HYPERPLANE ARRANGEMENTS AND COHOMOLOGY JUMP LOCI

Alex Suciu

Northeastern University

Workshop on Computational Geometric Topology in Arrangement Theory
ICERM, Brown University

July 6, 2015

1. HYPERPLANE ARRANGEMENTS
   - Complement and intersection lattice
   - Cohomology ring
   - Fundamental group
2. COHOMOLOGY JUMP LOCI
   - Characteristic varieties
   - Resonance varieties
   - The Tangent Cone theorem
3. JUMP LOCI OF ARRANGEMENTS
   - Resonance varieties
   - Multinets
   - Characteristic varieties
   - Propagation of jump loci

Hyperplane arrangements

- An *arrangement of hyperplanes* is a finite set $\mathcal{A}$ of codimension-1 linear subspaces in $\mathbb{C}^\ell$.
- *Intersection lattice* $L(\mathcal{A})$: poset of all intersections of $\mathcal{A}$, ordered by reverse inclusion, and ranked by codimension.
- *Complement*: $M(\mathcal{A}) = \mathbb{C}^\ell \setminus \bigcup_{H\in\mathcal{A}} H$.
- The Boolean arrangement $\mathcal{B}_n$
  - $\mathcal{B}_n$: all coordinate hyperplanes $z_i = 0$ in $\mathbb{C}^n$.
  - $L(\mathcal{B}_n)$: Boolean lattice of subsets of $\{0,1\}^n$.
  - $M(\mathcal{B}_n)$: complex algebraic torus $(\mathbb{C}^*)^n$.
- The braid arrangement $\mathcal{A}_n$ (or, reflection arr. of type A_{n-1})
  - $\mathcal{A}_n$: all diagonal hyperplanes $z_i - z_j = 0$ in $\mathbb{C}^n$.
  - $L(\mathcal{A}_n)$: lattice of partitions of $[n] = \{1,\dots,n\}$.
  - $M(\mathcal{A}_n)$: configuration space of n ordered points in $\mathbb{C}$ (a classifying space for P_n, the pure braid group on n strings).

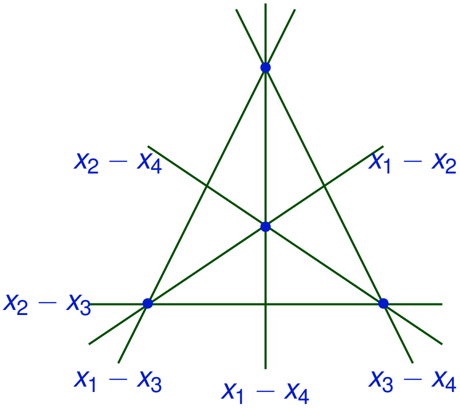


FIGURE : A planar slice of the braid arrangement $\mathcal{A}_4$

- We may assume that $\mathcal{A}$ is essential, i.e., $\bigcap_{H\in\mathcal{A}} H = \{0\}$.
- Fix an ordering $\mathcal{A} = \{H_1, \dots, H_n\}$, and choose linear forms $f_i \colon \mathbb{C}^\ell \to \mathbb{C}$ with $\ker(f_i) = H_i$. Define an injective linear map

$$\iota\colon \mathbb{C}^\ell \to \mathbb{C}^n, \quad z \mapsto (f_1(z), \dots, f_n(z)).$$

- This map restricts to an inclusion $\iota\colon M(\mathcal{A}) \hookrightarrow M(\mathcal{B}_n)$. Hence, $M(\mathcal{A}) = \iota(\mathbb{C}^\ell) \cap (\mathbb{C}^*)^n$, a "very affine" subvariety of $(\mathbb{C}^*)^n$, and thus, a Stein manifold.
- Therefore, $M = M(\mathcal{A})$ has the homotopy type of a connected, finite cell complex of dimension ℓ.
- In fact, M has a minimal cell structure (Dimca–Papadima, Randell, Salvetti, Adiprasito, …). Consequently, $H_*(M, \mathbb{Z})$ is torsion-free.

COHOMOLOGY RING

- The Betti numbers $b_q(M) := \operatorname{rank} H_q(M, \mathbb{Z})$ are given by

$$\sum_{q=0}^{\ell} b_q(M) t^q = \sum_{X \in L(\mathcal{A})} \mu(X)(-t)^{\operatorname{rank}(X)},$$

  with $\mu\colon L(\mathcal{A}) \to \mathbb{Z}$ given by $\mu(\mathbb{C}^\ell) = 1$ and $\mu(X) = -\sum_{Y \supsetneq X} \mu(Y)$.

- Let $E = \bigwedge(\mathcal{A})$ be the $\mathbb{Z}$-exterior algebra on degree-1 classes e_H dual to the meridians around the hyperplanes $H \in \mathcal{A}$.

- Let $\partial\colon E^\bullet \to E^{\bullet-1}$ be the differential given by $\partial(e_H) = 1$, and set $e_\mathcal{B} = \prod_{H \in \mathcal{B}} e_H$ for each $\mathcal{B} \subset \mathcal{A}$.

- The cohomology ring $H^*(M(\mathcal{A}), \mathbb{Z})$ is isomorphic to the Orlik–Solomon algebra $A(\mathcal{A}) = E/I$, where

$$I = \operatorname{ideal}\left\langle \partial e_\mathcal{B} \,\middle|\, \operatorname{codim} \bigcap_{H \in \mathcal{B}} H < |\mathcal{B}| \right\rangle.$$

FUNDAMENTAL GROUP

- Given a generic projection of a generic slice of $\mathcal{A}$ in $\mathbb{C}^2$, the fundamental group $\pi = \pi_1(M(\mathcal{A}))$ can be computed from the resulting braid monodromy $\alpha = (\alpha_1, \dots, \alpha_s)$, where $\alpha_r \in P_n$.
- π has a (minimal) finite presentation with
  - Meridional generators $x_1, \dots, x_n$, where $n = |\mathcal{A}|$.
  - Commutator relators $x_i \alpha_j (x_i)^{-1}$, where each α_j acts on F_n via the Artin representation.
- Let $\pi/\gamma_k(\pi)$ be the $(k-1)^{\text{th}}$ nilpotent quotient of π. Then:
  - $\pi_{\text{ab}} = \pi/\gamma_2$ equals $\mathbb{Z}^n$.
  - π/γ_3 is determined by $A^{\leqslant 2}(\mathcal{A})$, and thus by $L_{\leqslant 2}(\mathcal{A})$.
  - π/γ_4 (and thus, π) is *not* determined by $L(\mathcal{A})$. (Rybnikov).

CHARACTERISTIC VARIETIES

- Let X be a connected, finite cell complex, and let $\pi = \pi_1(X, x_0)$.
- Let $\Bbbk$ be an algebraically closed field, and let $\mathrm{Hom}(\pi, \Bbbk^*)$ be the affine algebraic group of $\Bbbk$-valued, multiplicative characters on π.
- The *characteristic varieties* of X are the jump loci for homology with coefficients in rank-1 local systems on X:

$$\mathcal{V}_s^q(X, \Bbbk) = \{\rho \in \mathrm{Hom}(\pi, \Bbbk^*) \mid \dim_{\Bbbk} H_q(X, \Bbbk_\rho) \geqslant s\}.$$

Here, $\Bbbk_\rho$ is the local system defined by ρ, i.e, $\Bbbk$ viewed as a $\Bbbk\pi$-module, via $g \cdot x = \rho(g)x$, and $H_i(X, \Bbbk_\rho) = H_i(C_*(\widetilde{X}, \Bbbk) \otimes_{\Bbbk\pi} \Bbbk_\rho)$.

- These loci are Zariski closed subsets of the character group.
- The sets $\mathcal{V}_s^1(X, \Bbbk)$ depend only on π/π''.

EXAMPLE (CIRCLE)

We have $\widetilde{S^1} = \mathbb{R}$. Identify $\pi_1(S^1, *) = \mathbb{Z} = \langle t \rangle$ and $\Bbbk\mathbb{Z} = \Bbbk[t^{\pm 1}]$. Then:

$$C_*(\widetilde{S^1}, \Bbbk) : \ 0 \longrightarrow \Bbbk[t^{\pm 1}] \xrightarrow{t-1} \Bbbk[t^{\pm 1}] \longrightarrow 0 \ .$$

For $\rho \in \mathrm{Hom}(\mathbb{Z}, \Bbbk^*) = \Bbbk^*$, we get

$$C_*(\widetilde{S^1}, \Bbbk) \otimes_{\Bbbk\mathbb{Z}} \Bbbk_\rho : \ 0 \longrightarrow \Bbbk \xrightarrow{\rho-1} \Bbbk \longrightarrow 0 \ ,$$

which is exact, except for $\rho = 1$, when $H_0(S^1, \Bbbk) = H_1(S^1, \Bbbk) = \Bbbk$.
Hence: $\mathcal{V}^0_1(S^1, \Bbbk) = \mathcal{V}^1_1(S^1, \Bbbk) = \{1\}$ and $\mathcal{V}^i_s(S^1, \Bbbk) = \emptyset$, otherwise.

EXAMPLE (PUNCTURED COMPLEX LINE)

Identify $\pi_1(\mathbb{C}\backslash\{n \text{ points}\}) = F_n$, and $\widehat{F_n} = (\Bbbk^*)^n$. Then:

$$\mathcal{V}^1_s(\mathbb{C}\backslash\{n \text{ points}\}, \Bbbk) = \begin{cases} (\Bbbk^*)^n & \text{if } s < n, \\ \{1\} & \text{if } s = n, \\ \emptyset & \text{if } s > n. \end{cases}$$

RESONANCE VARIETIES

- Let $A = H^*(X, \Bbbk)$, where $\operatorname{char} \Bbbk \neq 2$. Then: $a \in A^1 \Rightarrow a^2 = 0$.
- We thus get a cochain complex

$$(A, \cdot a)\colon \ A^0 \xrightarrow{\ a\ } A^1 \xrightarrow{\ a\ } A^2 \longrightarrow \cdots .$$

- The *resonance varieties* of X are the jump loci for the cohomology of this complex

$$\mathcal{R}^q_s(X, \Bbbk) = \{a \in A^1 \mid \dim_{\Bbbk} H^q(A, \cdot a) \geqslant s\}$$

- E.g., $\mathcal{R}^1_1(X, \Bbbk) = \{a \in A^1 \mid \exists b \in A^1,\ b \neq \lambda a,\ ab = 0\}$.
- These loci are *homogeneous* subvarieties of $A^1 = H^1(X, \Bbbk)$.

EXAMPLE

- $\mathcal{R}^1_1(T^n, \Bbbk) = \{0\}$, for all $n > 0$.
- $\mathcal{R}^1_1(\mathbb{C} \backslash \{n \text{ points}\}, \Bbbk) = \Bbbk^n$, for all $n > 1$.

THE TANGENT CONE THEOREM

- Given a subvariety $W \subset (\mathbb{C}^*)^n$, let $\tau_1(W) = \{z \in \mathbb{C}^n \mid \exp(\lambda z) \in W,\ \forall \lambda \in \mathbb{C}\}$.
- (Dimca–Papadima–S. 2009) $\tau_1(W)$ is a finite union of rationally defined linear subspaces, and $\tau_1(W) \subseteq \mathrm{TC}_1(W)$.
- (Libgober 2002/DPS 2009)
$$\tau_1(\mathcal{V}^i_s(X)) \subseteq \mathrm{TC}_1(\mathcal{V}^i_s(X)) \subseteq \mathcal{R}^i_s(X).$$
- (DPS 2009/DP 2014): Suppose X is a k-formal space. Then, for each $i \leqslant k$ and $s > 0$,
$$\tau_1(\mathcal{V}^i_s(X)) = \mathrm{TC}_1(\mathcal{V}^i_s(X)) = \mathcal{R}^i_s(X).$$
- Consequently, $\mathcal{R}^i_s(X, \mathbb{C})$ is a union of rationally defined linear subspaces in $H^1(X, \mathbb{C})$.

JUMP LOCI OF ARRANGEMENTS

Work of Arapura, Falk, D.Cohen–A.S., Libgober–Yuzvinsky, and Falk–Yuzvinsky completely describes the resonance varieties $\mathcal{R}_s(\mathcal{A}) = \mathcal{R}^1_s(M(\mathcal{A}), \mathbb{C})$:

- $\mathcal{R}_1(\mathcal{A})$ is a union of linear subspaces in $H^1(M(\mathcal{A}), \mathbb{C}) \cong \mathbb{C}^{|\mathcal{A}|}$.
- Each subspace has dimension at least 2, and each pair of subspaces meets transversely at 0.
- $\mathcal{R}_s(\mathcal{A})$ is the union of those linear subspaces that have dimension at least $s+1$.
- Each k-*multinet* on a sub-arrangement $\mathcal{B} \subseteq \mathcal{A}$ gives rise to a component of $\mathcal{R}_1(\mathcal{A})$ of dimension $k-1$. Moreover, all components of $\mathcal{R}_1(\mathcal{A})$ arise in this way.

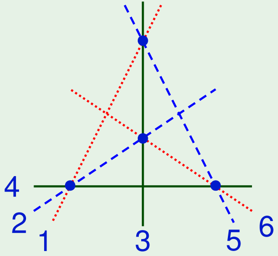
DEFINITION (FALK AND YUZVINSKY)

A *multinet* on $\mathcal{A}$ is a partition of the set $\mathcal{A}$ into $k \geqslant 3$ subsets $\mathcal{A}_1, \dots, \mathcal{A}_k$, together with an assignment of multiplicities, $m\colon \mathcal{A} \to \mathbb{N}$, and a subset $\mathcal{X} \subseteq L_2(\mathcal{A})$, called the base locus, such that:

1. $\exists\ d \in \mathbb{N}$ such that $\sum_{H\in\mathcal{A}_\alpha} m_H = d$, for all $\alpha \in [k]$.
2. If H and H' are in different classes, then $H \cap H' \in \mathcal{X}$.
3. $\forall\ X \in \mathcal{X}$, the sum $n_X = \sum_{H\in\mathcal{A}_\alpha : H\supset X} m_H$ is independent of α.
4. Each set $\left(\bigcup_{H\in\mathcal{A}_\alpha} H\right)\setminus\mathcal{X}$ is connected.

- A multinet as above is also called a (k, d)-multinet, or k-multinet.
- The multinet is *reduced* if $m_H = 1$, for all $H \in \mathcal{A}$.
- A *net* is a reduced multinet with $n_X = 1$, for all $X \in \mathcal{X}$.

EXAMPLE (BRAID ARRANGEMENT $\mathcal{A}_4$)



$\mathcal{R}_1(\mathcal{A}) \subset \mathbb{C}^6$ has 4 local components (from the triple points), and one essential component, from the above $(3,2)$-net:

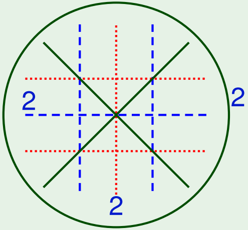
$$L_{124} = \{x_1 + x_2 + x_4 = x_3 = x_5 = x_6 = 0\},$$
$$L_{135} = \{x_1 + x_3 + x_5 = x_2 = x_4 = x_6 = 0\},$$
$$L_{236} = \{x_2 + x_3 + x_6 = x_1 = x_4 = x_5 = 0\},$$
$$L_{456} = \{x_4 + x_5 + x_6 = x_1 = x_2 = x_3 = 0\},$$
$$L = \{x_1 + x_2 + x_3 = x_1 - x_6 = x_2 - x_5 = x_3 - x_4 = 0\}.$$

- Let $\mathrm{Hom}(\pi_1(M(\mathcal{A})), \mathbb{C}^*) = (\mathbb{C}^*)^n$ be the character torus.
- The characteristic variety $\mathcal{V}_1(\mathcal{A}) := \mathcal{V}_1^1(M(\mathcal{A}), \mathbb{C})$ lies in the substorus $\{t \in (\mathbb{C}^*)^n \mid t_1 \cdots t_n = 1\}$.
- $\mathcal{V}_1(\mathcal{A})$ is a finite union of torsion-translates of algebraic subtori of $(\mathbb{C}^*)^n$.
- If a linear subspace $L \subset \mathbb{C}^n$ is a component of $\mathcal{R}_1(\mathcal{A})$, then the algebraic torus $T = \exp(L)$ is a component of $\mathcal{V}_1(\mathcal{A})$.
- All components of $\mathcal{V}_1(\mathcal{A})$ passing through the origin $1 \in (\mathbb{C}^*)^n$ arise in this way (and thus, are combinatorially determined).
- In general, though, there are translated subtori in $\mathcal{V}_1(\mathcal{A})$.

(Denham–S. 2014)

- Suppose there is a multinet $\mathcal{M}$ on $\mathcal{A}$, and there is a hyperplane H for which $m_H > 1$ and $m_H \mid n_X$ for each $X \in \mathcal{X}$ such that $X \subset H$.
- Then $\mathcal{V}_1(\mathcal{A} \setminus \{H\})$ has a component which is a 1-dimensional subtorus, translated by a character of order m_H.

EXAMPLE (THE DELETED B_3 ARRANGEMENT)



The B_3 arrangement supports a $(3,4)$-multinet; $\mathcal{X}$ consists of 4 triple points ($n_X = 1$) and 3 quadruple points ($n_X = 2$). So pick H with $m_H = 2$ to get a translated torus in $\mathcal{V}_1(\mathrm{B}_3 \setminus \{H\})$.

PROPAGATION OF CJLS

(Denham–S.–Yuzvinsky 2014/15)

- Suppose X is an *abelian duality space* of dimension n, i.e., $H^p(X, \mathbb{Z}\pi_{\text{ab}}) = 0$ for $p \neq n$ and $H^n(X, \mathbb{Z}\pi_{\text{ab}}) \neq 0$ and torsion-free.
- Let $B = H^n(X, \mathbb{Z}\pi_{\text{ab}})$ be the dualizing $\mathbb{Z}\pi_{\text{ab}}$-module. Given any $\mathbb{Z}\pi_{\text{ab}}$-module A, we have $H^i(X, A) \cong H_{n-i}(X, B \otimes A)$.
- Let $\rho\colon \pi \to \mathbb{C}^*$ be a character. If $H^p(X, \mathbb{C}_\rho) \neq 0$, then $H^q(X, \mathbb{C}_\rho) \neq 0$ for all $p \leqslant q \leqslant n$.
- Thus, the characteristic varieties of X "propagate":
$$\mathcal{V}_1^1(X) \subseteq \mathcal{V}_1^2(X) \subseteq \cdots \subseteq \mathcal{V}_1^n(X).$$
- If morever X admits a minimal cell structure (or X is formal), then
$$\mathcal{R}_1^1(X) \subseteq \mathcal{R}_1^2(X) \subseteq \cdots \subseteq \mathcal{R}_1^n(X).$$

- Let $\mathcal{A}$ be an arrangement of rank ℓ. Then its complement, $M(\mathcal{A})$, is an abelian duality space of dimension ℓ.
- Recall $M(\mathcal{A})$ is minimal (and formal). Thus, both the characteristic and the resonance varieties of $M(\mathcal{A})$ propagate.
- Propagation of resonance for arrangement complements was first established by Eisenbud–Popescu–Yuzvinsky, with further results by Budur.