Hyperplane arrangements and Milnor fibrations

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THE MILNOR FIBRATION(S) OF AN ARRANGEMENT

- Let $\mathcal{A}$ be a (central) hyperplane arrangement in $\mathbb{C}^\ell$.
- For each $H \in \mathcal{A}$, let $f_H\colon \mathbb{C}^\ell \to \mathbb{C}$ be a linear form with kernel H.
- For each choice of multiplicities $m = (m_H)_{H\in\mathcal{A}}$ with $m_H \in \mathbb{N}$, let

$$Q_m := Q_m(\mathcal{A}) = \prod_{H\in\mathcal{A}} f_H^{m_H},$$

a homogeneous polynomial of degree $N = \sum_{H\in\mathcal{A}} m_H$.

- The map $Q_m\colon \mathbb{C}^\ell \to \mathbb{C}$ restricts to a map $Q_m\colon M(\mathcal{A}) \to \mathbb{C}^*$.
- This is the projection of a smooth, locally trivial bundle, known as the *Milnor fibration* of the multi-arrangement $(\mathcal{A}, m)$,

$$F_m(\mathcal{A}) \longrightarrow M(\mathcal{A}) \xrightarrow{Q_m} \mathbb{C}^*.$$

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$$F_m(\mathcal{A}) \longrightarrow M(\mathcal{A}) \xrightarrow{\ Q_m\ } \mathbb{C}^*.$$

- The typical fiber, $F_m(\mathcal{A}) = Q_m^{-1}(1)$, is called the *Milnor fiber* of the multi-arrangement.
- $F_m(\mathcal{A})$ is a Stein manifold. It has the homotopy type of a finite cell complex, with $\gcd(m)$ connected components, of dim $\ell - 1$.
- The *(geometric) monodromy* is the diffeomorphism

$$h\colon F_m(\mathcal{A}) \to F_m(\mathcal{A}), \quad z \mapsto e^{2\pi \mathrm{i}/N} z.$$

- If all $m_H = 1$, the polynomial $Q = Q(\mathcal{A})$ is the usual defining polynomial, and $F(\mathcal{A})$ is the usual Milnor fiber of $\mathcal{A}$.

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EXAMPLE

Let $\mathcal{A}$ be the single hyperplane $\{0\}$ inside $\mathbb{C}$. Then $M(\mathcal{A}) = \mathbb{C}^*$, $Q_m(\mathcal{A}) = z^m$, and $F_m(\mathcal{A}) = m$-roots of 1.

EXAMPLE

Let $\mathcal{A}$ be a pencil of 3 lines through the origin of $\mathbb{C}^2$. Then $F(\mathcal{A})$ is a thrice-punctured torus, and h is an automorphism of order 3:

More generally, if $\mathcal{A}$ is a pencil of n lines in $\mathbb{C}^2$, then $F(\mathcal{A})$ is a Riemann surface of genus $\binom{n-1}{2}$, with n punctures.

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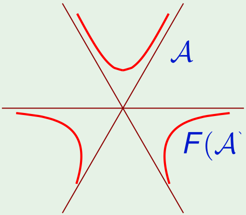

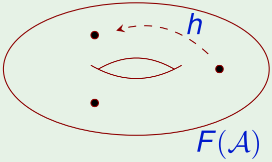

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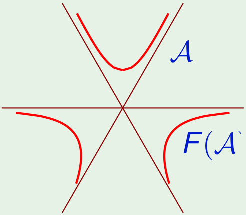

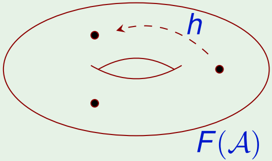

More generally, if $\mathcal{A}$ is a pencil of n lines in $\mathbb{C}^2$, then $F(\mathcal{A})$ is a Riemann surface of genus $\binom{n-1}{2}$, with n punctures.

- Let $\mathcal{B}_n$ be the Boolean arrangement, with $Q_m(\mathcal{B}_n) = z_1^{m_1} \cdots z_n^{m_n}$. Then $M(\mathcal{B}_n) = (\mathbb{C}^*)^n$ and

$$F_m(\mathcal{B}_n) = \ker(\mathbb{Q}_m) \cong (\mathbb{C}^*)^{n-1} \times \mathbb{Z}_{\gcd(m)}$$

- Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an essential arrangement. The inclusion $\iota\colon M(\mathcal{A}) \to M(\mathcal{B}_n)$ restricts to a bundle map

$$\begin{array}{ccccc} F_m(\mathcal{A}) & \longrightarrow & M(\mathcal{A}) & \xrightarrow{Q_m(\mathcal{A})} & \mathbb{C}^* \\ \downarrow & & \downarrow \iota & & \| \\ F_m(\mathcal{B}_n) & \longrightarrow & M(\mathcal{B}_n) & \xrightarrow{Q_m(\mathcal{B}_n)} & \mathbb{C}^* \end{array}$$

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$$F_m(\mathcal{A}) = M(\mathcal{A}) \cap F_m(\mathcal{B}_n)$$

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THE HOMOLOGY OF THE MILNOR FIBER

Some basic questions about the topology of the Milnor fibration:

(Q1) Are the homology groups $H_q(F_m(\mathcal{A}), \Bbbk)$ determined by $L(\mathcal{A})$? If so, is the characteristic polynomial of the algebraic monodromy, $h_*\colon H_q(F_m(\mathcal{A}), \Bbbk) \to H_q(F_m(\mathcal{A}), \Bbbk)$, also determined by $L(\mathcal{A})$?

(Q2) Are the homology groups $H_q(F_m(\mathcal{A}), \mathbb{Z})$ torsion-free? If so, does $F_m(\mathcal{A})$ admit a minimal cell structure?

(Q3) Is $F_m(\mathcal{A})$ a (partially) formal space?

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- Let $(\mathcal{A}, m)$ be a multi-arrangement with $\gcd\{m_H \mid H \in \mathcal{A}\} = 1$. Set $N = \sum_{H\in\mathcal{A}} m_H$.
- The Milnor fiber $F_m(\mathcal{A})$ is a regular $\mathbb{Z}_N$-cover of $U(\mathcal{A}) = \mathbb{P}(M(\mathcal{A}))$ defined by the homomorphism

$$\delta_m \colon \pi_1(U(\mathcal{A})) \twoheadrightarrow \mathbb{Z}_N, \quad x_H \mapsto m_H \bmod N$$

- Let $\widehat{\delta_m} \colon \operatorname{Hom}(\mathbb{Z}_N, \Bbbk^*) \to \operatorname{Hom}(\pi_1(U(\mathcal{A})), \Bbbk^*)$. If $\operatorname{char}(\Bbbk) \nmid N$, then

$$\dim_{\Bbbk} H_q(F_m(\mathcal{A}), \Bbbk) = \sum_{s\geqslant 1} \left| \mathcal{V}^q_s(U(\mathcal{A}), \Bbbk) \cap \operatorname{im}(\widehat{\delta_m}) \right|.$$

- This gives a formula for the polynomial $\Delta_q(t) = \det(t \cdot \mathrm{id} - h_*)$ in terms of the characteristic varieties of $U(\mathcal{A})$.

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- This gives a formula for the polynomial $\Delta_q(t) = \det(t \cdot \mathrm{id} - h_*)$ in terms of the characteristic varieties of $U(\mathcal{A})$.

- Write
$$\Delta(t) := \Delta_1(t) = \prod_{d|n} \Phi_d(t)^{e_d(\mathcal{A})},$$
where $\Phi_d(t)$ is the d-th cyclotomic polynomial, and $e_d(\mathcal{A}) \in \mathbb{Z}_{\geqslant 0}$.
- Transfer argument: $e_1(\mathcal{A}) = n - 1$.
- If there is a non-transverse multiple point on $\mathcal{A}$ of multiplicity not divisible by d, then $e_d(\mathcal{A}) = 0$. (Libgober 2002).
- In particular, if $\mathcal{A}$ has only points of multiplicity 2 and 3, then $\Delta(t) = (t-1)^{m-1}(t^2+t+1)^{e_3}$.
- If multiplicity 4 appears, then also get factor of $(t+1)^{e_2} \cdot (t^2+1)^{e_4}$.

EXAMPLE

Let $\mathcal{A}$ be the braid arrangement. $\mathcal{V}_1(\mathcal{A})$ has a single essential component, $T = \{t \in (\mathbb{C}^*)^6 \mid t_1t_2t_3 = t_1t_6^{-1} = t_2t_5^{-1} = t_3t_4^{-1} = 1\}$. Clearly, $\delta^2 \in T$, yet $\delta \notin T$; hence, $\Delta(t) = (t-1)^5(t^2+t+1)$.

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MODULAR INEQUALITIES

- Let $\sigma = \sum_{H\in\mathcal{A}} e_H \in A^1$ be the "diagonal" vector.
- Assume $\Bbbk$ has characteristic $p > 0$, and define

$$\beta_p(\mathcal{A}) = \dim_{\Bbbk} H^1(A, \cdot\sigma).$$

That is, $\beta_p(\mathcal{A}) = \max\{s \mid \sigma \in \mathcal{R}^1_s(A, \Bbbk)\}$.

THEOREM (COHEN–ORLIK 2000, PAPADIMA–S. 2010)

$e_{p^s}(\mathcal{A}) \leqslant \beta_p(\mathcal{A})$, for all $s \geqslant 1$.

THEOREM

1. *Suppose $\mathcal{A}$ admits a k-net. Then $\beta_p(\mathcal{A}) = 0$ if $p \nmid k$ and $\beta_p(\mathcal{A}) \geqslant k-2$, otherwise.*
2. *If $\mathcal{A}$ admits a reduced k-multinet, then $e_k(\mathcal{A}) \geqslant k-2$.*

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COMBINATORICS AND MONODROMY

THEOREM (PAPADIMA–S. 2014)

Suppose $\mathcal{A}$ has no points of multiplicity $3r$ with $r > 1$. Then $\mathcal{A}$ admits a reduced 3-multinet iff $\mathcal{A}$ admits a 3-net iff $\beta_3(\mathcal{A}) \neq 0$. Moreover,

- $\beta_3(\mathcal{A}) \leqslant 2$.
- $e_3(\mathcal{A}) = \beta_3(\mathcal{A})$, *and thus* $e_3(\mathcal{A})$ *is combinatorially determined.*

COROLLARY (PS)

Suppose all flats $X \in L_2(\mathcal{A})$ have multiplicity 2 or 3. Then $\Delta(t)$, and thus $b_1(F(\mathcal{A}))$, are combinatorially determined.

THEOREM (PS)

Suppose $\mathcal{A}$ supports a 4-net and $\beta_2(\mathcal{A}) \leqslant 2$. Then

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CONJECTURE (PS)

Let $\mathcal{A}$ be an arrangement which is not a pencil. Then $e_{p^s}(\mathcal{A}) = 0$ for all primes p and integers $s \geqslant 1$, with two possible exceptions:

$$e_2(\mathcal{A}) = e_4(\mathcal{A}) = \beta_2(\mathcal{A}) \text{ and } e_3(\mathcal{A}) = \beta_3(\mathcal{A}).$$

If $e_d(\mathcal{A}) = 0$ for all divisors d of $|\mathcal{A}|$ which are not prime powers, this conjecture would give:

$$\Delta_{\mathcal{A}}(t) = (t-1)^{|\mathcal{A}|-1}((t+1)(t^2+1))^{\beta_2(\mathcal{A})}(t^2+t+1)^{\beta_3(\mathcal{A})}.$$

The conjecture has been verified for several classes of arrangements:

- Complex reflection arrangements (Măcinic–Papadima–Popescu).
- Certain types of real arrangements (Yoshinaga, Bailet, Torielli).
- Arrangements w/ connected multiplicity graph (Salvetti–Serventi).

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TORSION IN HOMOLOGY

THEOREM (COHEN–DENHAM–S. 2003)

For every prime $p \geqslant 2$, there is a multi-arrangement $(\mathcal{A}, m)$ such that $H_1(F_m(\mathcal{A}), \mathbb{Z})$ has non-zero p-torsion.

Simplest example: the arrangement of 8 hyperplanes in $\mathbb{C}^3$ with

$$Q_m(\mathcal{A}) = x^2y(x^2 - y^2)^3(x^2 - z^2)^2(y^2 - z^2)$$

Then $H_1(F_m(\mathcal{A}), \mathbb{Z}) = \mathbb{Z}^7 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

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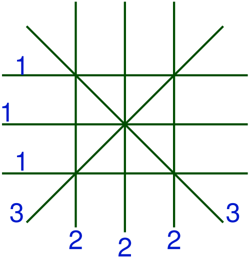

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We now can generalize and reinterpret these examples, as follows.

A *pointed multinet* on an arrangement $\mathcal{A}$ is a multinet structure, together with a distinguished hyperplane $H \in \mathcal{A}$ for which $m_H > 1$ and $m_H \mid n_X$ for each $X \in \mathcal{X}$ such that $X \subset H$.

THEOREM (DENHAM–S. 2014)

Suppose $\mathcal{A}$ admits a pointed multinet, with distinguished hyperplane H and multiplicity m. Let p be a prime dividing m_H. There is then a choice of multiplicities m' on the deletion $\mathcal{A}' = \mathcal{A} \backslash \{H\}$ such that $H_1(F_{m'}(\mathcal{A}'), \mathbb{Z})$ has non-zero p-torsion.

This torsion is explained by the fact that the geometry of $\mathcal{V}_1^1(M(\mathcal{A}'), \Bbbk)$ varies with $\operatorname{char}(\Bbbk)$.

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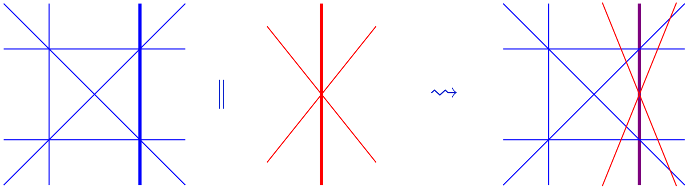
To produce p-torsion in the homology of the usual Milnor fiber, we use a “polarization" construction:



$(\mathcal{A}, m) \rightsquigarrow \mathcal{A} \| m$, an arrangement of $N = \sum_{H \in \mathcal{A}} m_H$ hyperplanes, of rank equal to $\operatorname{rank} \mathcal{A} + |\{H \in \mathcal{A} \colon m_H \geqslant 2\}|$.

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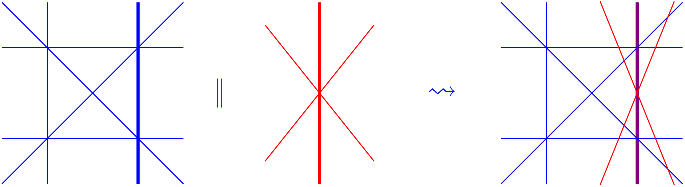

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COROLLARY (DS)

For every prime $p \geqslant 2$, there is an arrangement $\mathcal{A}$ such that $H_q(F(\mathcal{A}), \mathbb{Z})$ has non-zero p-torsion, for some $q > 1$.

Simplest example: the arrangement of 27 hyperplanes in $\mathbb{C}^8$ with

$$Q(\mathcal{A}) = xy(x^2-y^2)(x^2-z^2)(y^2-z^2)w_1w_2w_3w_4w_5(x^2-w_1^2)(x^2-2w_1^2)(x^2-3w_1^2)(x-4w_1)\cdot$$

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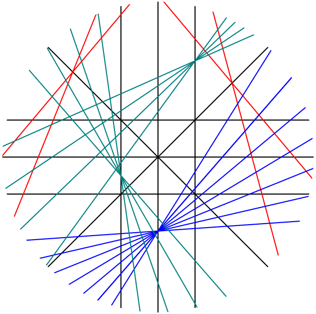

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THE FORMALITY PROBLEM

EXAMPLE (ZUBER 2010)

- Let $\mathcal{A}$ be the arrangement in $\mathbb{C}^3$ defined by
$$Q = (z_1^3 - z_2^3)(z_1^3 - z_3^3)(z_2^3 - z_3^3).$$
- The variety $\mathcal{R}^1(M) \subset \mathbb{C}^9$ has 12 local components (from triple points), and 4 essential components (from 3-nets).
- One of these 3-nets corresponds to the rational map $\mathbb{CP}^2 \dashrightarrow \mathbb{CP}^1$, $(z_1, z_2, z_3) \mapsto (z_1^3 - z_2^3, z_2^3 - z_3^3)$.
- This map can be used to construct a 4-dimensional subtorus $T = \exp(L)$ inside $\mathrm{Hom}(\pi_1(F(\mathcal{A})), \mathbb{C}^*) = (\mathbb{C}^*)^{12}$.
- The subspace $L \subset H^1(F(\mathcal{A}), \mathbb{C})$ is *not* a component of $\mathcal{R}^1(F(\mathcal{A}))$.
- Thus, the tangent cone formula is violated, and so the Milnor fiber $F(\mathcal{A})$ is not 1-formal.

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