu(X)-1}.$$

- This shows that $b_1(\partial\overline{F})$ is a much less subtle invariant than $b_1(F)$: it depends only on the number and type of multiple points of $\mathbb{P}(\mathcal{A})$, but not on their relative position.
- On the other hand, the torsion in $H_1(\partial\overline{F}, \mathbb{Z})$ is still not understood.
- For a generic arrangement of n planes in $\mathbb{C}^3$, I expect that $H_1(\partial\overline{F}, \mathbb{Z}) = \mathbb{Z}^{n(n-1)/2} \oplus \mathbb{Z}_n^{(n-2)(n-3)/2}$.
- In general, it would be interesting to see whether all the torsion in $H_1(\partial\overline{F}(\mathcal{A}), \mathbb{Z})$ consists of $\mathbb{Z}_n$-summands, where $n = |\mathcal{A}|$.